$\boxed{\mathbb{EXP}}$ & $\mathbb{NEXP}$

$\mathbb{P}$ vs $\mathbb{NP}$ $\qquad$ $\mathbb{NP}$ vs co-$\mathbb{NP}$

$$\mathbb{EXP} = \bigcup_{c \geq 1} \mathrm{DTIME}(2^{n^c}) \qquad \mathbb{E} = \bigcup_{c\geq 1} \mathrm{DTIME}(2^{cn})$$

$$\mathbb{NEXP} = \bigcup_{c \geq 1} \mathrm{NTIME}(2^{n^c}) \qquad \mathcal{L} \overset{f}{\leq_p} \mathcal{L}'$$

$|x|$ $\quad$ $|f(x)| = \mathrm{poly}(|x|)$

<u>Clm</u>: $\mathbb{NP} \subseteq \mathbb{EXP}$

<u>Proof idea</u>: For $\mathcal{L} \in \mathbb{NP}$, we have an exponential time algorithm that simply tries all certificates.

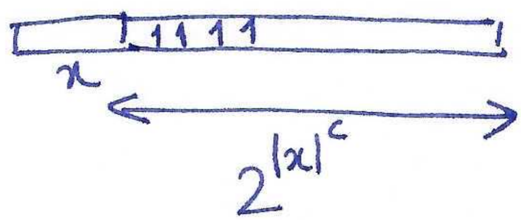
<u>Thm</u>: If $\mathbb{EXP} \neq \mathbb{NEXP}$, then $\mathbb{P} \neq \mathbb{NP}$.

<u>Proof</u>: Padding argument.

Assume $\mathbb{P} = \mathbb{NP}$. Consider $\mathcal{L} \in \mathbb{NEXP}$.

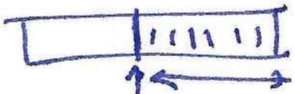
$\exists c \geq 1$, $\mathcal{L} \in \mathrm{NTIME}(2^{n^c})$. There is an NTM M that runs in $O(2^{n^c})$ time and decides $\mathcal{L}$.

$$\mathcal{L}_{pad} = \left\{ \langle x, 1^{2^{|x|^c}} \rangle \mid x \in \mathcal{L} \right\}$$



Let us design a NTM M' that decides $\mathcal{L}_{pad}$.

M' (on input w)



1. Checks if $w = \langle x, 1^{2^{|x|^c}} \rangle$. If not, REJECT. $O(n^2)$

simulation

2. Runs M on x, and ~~fl~~ follow output. $O(n)$

Q. What is running time of M'?

✓(A) $O(n^2)$ (B) poly(n) (C) $O(2^{n^c})$

$n = |w|$

$L_{pad} \in NP$. We assume $P = NP$

Therefore, $L_{pad} \in P$. Let us show that $L \in EXP$.

$L_{pad} = \{ \langle x, 1^{2^{|x|^c}} \rangle \mid x \in L \}$

$\exists$ polytime TM N that decides L_{pad}.

Consider N' (on input x)

1. Constructs $\langle x, 1^{2^{|x|^c}} \rangle$.

2. Feeds this to N and follows output.

N' runs in poly(2^{n^c}) time, so $L \in EXP$.

$$P = NP \Longrightarrow EXP = NEXP \qquad \blacksquare$$

Time Hierarchy Theorem

$\mathrm{DTIME}(n^c) \subsetneq \mathrm{DTIME}(n^{c+1})$

Thm: Let $f, g : \mathbb{N} \to \mathbb{N}$ be time constructible.

such that $\lim_{n\to\infty} \frac{f(n)\log f(n)}{g(n)} \to 0$ $\left(f(n)\log f(n) = o(g(n)) \right)$

$\boxed{\forall n \quad f(n)\log f(n) < g(n)}$

then $\mathrm{DTIME}(f(n)) \subsetneq \mathrm{DTIME}(g(n))$

Proof: Diagonalization!

~~For any string x~~ Construct machine M

M (on input $\langle N \rangle$)

M is like the MOST efficient TM for L(M).

