The Chromatic Number of Random Regular Graphs

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Abstract. Given any integer $d \geq 3$, let k be the smallest integer such that $d < 2k \log k$. We prove that with high probability the chromatic number of a random d-regular graph is k, $k+1$, or $k+2$.

1 Introduction

In [10], Łuczak proved that for every real $d > 0$ there exists an integer $k = k(d)$ such that w.h.p.[1] $\chi(\mathcal{G}(n, d/n))$ is either k or $k+1$. Recently, these two possible values were determined by the first author and Naor [4].

Significantly less is known for random d-regular graphs $\mathcal{G}_{n,d}$. In [6], Frieze and Łuczak extended the results of [9] for $\chi(\mathcal{G}(n,p))$ to random d-regular graphs, proving that for all integers $d > d_0$, w.h.p.

$$\left| \chi(\mathcal{G}_{n,d}) - \frac{d}{2 \log d} \right| = \Theta\left(\frac{d \log \log d}{(\log d)^2} \right) \ .$$

Here we determine $\chi(\mathcal{G}_{n,d})$ up to three possible values for all integers. Moreover, for roughly half of all integers we determine $\chi(\mathcal{G}_{n,d})$ up to two possible values. We first replicate the argument in [10] to prove

Theorem 1. *For every integer d, there exists an integer $k = k(d)$ such that w.h.p. the chromatic number of $\mathcal{G}_{n,d}$ is either k or $k+1$.*

We then use the second moment method to prove the following.

Theorem 2. *For every integer d, w.h.p. $\chi(\mathcal{G}_{n,d})$ is either k, $k+1$, or $k+2$, where k is the smallest integer such that $d < 2k \log k$. If, furthermore, $d > (2k-1) \log k$, then w.h.p. $\chi(\mathcal{G}_{n,d})$ is either $k+1$ or $k+2$.*

The table below gives the possible values of $\chi(\mathcal{G}_{n,d})$ for some values of d.

d	4	5	6	7, 8, 9	10	100	1,000,000
$\chi(\mathcal{G}_{n,d})$	3, 4	3, 4, 5	4, 5	4, 5, 6	5, 6	18, 19, 20	46523, 46524

[1] Given a sequence of events $\mathcal{E}_n$, we say that $\mathcal{E}$ holds *with positive probability* (w.p.p.) if $\liminf_{n \to / \infty} \Pr[\mathcal{E}_n] > 0$, and *with high probability* (w.h.p.) if $\liminf_{n \to \infty} \Pr[\mathcal{E}_n] = 1$.

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1.1 Preliminaries and Outline of the Proof

Rather than proving our results for $\mathcal{G}_{n,d}$ directly, it will be convenient to work with random d-regular multigraphs, in the sense of the configuration model [5]; that is, multigraphs $\mathcal{C}_{n,d}$ generated by selecting a uniformly random configuration (matching) on dn "vertex copies." It is well-known that for any fixed integer d, a random such multigraph is simple w.p.p. As a result, to prove Theorem 1 we simply establish its assertion for $\mathcal{C}_{n,d}$.

To prove Theorem 2 we use the second moment method to show

Theorem 3. *If $d < 2k \log k$, then w.p.p. $\chi(\mathcal{C}_{n,d}) \le k+1$.*

Proof of Theorem 2. For integer k let $u_k = (2k-1)\log k$ and $c_k = 2k \log k$. Observe that $c_{k-1} < u_k < c_k$. Thus, if k is the smallest integer such that $d < c_k$, then either i) $u_k < d < c_k$ or ii) $u_{k-1} < c_{k-1} < d \le u_k < c_k$.

