READ ME FIRST
- Don’t spend too much time on any one problem. This exam will take approximately 60 minutes.
- Amount of time spent on a problem is not necessarily proportional to the points.
- Scan through the entire test and do the easy problems first.
- If something is not clear, ASK.
- BE NEAT. We cannot give you points for something that we can’t read.
- Write down your assumptions.
- Don’t just write your answer, show how you got them.
- This is a CLOSED BOOK, CLOSED NOTES exam.

<table>
<thead>
<tr>
<th></th>
<th>25 points</th>
<th>Quaternions</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>25 points</td>
<td>Behavioral Anim.</td>
</tr>
<tr>
<td>3</td>
<td>25 points</td>
<td>Morphing</td>
</tr>
<tr>
<td>4</td>
<td>25 points</td>
<td>Inverse Kinematics</td>
</tr>
<tr>
<td></td>
<td>100 points</td>
<td>GRAND TOTAL</td>
</tr>
</tbody>
</table>
1. Quaternions (25 points)

A stretchable stick initially oriented so that one end is at $P_A(1,1,1)$ and the other
end is at $P_B(2,2,2)$ is to be rotated (and translated) to a new orientation such
that the first end will be at $P'_A(1,2,3)$ and the other end will be at $P'_B(3,2,1)$.
Ignoring the translation amounts, answer the following questions:

(a) 10 points

What is the axis of rotation?

$$\vec{v}_1 : P_A - P_B = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{v}_2 : P_A - P'_B = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$$

$$\vec{A} = \vec{v}_1 \times \vec{v}_2 = \begin{bmatrix} 1 & j & k \\ 1 & 1 & 1 \\ 2 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix}$$

(b) 5 points

What is the angle of rotation?

$$\vec{v}_1 \cdot \vec{v}_2 = |\vec{v}_1||\vec{v}_2| \cos \theta$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} = \sqrt{3} \sqrt{3} \cos \theta$$

$$\theta = \arccos \left( \frac{1}{\sqrt{12}} \right)$$

(c) 10 points

Represent this rotation using a quaternion.

$$\hat{q} = R_{\hat{A}}^{\hat{B}} = \begin{bmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} \hat{x} y z \end{bmatrix}$$

$$= \begin{bmatrix} \cos 45 \sin 45 (\frac{\sqrt{2}}{2} \frac{\sqrt{6}}{2} \frac{\sqrt{16}}{2}) \\ \frac{\sqrt{2}}{2} \frac{\sqrt{6}}{2} \frac{\sqrt{16}}{2} \frac{\sqrt{16}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \frac{\sqrt{6}}{2} \frac{\sqrt{16}}{2} \frac{\sqrt{16}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \frac{\sqrt{6}}{2} \frac{\sqrt{16}}{2} \frac{\sqrt{16}}{2} \end{bmatrix}$$

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2. Behavioral Animation (25 points)
A boid is moving through a force field. Assume that the boid's next trajectory is determined according to: 90% based on its current trajectory, and 10% based on external forces. If the boid's current velocity is $[10,10,0]$ and the uniform force field is $[0,0,-1]$, what will be the boid's new trajectory?

\[
\vec{R} = 0.9\vec{V} + 0.1\vec{F} = \begin{bmatrix} 9 \\ 9 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -0.1 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ -0.1 \end{bmatrix}
\]

After one time step.

\[
\vec{V} = \begin{bmatrix} 10 \\ 10 \\ 0 \end{bmatrix}
\]
\[
\vec{F} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}
\]

\[
\vec{R} \quad \begin{array}{c} \text{scaled vector addition} \\ \text{20 pts} \\ \text{5 pts} \end{array}
\]

\[
\text{partial credit} \quad \begin{array}{c} \text{if some reasonable attempt} \\ \text{reasonable attempt} \end{array}
\]

Score: ______________________
3. Morphing (25 points)
Analyze the line-pair using the Beier and Neely algorithm. At time=0, a line goes from (10, 10) to (10, 70). The corresponding line at time=10 goes from (30, 30) to (30, 70).

(a) 5 points
Where is the corresponding line for the intermediate grid at time=5?

(b) 10 points
Which pixel in the initial (time=0) frame corresponds to the intermediate grid pixel at (30, 45)?

\[
u = \frac{45 - 20}{70 - 20} = \frac{25}{50} = \frac{1}{2}\\

v = 30 - 20 = 10\\

x = 10 + v = 20\\
y = 10 + u(70 - 10) = 40\]

(c) 10 points
Which pixel in the final (time=10) frame corresponds to the intermediate grid pixel at (30, 45)?

\[
u = \frac{1}{2}\\
v = 10\\

x = 30 + v = 40\\
y = 30 + u(70 - 30) = 50\]
4. **Inverse Kinematics (25 points)**

In the figure below, joint A is at (0, 0), the end effector is at E (15, 5). The links L1 and L2 have link lengths of $10\sqrt{2}$ and $5\sqrt{2}$ respectively. Currently, $\theta_A$ is 45 and $\theta_B$ is 90. If we want to move the end effector to a new position G at (16, 6), find the instantaneous angular velocities for $\theta_A$ and $\theta_B$. Solve the problem as completely as you can.

\[
\begin{bmatrix}
\dot{\theta}_A \\
\dot{\theta}_B
\end{bmatrix} = \begin{bmatrix}
1 & -5 & 5 \\
15 & 5 &
\end{bmatrix}^{-1} \begin{bmatrix}
-5 \\
15
\end{bmatrix}
\]

\[
\begin{bmatrix}
\dot{\theta}_A \\
\dot{\theta}_B
\end{bmatrix} = \frac{1}{20} \begin{bmatrix}
-1 \\
3
\end{bmatrix}
\]

\[
\begin{bmatrix}
\dot{\theta}_A \\
\dot{\theta}_B
\end{bmatrix} = \frac{1}{20} \begin{bmatrix}
0 \\
4
\end{bmatrix} = \begin{bmatrix}
0 \\
\frac{1}{5}
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 1 & \hat{k} \\
0 & 0 & 1 \\
15 & 5 & 0
\end{bmatrix} = \begin{bmatrix}
-5 \\
15 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 1 & \hat{k} \\
0 & 0 & 1 \\
5 & -5 & 0
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
5 & -5 & 0
\end{bmatrix} = \begin{bmatrix}
5 \\
5 \\
0
\end{bmatrix}
\]