(1) Simulate N on $\langle N \rangle$ (using efficient simulation of Hennie-Stearns) for $g(n)$ steps

$\mathcal{U}_{TM}$

$n = |\langle N \rangle|$

(2) If simulation halts, flip output.

Else, reject.

$L(M) \in \mathrm{DTIME}(g(n))$

(We will show that $L(M) \notin \mathrm{DTIME}(f(n))$).

Suppose, for contradiction's sake, that

$L(M) \in \mathrm{DTIME}(f(n))$, decided by TM M'.

M' (on input x) halts in $f(|x|)$ time

Consider some M'' that is equivalent to M' but has large enough encoding length to ensure that ~~$f(x$~~ $f(|\langle M''\rangle|)\log(|\langle M''\rangle|) < g(|\langle M''\rangle|)$

M ~~is equivalent to~~ M'. M' is equivalent to M''.

$\boxed{L(M) = L(M')}$

M runs in $g(n)$ $\qquad$ M' runs in $f(n)$

$$L(M'') = L(M') = L(M)$$

Run $M(\underbrace{\langle M''\rangle}_{n})$ and see what happens.

$M''(\langle M''\rangle)$ runs in $f(|\langle M''\rangle|)$ time

The simulation (by HS) runs in $f(|\langle M''\rangle|)\log(\ldots) < g(n)$.

So the simulation halts.

Output of $M(\langle M''\rangle)$ is the opposite of $M''(\langle M''\rangle)$.

Contradiction! So $L(M) \notin \mathrm{DTIME}(f(n))$. $\blacksquare$

<u>Thm</u>: $\mathbb{P} \neq \mathbb{EXP}$

$\bigcup_{c \geq 1} \mathrm{DTIME}(n^c)$

$g(n) = 2^n$

$\forall c \quad \mathrm{DTIME}(n^c) \subsetneq \mathrm{DTIME}(2^n)$

$\not\Rightarrow \bigcup_{c \geq 1} \mathrm{DTIME}(n^c) \subsetneq \mathrm{DTIME}(2^n)$

$\mathbb{P} \subseteq \mathbb{NP} \subseteq \mathbb{EXP}$

$\mathbb{P} \neq \mathbb{EXP}$

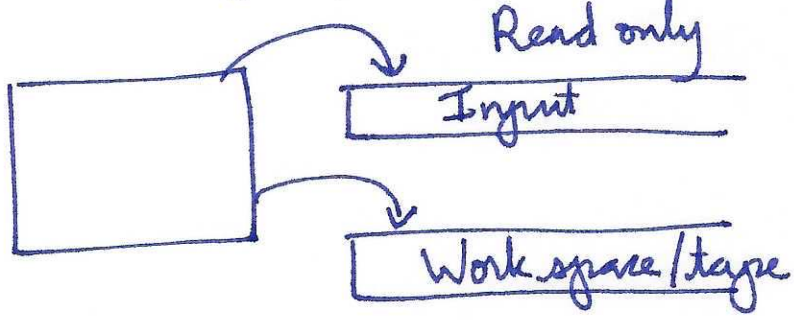
so ~~either~~ $\mathbb{P} \neq \mathbb{NP}$ or $\mathbb{NP} \neq \mathbb{EXP}$.

(We think both.)

Space complexity

To define space complexity, we will use more tapes

(Input tape : read only
work tape : usual
Output tape : final result)



Def: Let $s : \mathbb{N} \to \mathbb{N}$ be space-constructible.
$SPACE(s(n))$ is the class of languages decided by TMs using at most $\underbrace{c}_{\text{constant}} \cdot s(n)$ work space.

$s(n)$ can be less than n !

$$\mathbb{PSPACE} = \bigcup_{c \geq 1} SPACE(n^c)$$

$$\mathbb{NPSPACE} = \bigcup_{c \geq 1} NSPACE(n^c)$$

$\mathbb{L} = SPACE(\log n)$ $\quad$ $\mathbb{NL} = NSPACE(\log n)$

You need $O(\log n)$ space just to write down the input length.

In $< \log n$ space, it's hard to even decide very simple language.