A simple first moment argument (see e.g. [11]) implies that if $d > u_k$ then w.h.p. $\chi(\mathcal{C}_{n,d}) > k$. Thus, if $u_k < d < c_k$, then w.h.p. $\mathcal{C}_{n,d}$ is non-k-colorable while w.p.p. it is $(k+1)$-colorable. Therefore, by Theorem 1, w.h.p. the chromatic number of $\mathcal{C}_{n,d}$ (and therefore $\mathcal{G}_{n,d}$) is either $k+1$ or $k+2$. In the second case, we cannot eliminate the possibility that $\mathcal{G}_{n,d}$ is w.p.p. k-colorable, but we do know that it is w.h.p. non-$(k-1)$-colorable. Thus, similarly, it follows that $\chi(\mathcal{G}_{n,d})$ is w.h.p. k, $k+1$ or $k+2$. □

Throughout the rest of the paper, unless we explicitly say otherwise, we are referring to random multigraphs $\mathcal{C}_{n,d}$. We will say that a multigraph is k-colorable iff the underlying simple graph is k-colorable. Also, we will refer to multigraphs and configurations interchangeably using whichever form is most convenient.

2 2-Point Concentration

In [10], Łuczak in fact established two-point concentration for $\chi(\mathcal{G}(n, d/n))$ for all $\epsilon > 0$ and $d = O(n^{1/6-\epsilon})$. Here, mimicking his proof, we establish two-point concentration for $\chi(\mathcal{G}_{n,d})$ for all $\epsilon > 0$ and $d = O(n^{1/7-\epsilon})$.

Our main technical tool is the following martingale-based concentration inequality for random variables defined on $\mathcal{C}_{n,d}$ [12, Thm 2.19]. Given a configuration C, we define a *switching* in C to be the replacement of two pairs $\{e_1, e_2\}$, $\{e_3, e_4\}$ by $\{e_1, e_3\}$, $\{e_2, e_4\}$ or $\{e_1, e_4\}$, $\{e_3, e_2\}$.

Theorem 4. *Let X_n be a random variable defined on $\mathcal{C}_{n,d}$ such that for any configurations C, C' that differ by a switching*

$$|X_n(C) - X_n(C')| \le b \ ,$$

for some constant $b > 0$. Then for every $t > 0$,

$$\Pr\big[X_n \le \mathbf{E}[X_n] - t\big] < \mathrm{e}^{-\frac{t^2}{dnb^2}} \quad \text{and} \quad \Pr\big[X_n \ge \mathbf{E}[X_n] + t\big] < \mathrm{e}^{-\frac{t^2}{dnb^2}} \ .$$

Theorem 1 will follow from the following two lemmata. The proof of Lemma 1 is a straightforward union bound argument and is relegated to the full paper.

Lemma 1. *For any $0 < \epsilon < 1/6$ and $d < n^{1/6-\epsilon}$, w.h.p. every subgraph induced by $s \le nd^{-3(1+2\epsilon)}$ vertices contains at most $(3/2-\epsilon)s$ edges.*

Lemma 2. *For a given function $\omega(n)$, let $k = k(\omega, n, p)$ be the smallest k such that*

$$\Pr[\chi(\mathcal{C}_{n,d}) \le k] \ge 1/\omega(n) \ .$$

With probability greater than $1 - 1/\omega(n)$, all but $8\sqrt{nd \log \omega(n)}$ vertices of $\mathcal{C}_{n,d}$ can be properly colored using k colors.

Proof. For a multigraph G, let $Y_k(G)$ be the minimal size of a set of vertices S for which $G - S$ is k-colorable. Clearly, for any k and G, switching two edges of G can affect $Y_k(G)$ by at most 4, as a vertex cannot contribute more than itself to $Y_k(G)$. Thus, if $\mu_k = \mathbf{E}[Y_k(\mathcal{C}_{n,d})]$, Theorem 4 implies

$$\Pr[Y_k \le \mu_k - \lambda\sqrt{n}] < \mathrm{e}^{-\frac{\lambda^2}{16d}} \quad \text{and} \quad \Pr[Y_k \ge \mu_k + \lambda\sqrt{n}] < \mathrm{e}^{-\frac{\lambda^2}{16d}} \ . \qquad (1)$$

Define now $u = u(n, p, \omega(n))$ to be the least integer for which $\Pr[\chi(G) \le u] \ge 1/\omega(n)$. Choosing $\lambda = \lambda(n)$ so as to satisfy $e^{-\lambda^2/(16d)} = 1/\omega(n)$, the first inequality in (1) yields

$$\Pr[Y_u \le \mu_u - \lambda\sqrt{n}] < 1/\omega(n) \le \Pr[\chi(G) \le u] = \Pr[Y_u = 0] \ .$$

