CMPS290A: Sublinear algorithms for graphs	**Spring 2017**

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6.1 General Monotonicity Testers Through Ramsey Theory

Theorem 6.1 *Any monotonicity tester for function $f : [n] \to \mathbb{N}$ requires $\Omega(\log(n))$ queires.*

Definition 6.2 *In [F02], a (t,ε,δ)-randomized tester is a monotonicity tester with the proximity parameter ε that:*

1. *makes $\leq t$ queries on an input function*
2. *makes $\leq \delta$ error on any input*

Theorem 6.3 *In [F02], If ∃ a (t,ε,δ)-tester, there exists a (t,ε,2δ) comparison-based tester.*

Theorem 6.4 *In [R30], given any finite coloring of $\mathbb{N}^s$, there exists an infinite monochromatic set where:*

- *Finite coloring: $\chi : \mathbb{N}^s \to \{1, 2, ..., c\}$*
- *Monochromatic set: $M \subseteq \mathbb{N}$ s.t. $\chi|_{M^s}$ is constant*
- *$\mathbb{N}$-Natural numbers*
- *$\forall s \in \mathbb{N}$, $\mathbb{N}^{(s)}$: All subsets of $\mathbb{N}$ with cardinality s, where $s \in \mathbb{N}$*

Theorem 6.5 *In section 2 of [CS14], a theorem states that if we fix $t \in \mathbb{N}$, given a collection of finite coloring for $\mathbb{N}^{(2)}, \mathbb{N}^{(3)}, ..., \mathbb{N}^{(t)}$, ∃ an infinite monochromatic set w.r.t. all these colorings.*

We can define a monotonicity tester by considering a function P, and for every sequence $x_1, x_2, ..., x_s$ in $[n](s \leq t)$ and x_{s+1} in $[n]$, then $P^{x_s+1}_{x_1,x_2,...,x_s}$: $\mathbb{N}^{(s)} \to [0, 1]$ s.t. $\forall T \in \mathbb{N}^{(s)}$:

$\sum_{x\in[n]} P^{x}_{x_1,x_2,...,x_s}(T) + P^{ACC}_{x_1,x_2,...,x_s}(T) + P^{REJ}_{x_1,x_2,...,x_s}(T) = 1$, where:

- $P^{ACC}_{x_1,x_2,...,x_s}$: $\mathbb{N}^{(s)} \to [0, 1]$
- $P^{REJ}_{x_1,x_2,...,x_s}$: $\mathbb{N}^{(s)} \to [0, 1]$
- T: Values in query order

It is worth mentioning that the above equation specifies that the next query is a probability distribution.

Now, consider the function $q^{x_{s+1}}_{x_1,x_2,...,x_s,\sigma}$: $\mathbb{N}^{(s)} \to [0,1]$, where σ: permutations of $[s] \in S_s$, then:

$q^{x_{s+1}}_{x_1,x_2,...,x_s,\sigma}(A) = P^{x_{s+1}}_{x_1,x_2,...,x_s}(\sigma(a_1), \sigma(a_2), ..., \sigma(a_s))$, where $A = a_1 < a_2 < a_3 < ... < a_s$ are the values stored in sorted order. Therefore, in this case: $T \equiv (A, \sigma)$

To illustrate more, assume that the tester has queried: $x_1, x_2, ..., x_s$ and gets back the values: $v_1, v_2, ..., v_s$, then we get $P^s_{x_1,x_2,...,x_s}(v_1, v_2, ..., v_s)$. In q-world, we first sort $(v_1, v_2, ..., v_s)$ to (σ, A) and therefore $v_i = \sigma(a_i)$

Claim 6.6 *Tester is comparison-based $\equiv$ All q functions are constant*

Definition 6.7 *A tester is discrete if $\exists K \in \mathbb{N}$ s.t. all q-values (p-values) are multiples of $\frac{1}{K}$*

Claim 6.8 *If $\exists$ a (t,ε,δ)-tester, there exists a t,ε,2δ)-discrete tester.*

Lemma 6.9 *If $\exists$ a (t,ε,2δ)-discrete tester, then there exists a t,ε,2δ)-comparison-based tester.*

Proof: Define the following colorings of $\mathbb{N}^{(s)}$:

- χ: $\mathbb{N}^{(s)} \to \{0, \frac{1}{K}, \frac{2}{K}, \frac{3}{K}, ..., 1\}^{s!(n+2)n^s}$
- $\chi(A) = (q^{x_{s+1}}_{x_1,x_2,...,x_s,\sigma}(A) \forall x_1, x_2, ..., x_s, \sigma, x_{s+1})$

Then, $\exists$ an infinite set $M \subseteq \mathbb{N}$ (according to Ramsey) that is monochromatic.

Fix any s, and A_1, $A_2 \in M^s$, then:

- $\chi(A_1) = (q^{x_{s+1}}_{x_1,x_2,...,x_s,\sigma}(A1) \forall x_1, x_2, ..., x_s, \sigma, x_{s+1})$
- $\chi(A_2) = (q^{x_{s+1}}_{x_1,x_2,...,x_s,\sigma}(A2) \forall x_1, x_2, ..., x_s, \sigma, x_{s+1})$

■

$\exists$ infinite $M \subseteq \mathbb{N}$ s.t. $\forall f : [n] \to M$, the tester $\mathscr{T}$ is comparison-based.

Let $M = \{m_1, m_2, ...\}$

Define bijection: ϕ: $\mathbb{N} \to M$, then $\phi(i) = m_i$

Given any f, $\phi \circ f$ has the same distance of monotonicity as f. Then the final tester $\mathscr{C}$ invokes $\mathscr{T}$ on $\phi \circ f$ and $\mathscr{C}$ is comparison-based. It is worth mentioning that $\phi \circ f$ has range M.

If the range is finite or small (i.e. $f : [n] \to [r]$), [Pallavoor-Rakhodmikova-Varma17] give an $bigO(\frac{\log(r)}{\varepsilon})$ tester. Also, [Blais-Rakhodmikova-Yaroslartsev14] Non-adaptive testers require $\Omega(\log(r))$ queries.

6.2 Proof of Ramsey's Theory

Theorem 6.10 *Given finite coloring of $\mathbb{N}^{(\sim)}$, $\exists$ infinite monochromatic subset of $\mathbb{N}$.*

Proof: for $s = 2$, let a_0 be the minimum of $\mathbb{N}$. Pick the infinite color class, say color is c_0, and call that set S_1.

In S_1, let infinte color class gave color c_1, call that set S_2.

Hence, some color c appears infinitely in $c_0, c_1, ...$, hence that color class in monochromatic.

Given infinite set S_i:

1. Let a_i be the minimum of S_i
2. There exists infinite set $S_{i+1} \subseteq S_i$ and color c_i s.t. $\forall\ s \in S_i$, $\chi((a, s)) = c_i$.
   Therefore by induction on s, we can continue the rest of the proof.

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References

[F02] E. Fischer, On the strength of comparisons in property testing, *Information and Computation Journal* **189** (2004), pp. 107–116.

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