CMPS290A: Sublinear algorithms for graphs **Spring 2017**

Lecture 10: 5/4/2017

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NB *These notes only cover the second part of lecture 10*

10.1 Proof the general tester for monotonicity on the hypercube works

Lemma 10.1 $\sum_{1=d}^{\infty} \varepsilon_f, i = \Omega(\varepsilon_f)$

We seek to prove that the average distance of the line is at least ε_f. To do this we will implement the sorting operator $S_i(g)$. This sorts g along every $i - line$.

Lemma 10.2 *Suppose g is monotone along dimension $1, \ldots, i-1$.*
Examine $S_i(g)$

1. *$S_i(g)$ is monotone along $1, \ldots, i$*
2. *For $j > i, \varepsilon_{S_{i(g)},j} \leq \varepsilon_{g,j}$*

In plain English, that the average distance along the j line in $S_{i(g)}$ is less than or equal to the average distance along the j line in g.

Sorting in the i-th dimension will not make things worse in the j-th dimension.

Now we must prove the correctness of the above lemma. We will do this by looking at each of the two parts individually. First we will prove that $S_i(g)$ is monotone along $1, \ldots, i$

Proof: Our goal is to sort columns in a $n \times n$ matrix and observe the subsequent changes in the rows. To do this we will use the following technique,

1. Select a sorting algorithm such as bubble sort
2. We only need to take two adjacent rows $(r, r+1)$ and apply bubble sort. This gives us a $2 \times n$ sub-matrix
3. Sort columns in the sub-matrix and prove lemma sub properties hold

■

This allows us to concentrate on a simpler $2 \times n$ sub-matrix where there are only four possible choices, rather than a more complex $n \times n$ matrix.

The case analysis is left as an exercise.

We also still need to prove the second part of the lemma, "For $j > i, \varepsilon_{S_{i(g)},j} \leq \varepsilon_{g,j}$".

Proof:
Consider that we have two rows, r and $r+1$. We will then sort the columns.

The functions are initially f_r and f_{r+1}. After the transformation the functions become f'_r and f'_{r+1}.

We want to prove that,

$$\varepsilon_{f'_r} + \varepsilon_{f'_{r+1}} \leq \varepsilon_{f_r} + \varepsilon_{f'_{r+1}}$$

Let A be the to closest monotone function to f_r, B the closest monotone function to f_{r+1}. We will also mark the point where the zeros change to ones as a and b respectively.

Suppose,

$$dist(f'_r, A) + dist(f'_{r+1}, B) \leq dist(f_r, A) + dist(f_{r+1}, B)$$

By definition, $dist(f_r, A) + dist(fr+1, B) = \varepsilon_{f_r} + \varepsilon_{f_{r+1}}$ and $dist(f'_r, A) + dist(f'_{r+1}, B) \leq \varepsilon_{f'_r} + \varepsilon_{f'_{r+1}}$ since this is the maximum distance to monotonicity. ■

Case analysis:
If $a < b$, then $dist(f'_r, A) + dist(f_{r+1}, B) \leq dist(f_r, A) + dist(f'_{r+1}, B)$
If $a > b$, then $dist(f'_r, B) + dist(f_{r+1}, A) \leq dist(f_r, B) + dist(f'_{r+1}, B)$

References

[GGLRS00] O. Goldreich, S. Goldwasser, E. Lehman, et al., Testing Monotonicity, *Combinatorica* **20** (2000), pp. 301–337.

[DGL+00] Y. Dodis, O. Goldreich, E. Lehman, et al. Improved Testing Algorithms for Monotonicity, (2000), pp. 1-19.