Clearly, if $\Pr[Y_u \le \mu_u - \lambda\sqrt{n}] < \Pr[Y_u = 0]$ then $\mu_u < \lambda\sqrt{n}$. Thus, the second inequality in (1) implies $\Pr[Y \ge 2\lambda\sqrt{n}] < 1/\omega(n)$ and, by our choice, $\lambda = 4\sqrt{d \log \omega(n)}$. □

Proof of Theorem 1. The result is trivial for $d = 1, 2$. Given $d \ge 3$, let $k = k(d, n) \ge 3$ be the smallest integer for which the probability that $\mathcal{C}_{n,d}$ is k-colorable is at least $1/\log\log n$. By Lemma 2, w.h.p. there exists a set of vertices S such that all vertices outside S can be colored using k colors and $|S| < 8\sqrt{nd \log\log\log n} < \sqrt{nd}\log n \equiv s_0$. From S, we will construct an increasing sequence of sets of vertices $\{U_i\}$ as follows. $U_0 = S$; for $i \ge 0$, $U_{i+1} = U_i \cup \{w_1, w_2\}$, where $w_1, w_2 \notin U_i$ are adjacent and each of them has some neighbor in U_i. The construction ends, with U_t, when no such pair exists.

Observe that the neighborhood of U_t in the rest of the graph, $N(U_t)$, is always an independent set, since otherwise the construction would have gone on. We further claim that w.h.p. the graph induced by the vertices in U_t is k-colorable. Thus, using an additional color for $N(U_t)$ yields a $(k+1)$-coloring of the entire multigraph, concluding the proof.

We will prove that U_t is, in fact, 3-colorable by proving that $|U_t| \le s_0/\epsilon$. This suffices since by Lemma 1 w.h.p. every subgraph H of b or fewer vertices has average degree less than 3 and hence contains a vertex v with $\deg(v) \le 2$. Repeatedly invoking Lemma 1 yields an ordering of the vertices in H such that each vertex is adjacent to no more than 2 of its successors. Thus, we can start with the last vertex in the ordering and proceed backwards; there will always be at least one available color for the current vertex. To prove $|U_t| \leq 2s_0 \log n$ we observe that each pair of vertices entering U "brings in" with it at least 3 new edges. Therefore, for every $j \geq 0$, U_j has at most $s_0 + 2j$ vertices and at least $3j$ edges. Thus, by Lemma 1, w.h.p. $t < 3s_0/(4\epsilon)$. □

3 Establishing Colorability in Two Moments

Let us say that a coloring σ is *nearly–balanced* if its color classes differ in size by at most 1, and let X be the number of nearly–balanced k-colorings of $\mathcal{C}_{n,d}$. Recall that $c_k = 2k \log k$. We will prove that for all $k \geq 3$ and $d < c_{k-1}$ there exist constants $C_1, C_2 > 0$ such that for all sufficiently large n (when dn is even),

$$\mathbf{E}[X] > C_1\, n^{-(k-1)/2}\, k^n \left(1 - \frac{1}{k}\right)^{dn/2} , \tag{2}$$

$$\mathbf{E}[X^2] < C_2\, n^{-(k-1)}\, k^{2n} \left(1 - \frac{1}{k}\right)^{dn} . \tag{3}$$

By the Cauchy-Schwartz inequality (see e.g. [7, Remark 3.1]), we have $\Pr[X > 0] > \mathbf{E}[X]^2/\mathbf{E}[X^2] > C_1^2/C_2 > 0$, and thus Theorem 3.

To prove (2), (3) we will need to bound certain combinatorial sums up to constant factors. To achieve this we will use the following Laplace-type lemma, which generalizes a series of lemmas in [2–4]. Its proof is standard but somewhat tedious, and is relegated to the full paper.

