Assignment 3

1) Use the same diagonalization argument as the time hierarchy theorem. Just note that an $f(n)$-space machine can be simulated in $g(n)$ space for any $g(n) = o(f(n))$. (There is no $(\log n)$-less, as in the case of time simulations.)

2) We just need to check the distance condition for all pairs s.t. The logspace machine can iterate over these pairs. Starting from s, the machine can try out every possible path of length 6, using $6(\log n)$ space to store the path.

3) Given 2-SAT instance $\Phi$, construct $G_\Phi$ as follows.

   Create vertices $x_i$, $\overline{x_i}$ for each variable.
   For clause $y_a \lor y_b$, create edges $\overline{y_a} \rightarrow y_b$ and $\overline{y_b} \rightarrow y_a$ (so for clause $x_i$, we create edges $\overline{x_i} \rightarrow \overline{x_i}$ and $x_i \rightarrow x_i$.)

$G_\Phi$ is the graph of implications. If $\exists$ a path from s to t, that means: setting literal $i$ to true will imply setting t to true (in any valid assignment).

Consider strongly connected components of $G_\Phi$.

Claim: $\Phi$ is satisfiable iff $\forall i$, $x_i$ and $\overline{x_i}$ are not in the same strongly connected component.

Proof: Suppose for some i, $x_i$ and $\overline{x_i}$ are in the same component. Then, regardless of whether we set $x_i$ to true or false, we get a contradiction. So $\Phi$ is not satisfiable.
Suppose $x_i$, $x_i$, and $\overline{x}_i$ are not in the same connected component. We show $\Phi$ is satisfiable by induction on the number of clauses $m$.

(If $m=1$, claim is trivially true.)

Induction step: take the DAG of strongly connected components, and consider some sink. This is a set of strongly connected vertices, such that $\forall v \in S$, $S$ no vertex in $S$ has a path to $v$.

Let all literals in $S$ to true. Observe that all clauses involving any "variable" corresponding to these literals is satisfied. (Why?) Suppose $x_i \in S$, so $x_i$ is set to true. Any clause with $x_i$ is satisfied. If there is a clause $\overline{x}_i \lor x_j$, then there is an edge $x_i \rightarrow \overline{x}_j$. So $\overline{x}_j$ is also in $S$, and is set to true.

Now remove all variables corresponding to $S$ and all clauses involving these variables (which are all true). We have a "and" 2-SAT instance $\Phi$. If $\Phi$ is satisfiable, $\Phi$ is satisfiable. By the inductive hypothesis, $\Phi$ is satisfiable.

So $\Phi \in 2SAT$ iff $\forall i \left[ \langle G, x_i, \overline{x}_i \rangle \land PATH \lor \langle G, \overline{x}_i, x_i \rangle \land PATH \right]$

Given $\Phi$, we can construct $G$ is logspace.

Gloriously (most of you got this wrong)

$PATH \in NL$ by Immnerman-Szelepcsenyi's Theorem.

So the problem $\forall i \left[ \langle G, x_i, \overline{x}_i \rangle \land PATH \lor \langle G, \overline{x}_i, x_i \rangle \land PATH \right]$ is in $NL$
4) The easiest proof is to go back to our reduction of $\text{SAT} \leq_p \text{3SAT}$.

Basically, given any formula $\Phi(x)$, we constructed $\Phi(x,y)$
\begin{align*}
\text{s.t. } & \Phi(x) \text{ is true } \iff \exists y \big[ \Phi(x,y) \text{ is true}\big] \\
(\text{Think of } x \text{ as some fixed assignment.})
\end{align*}

in poly time

So
\begin{align*}
Q_1 x_1, Q_2 x_2 & \iff Q_n x_n \overline{\Phi(x)} \\
\iff Q_1 x, Q_2 x & \iff Q_n x \forall y, \exists y - \exists y \Phi(x,y)
\end{align*}