CMPS 290 A: Sublinear algorithms for graphs **Spring 2017**

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3.1 Yao's Minimax Principle for Property Testing

Any monotonicity tester for fn function $f : [n] \to N$ requires $\Omega(logn/\ \epsilon)$ queries.
Where,
Algorithm Upper bound : One algorithm sided, non-adaptive.
Algorithm Lower bound : Two-sided, adaptive.

Yao's minimax principle is a generic tool for proving lower bounds on randomized algorithms. By proving lower bounds of deterministic algorithms we prove lower bounds of randomized algorithms.The core idea of Yao's principle comes from Von Neumann's theorem. which states that:

Theorem 3.1 *for any two person zero sum game specified by a matrix T -*

$$\min_p \max_q p^\intercal T q = \max_q \min_p p^\intercal T q$$

where,
$p \to$ probability distribution over rows of T, representing mixed strategy.
$q \to$ probability distribution given over columns of T
$T \to$ Pay off matrix of game.

Given Von Neumann's theorem and employing observations from Loomi's theorem we adapt to Yao's min-max principle.

If p and q represent probability distributions over the rows and columns, respectively of the pay off matrix T, then for a fixed p, the choice of q that maximises $p^\intercal T q$ will be a pure strategy defined by e_i which always chooses the same column i.

Similarly for a fixed q the choice of p that minimises $p^\intercal T q$ will be developed by e_j This observation implies the following theorem of Loomi's :

$$\min_p \max_i p^\intercal T e_i = \max_q \min_j e_j^\intercal T q$$

Defining the pay off matrix where for $A \in \mathcal{A} and x \in I$, the entry T(x, A) is the run time of A o x.

Note : T is the pay off matrix of a two player zero sum game and thus we may apply Loomi's theorem to T and this yields Yao's mi-max principle.

$$\max_q \min_A \mathbf{1}_A^\intercal T q \geq \min_A \mathbf{1}_A^\intercal T q \geq \tau$$

3.1.1 Zero-Sum Game

From game theory a zero-sum game is a mathematical representation of a situation in which each participant's gain or loss of utility is exactly balanced by the losses or gains of the utility of the other participants.

p_i Using principles from game theory and and Yao's minimax principle to view computaion we can use proving lower bounds of deterministic algorithms to prove lower bounds of randomised algorithms.
We understand that a randomised algorithm is basically a distribution over a deterministic algorith Label the vertices $1, 2, \ldots, n$. Define $d^{(k)}(i, j)$ to be the length of a shortest path from i to j, using intermediate vertices from $\{1, 2, \ldots, k\}$ only. Obviously, $d^{(n)}(i, j)$ is the full problem.

3.1.2 Yao's min-max principle:

For all distributions p over $\mathscr{A}$ and q over I and fixed elements $X \in I$, $A \in \mathscr{A}$

$$\min_p \max_x p^\intercal T \mathbf{1}_x = \max_q \min_A \mathbf{1}_A^\intercal T q$$

where $\mathbf{1}_x$ and $\mathbf{1}_A$represent the pure strategies corresponding to x and A respectively.

If there exists a distribution q^* over inputs such that every deterministic algoritm requires τ expected runtime on input chosen from q^* then:

$$\max_q \min_A \mathbf{1}_A^\intercal T q \geq \min_A \mathbf{1}_A^\intercal T q^* \geq \tau$$

Therefore by Yao's min-max principle :

$$\min_p \max_x p^\intercal T \mathbf{1}_x \geq \tau$$

3.1.3 Yao's min-max promciple for property testing

Recall the following definition of testable properties -

Property P is testable if there exists a randomized algorithm and a function $q : R \to N$ such that $\forall$ $\epsilon \in$ (0,1) and all inputs f:D→R, A(f) makes q($\epsilon|D|$) queries on f where q($\epsilon|D|$) = o($|D|$) and

- if f $\in$ P then Pr [A(f) accepting] >2/3
- if dist (f,P)¿ ϵ, then Pr[A(f) rejecting] >2/3.

We now state Yao's principle for property testing :

Theorem 3.2 *For a fix property P and ϵ >0, suppose there exists distribution $\mathscr{Y}$ and $\mathscr{N}$ on functions such that :*

1. $Pr_{f\sim\mathscr{Y}}[f \in P] = 1$
2. $Pr_{f\sim\mathscr{Y}}[fis\epsilon farfromP] = 1$

Defining the distribution I to be f $\mathscr{Y}$ *with probability 1/2 and f* $\mathscr{N}$ *with probability 1/2.*

Suppose for all q deterministic algorithms A - $Pr_{f\sim l}[Aerrsonf]$>1/3*, then there is no q query property tester for proximity parameter* ϵ*.*

Proof:

Consider a q-query randomized algorithm $\mathscr{A}$ ($\mathscr{A}$ is a distribution over deterministic algorithms).
For all q-query algorithms A -

$$\mathbb{E}_{f\sim d}[\chi(A, f)] > 1/3$$

$$\mathbb{E}_{A\sim\mathscr{A}}[\mathbb{E}_{f\sim d}[X(A, f)]] > 1/3$$

$\chi(A, f)$ indicates for A being wrong on f .

$$\mathbb{E}_{f\sim p}[\mathbb{E}_{A\sim\mathscr{A}}[\chi(A, f)]] > 1/3$$

$$\mathbb{E}_{f\sim d}[Pr[A\text{is correct on f}] > 1/3$$

Therefore, $\exists$ f s.t $Pr[A\text{is incorrect on}f] > 1/3$.

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References

[Yao77] Yao's min-max principle

[F76] M. L. Fredman, New Bounds on the Complexity of the Shortest Path Problem, *SIAM Journal on Computing* **5** (1976), pp. 83-89.