Lemma 3. *Let ℓ, m be positive integers. Let $\mathbf{y} \in \mathbb{Q}^m$, and let M be a $m \times \ell$ matrix of rank r with integer entries whose top row consists entirely of 1's. Let s, t be nonnegative integers, and let $\mathbf{v}_i, \mathbf{w}_j \in \mathbb{N}^\ell$ for $1 \leq i \leq s$ and $1 \leq j \leq t$, where each $\mathbf{v}_i$ and $\mathbf{w}_j$ has at least one nonzero component, and where moreover $\sum_{i=1}^s \mathbf{v}_i = \sum_{j=1}^t \mathbf{w}_j$. Let $f : \mathbb{R}^\ell \to \mathbb{R}$ be a positive twice-differentiable function. For $n \in \mathbb{N}$, define*

$$S_n = \sum_{\{\mathbf{z} \in \mathbb{N}^\ell : M \cdot \mathbf{z} = \mathbf{y}n\}} \frac{\prod_{i=1}^s (\mathbf{v}_i \cdot \mathbf{z})!}{\prod_{j=1}^t (\mathbf{w}_j \cdot \mathbf{z})!}\, f(\mathbf{z}/n)^n$$

and define $g : \mathbb{R}^\ell \to \mathbb{R}$ as

$$g(\zeta) = \frac{\prod_{i=1}^s (\mathbf{v}_i \cdot \zeta)^{(\mathbf{v}_i \cdot \zeta)}}{\prod_{j=1}^t (\mathbf{w}_j \cdot \zeta)^{(\mathbf{w}_j \cdot \zeta)}}\, f(\zeta)$$

where $0^0 \equiv 1$. Now suppose that, conditioned on $M \cdot \zeta = \mathbf{y}$, g is maximized at some ζ^* with $\zeta_i^* > 0$ for all i, and write $g_{\max} = g(\zeta^*)$. Furthermore, suppose that the matrix of second derivatives $g'' = \partial^2 g / \partial \zeta_i\, \partial \zeta_j$ is nonsingular at ζ^*.

Then there exist constants $A, B > 0$, such that for any sufficiently large n for which there exist integer solutions $\mathbf{z}$ to $M \cdot \mathbf{z} = \mathbf{y}n$, we have

$$A \leq \frac{S_n}{n^{-(\ell+s-t-r)/2}\, g_{\max}^n} \leq B \ .$$

For simplicity, in the proofs of (2) and (3) below we will assume that n is a multiple of k, so that nearly–balanced colorings are in fact exactly balanced, with n/k vertices in each color class. The calculations for other values of n differ by at most a multiplicative constant.

4 The First Moment

Clearly, all (exactly) balanced k-partitions of the n vertices are equally likely to be proper k-colorings. Therefore, $\mathbf{E}[X]$ is the number of balanced k-partitions, $n!/(n/k)!^k$, times the probability that a random d-regular configuration is properly colored by a fixed balanced k-partition.

To estimate this probability we will label the d copies of each vertex, thus giving us $(dn-1)!!$ distinct configurations, and count the number of such configurations that are properly colored by a fixed balanced k-partition. To generate such a configuration we first determine the number of edges between each pair of color classes. Suppose there are b_{ij} edges between vertices of colors i and j for each $i \neq j$. Then a properly colored configuration can be generated by i) choosing which b_{ij} of the dn/k copies in each color class i are matched with copies in each color class $j \neq i$, and then ii) choosing one of the $b_{ij}!$ matchings for each unordered pair $i < j$. Therefore, the total number of properly colored configurations is

$$\prod_{i=1}^{k} \frac{(dn/k)!}{\prod_{j\neq i} b_{ij}!} \cdot \prod_{i<j} b_{ij}! = \frac{(dn/k)!^k}{\prod_{i<j} b_{ij}!} \ .$$

Summing over all choices of the $\{b_{ij}\}$ that satisfy the constraints

$$\forall i : \sum_j b_{ij} = dn/k \ , \tag{4}$$

we get

$$\begin{aligned} \mathbf{E}[X] &= \frac{n!}{(n/k)!^k} \frac{1}{(dn-1)!!} \sum_{\{b_{ij}\}} \frac{(dn/k)!^k}{\prod_{i<j} b_{ij}!} \\ &= 2^{dn/2} \frac{n!}{(n/k)!^k} \frac{(dn/k)!^k}{(dn)!} \sum_{\{b_{ij}\}} \frac{(dn/2)!}{\prod_{i<j} b_{ij}!} \ . \end{aligned}$$

By Stirling's approximation $\sqrt{2\pi n}\,(n/\mathrm{e})^n < n! < \sqrt{4\pi n}\,(n/\mathrm{e})^n$ we get

$$\mathbf{E}[X] > D_1 \frac{2^{dn/2}}{k^{(d-1)n}} \sum_{\{b_{ij}\}} \frac{(dn/2)!}{\prod_{i<j} b_{ij}!} \ , \tag{5}$$

where $D_1 = 2^{-(k+1)/2}\, d^{(k-1)/2}$.

