**CMPS290A: Sublinear algorithms for graphs** **Spring 2017**

# Lecture 1: 4/13/17

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## 1.1 The Domain: Posets and DAGS

A Monotonic function is any strictly increasing function who's domain is a Poset (Partially Ordered Set). Partially ordered sets have many representations. We'll be using DAGs (Directional Acyclic Graphs) because they give us strong intuition in many cases.

Given any two elements $u$ and $v$ in a DAG one of three things is true.

1. there is a path from $u$ to $v$
2. there is a path from $v$ to $u$
3. there is no path between them

If 1 is true than we say that $u$ is less than $v$ or $u < v$.
If 2 is true than we say that $v$ is less than $u$ or $v < u$.
If 3 is true than we say $u$ and $v$ are incomparable or $v \nsim u$.

### 1.1.1 Special Posets

There are three special classes of Posets that deserve special mention.

The first of these are referred to as the "Lines" or the Totally Ordered Sets. The name "Line" comes from this Posets's DAG which resembles a line. The first element has one exiting arrow, all intermediate elements have 1 entering and 1 exiting arrow, and the final element has a single entering arrow. All elements are connected and are comparable. This is why it also referred to as the Totally Ordered set, for any two elements one is greater and one is lesser. A classic line would be the natural numbers less than $n$ and would be written as $[n]$.

The second of these are the Hypercube(s). They are often represented as bit-strings and have bitwise or coordinate-wise partial order. Formally the hypercube $d$ is the set $\{0,1\}^d$. The ordering is defined by $x < y$: if $\forall i \leq d, x_i \leq y_i$. $x \in \{0,1\}^d$. Intuitively: if $x$ is less than $y$, than $y$ is $x$ with additional 1s. An element of Hypercube $d$ can also be thought of as a subset of $[d]$. The hypercubes' ordering is sometimes called order by containment because a greater element $u$ will contain every element of $d$ that a lesser element $v$ does.

The third is are Hyper-grids $[n]^d$, they like the hyper cube have coordinate-wise partial order. Formally the hyper-grid $d$ is the set of $d$-tuples of elements of $[n]$. The ordering is defined by $x < y$: if $\forall i \leq d, x_i \leq y_i$. As the name implies a member of the hyper grid can be thought of as a point in the $d$-dimension Cartesian product space of $\mathbb{N}$. Element $u$ is greater than all points $v$ that have lesser components for each dimension.

## 1.2 Monotonic functions

$f : D \to R$ is monotone or monotonic if $\forall x < y : f(x) \leq (y)$. In other words, monotonic functions are always increasing.

### 1.2.1 Monotonic functions on the Special Posets

The range of a Monotonic function on a line indexed in the order of the domain is a sorted array.

Another example of monotonic functions are spam filters, which are monotonic functions on hypercubes. The hypercube has a bit for each flag we think might increase the likelihood of the email being spam. Things like email addresses that haven't been seen before. And the chance of a particular email being spam should always be higher if it sets an additional flag. The hypercube has partial ordered by containment so if adding flags always increases the chance of an email being spam than the filter would be a monotonic function.

[A88] In the early days of machine learning it was found that monotonic functions were easier to learn than most arbitrary functions.

[A88] D. Angluin, Queries and Concept Learning, *Information and Computation Journal, Machine Learning 2* (1988), pp. 319–342.