To bound the sum in (5) from below we use Lemma 3. Specifically, $\mathbf{z}$ consists of the variables b_{ij} with $i < j$, so $\ell = k(k-1)/2$. For $k \geq 3$, the k constraints (4) are linearly independent, so representing them as $M \cdot \mathbf{z} = \mathbf{y}n$ gives a matrix M of rank k. Moreover, they imply $\sum_{i<j} b_{ij} = dn/2$, so adding a row of 1's to the top of M and setting $y_1 = d/2$ does not increase its rank. Integer solutions $\mathbf{z}$ exist whenever n is a multiple of k and dn is even. We set $s = 1$ and $t = \ell$; the vector $\mathbf{v}_1$ consists of 1's and the $\mathbf{w}_j$ are the ℓ basis vectors. Finally, $f(\zeta) = 1$. Thus, $\ell + s - t - r = -(k-1)$ and

$$g(\zeta) = \frac{(d/2)^{d/2}}{\prod_{j=1}^{\ell} \zeta_k^{\zeta_k}} = \frac{1}{\prod_{j=1}^{\ell} (2\zeta_j/d)^{\zeta_j}} = \mathrm{e}^{(d/2)H(2\zeta/d)} ,$$

where H is the entropy function $H(\mathbf{x}) = -\sum_{j=1}^{\ell} x_j \log x_j$.

Since g is convex it is maximized when $\zeta_j^* = d/(2\ell)$ for all $1 \leq j \leq \ell$, and g'' is nonsingular. Thus, $g_{\max} = (k(k-1)/2)^{d/2}$ implying that for some $A > 0$ and all sufficiently large n

$$\begin{aligned} \mathbf{E}[X] &> D_1 \frac{2^{dn/2}}{k^{(d-1)n}} \times A\, n^{-(k-1)/2} \left(\frac{k(k-1)}{2}\right)^{dn/2} \\ &= D_1 A\, n^{-(k-1)/2}\, k^n \left(1 - \frac{1}{k}\right)^{dn/2} . \end{aligned}$$

Setting $C_1 = D_1 A$ completes the the proof.

5 The Second Moment

Recall that X is the sum over all balanced k-partitions of the indicators that each partition is a proper coloring. Therefore, $\mathbf{E}[X^2]$ is the sum over all pairs of balanced k-partitions of the probability that both partitions properly color a random d-regular configuration. Given a pair of partitions σ, τ, let us say that a vertex v is in class (i, j) if $\sigma(v) = i$ and $\tau(v) = j$. Also, let a_{ij} denote the number of vertices in each class (i, j). We call $A = (a_{ij})$ the *overlap matrix* of the pair σ, τ. Note that since both σ and τ are balanced

$$\forall i : \sum_j a_{ij} = \sum_j a_{ji} = n/k \ . \tag{6}$$

We will show that for any fixed pair of k-partitions, the probability that they both properly color a random d-regular configuration depends only on their overlap matrix A. Denoting this probability by $q(A)$, since there are $n!/\prod_{ij} a_{ij}!$ pairs of partitions giving rise to A, we have

$$\mathbf{E}[X^2] = \sum_A \frac{n!}{\prod_{ij} a_{ij}!}\, q(A) \tag{7}$$

where the sum is over matrices A satisfying (6).

Fixing a pair of partitions σ and τ with overlap matrix A, similarly to the first moment, we label the d copies of each vertex thus getting $(dn-1)!!$ distinct configurations. To generate configurations properly colored by both σ and τ we first determine the number of edges between each pair of vertex classes. Let us say that there are $b_{ijk\ell}$ edges connecting vertices in class (i,j) to vertices in class (k,ℓ). By definition, $b_{ijk\ell} = b_{k\ell ij}$, and if both colorings are proper, $b_{ijk\ell} = 0$ unless $i \neq k$ and $j \neq \ell$. Since the configuration is d-regular, we also have

$$\forall i,j : \sum_{k \neq i, \ell \neq j} b_{ijk\ell} = d a_{ij} \ . \tag{8}$$

To generate a configuration consistent with A and $\{b_{ijk\ell}\}$ we now i) choose for each class (i,j), which $b_{ijk\ell}$ of its da_{ij} copies are to be matched with copies in each class (k,ℓ) with $k \neq i$ and $\ell \neq j$, and then ii) choose one of the $b_{ijk\ell}!$ matchings for each unordered pair of classes $i < k$, $j \neq \ell$. Thus,

$$\begin{aligned} q(A) &= \frac{1}{(dn-1)!!} \sum_{\{b_{ijk\ell}\}} \left(\prod_{ij} \frac{(da_{ij})!}{\prod_{k \neq i, \ell \neq j} b_{ijk\ell}!} \cdot \prod_{i<k, j \neq \ell} b_{ijk\ell}! \right) \\ &= 2^{dn/2} \frac{\prod_{ij} (da_{ij})!}{(dn)!} \sum_{\{b_{ijk\ell}\}} \frac{(dn/2)!}{\prod_{i<k, j\neq \ell} b_{ijk\ell}!} \ , \end{aligned} \tag{9}$$

where the sum is over the $\{b_{ijk\ell}\}$ satisfying (8). Combining (9) with (7) gives

$$\mathbf{E}[X^2] = 2^{dn/2} \sum_{\{a_{ij}\}} \sum_{\{b_{ijk\ell}\}} \frac{n!}{\prod_{ij} a_{ij}!} \frac{\prod_{ij} (da_{ij})!}{(dn)!} \frac{(dn/2)!}{\prod_{i<k, j \neq \ell} b_{ijk\ell}!} \ . \tag{10}$$

To bound the sum in (10) from above we use Lemma 3. We let $\mathbf{z}$ consist of the combined set of variables $\{a_{ij}\} \cup \{b_{ijk\ell} : i < k, j \neq \ell\}$, in which case its dimensionality ℓ (not to be confused with the color ℓ) is $k^2 + (k(k-1))^2/2$. We represent the combined system of constraints (6), (8) as $M \cdot \mathbf{z} = \mathbf{y}n$. The k^2 constraints (8) are, clearly, linearly independent while the $2k$ constraints (6) have rank $2k-1$. Together these imply $\sum_{ij} a_{ij} = 1$ and $\sum_{i<k, j \neq \ell} b_{ijk\ell} = d/2$, so adding a row of 1's to the top of M does not change its rank from $r = k^2+2k-1$. Integer solutions $\mathbf{z}$ exist whenever n is a multiple of k and dn is even. Finally, $f(\zeta) = 2^{d/2}$, $s = k^2+2$ and $t = k^2+1+(k(k-1))^2/2$, so $\ell+s-t-r = -2(k-1)$.

Writing α_{ij} and $\beta_{ijk\ell}$ for the components of ζ corresponding to a_{ij}/n and $b_{ijk\ell}/n$, respectively, we thus have

$$\begin{aligned} g(\zeta) &= 2^{d/2} \frac{1}{\prod_{ij} \alpha_{ij}^{\alpha_{ij}}} \frac{\prod_{ij} (d\alpha_{ij})^{d\alpha_{ij}}}{d^d} \frac{(d/2)^{d/2}}{\prod_{i<k, j \neq \ell} \beta_{ijk\ell}^{\beta_{ijk\ell}}} \\ &= \frac{1}{\prod_{ij} \alpha_{ij}^{\alpha_{ij}}} \frac{d^{d/2} \prod_{ij} \alpha_{ij}^{d\alpha_{ij}}}{\prod_{i<k, j\neq\ell} \beta_{ijk\ell}^{\beta_{ijk\ell}}} \ . \end{aligned} \tag{11}$$

In the next section we maximize $g(\zeta)$ over $\zeta \in \mathbb{R}^\ell$ satisfying $M \cdot \zeta = \mathbf{y}$. We note that g'' is nonsingular at the maximizer we find below, but we relegate the proof of this fact to the full paper.

6 A Tight Relaxation

Maximizing $g(\zeta)$ over $\zeta \in \mathbb{R}^\ell$ satisfying $M \cdot \zeta = \mathbf{y}$ is greatly complicated by the constraints

$$\forall i,j : \sum_{k \neq i, \ell \neq j} \beta_{ijk\ell} = d\alpha_{ij} \ . \tag{12}$$

To overcome this issue we i) reformulate $g(\zeta)$ and ii) relax the constraints, in a manner such that the maximum value remains unchanged while the optimization becomes much easier.

The relaxation amounts to replacing the k^2 constraints (12) with their sum divided by 2, i.e., with the single constraint

$$\sum_{i<k, j \neq \ell} \beta_{ijk\ell} = d/2 \ . \tag{13}$$

But attempting to maximize (11) under this single constraint is, in fact, a bad idea since the new maximum is much greater. Instead, we maximize the following equivalent form of $g(\zeta)$

$$g(\zeta) = \frac{1}{\prod_{ij} \alpha_{ij}^{\alpha_{ij}}} \frac{d^{d/2} \prod_{ij} \alpha_{ij}^{\sum_{k \neq i, \ell \neq j} \beta_{ijk\ell}}}{\prod_{i<k, j \neq \ell} \beta_{ijk\ell}^{\beta_{ijk\ell}}} \ , \tag{14}$$

derived by using (12) to substitute for the exponents $d\alpha_{ij}$ in the numerator of (11). This turns out to be enough to drive the maximizer back to the subspace $M \cdot \zeta = \mathbf{y}$.

Specifically, let us hold $\{\alpha_{ij}\}$ fixed and maximize $g(\zeta)$ with respect to $\{\beta_{ijk\ell}\}$ using the method of Lagrange multipliers. Since $\log g$ is monotonically increasing in g, it is convenient to maximize $\log g$ instead. If λ is the Lagrange multiplier corresponding to the constraint (13), we have for all $i < k, j \neq \ell$:

$$\begin{aligned}\lambda &= \frac{\partial}{\partial \beta_{ijk\ell}} \log g(\zeta) = \frac{\partial}{\partial \beta_{ijk\ell}} \left(\beta_{ijk\ell} \log(\alpha_{ij}\alpha_{k\ell}) - \beta_{ijk\ell} \log \beta_{ijk\ell} \right) \\ &= \log \alpha_{ij} + \log \alpha_{k\ell} - \log \beta_{ijk\ell} - 1\end{aligned}$$

and so

$$\forall i < k, j \neq l : \ \beta_{ijk\ell} = C \alpha_{ij} \alpha_{k\ell}, \text{ where } C = \mathrm{e}^{-\lambda - 1} \ . \tag{15}$$

Clearly, such $\beta_{ijk\ell}$ also satisfy the original constraints (12), and therefore the upper bound we obtain from this relaxation is in fact tight.

To solve for C we sum (15) and use (13), getting

$$\frac{2}{C} \sum_{i<k, j \neq \ell} \beta_{ijk\ell} = \frac{d}{C} = \sum_{i \neq k, j \neq \ell} \alpha_{ij} \alpha_{k\ell} = 1 - \frac{2}{k} + \sum_{ij} \alpha_{ij}^2 \equiv p \ .$$

Thus $C = d/p$ and (15) becomes

$$\forall i < k, j \neq l : \ \beta_{ijk\ell} = \frac{d \alpha_{ij} \alpha_{k\ell}}{p} \tag{16}$$

Observe that $p = p(\{a_{ij}\})$ is the probability that a single edge whose endpoints are chosen uniformly at random is properly colored by both σ and τ, if the overlap matrix is $a_{ij} = \alpha_{ij} n$. Moreover, the values for the $b_{ijk\ell}$ are exactly what we would obtain, in expectation, if we chose from among the $\binom{n}{2}$ edges with replacement, rejecting those improperly colored by σ or τ, until we had $dn/2$ edges – in other words, if our graph model was $G(n, m)$ with replacement, rather than $\mathcal{G}_{n,d}$.

Substituting the values (16) in (14) and applying (13) yields the following upper bound on $g(\zeta)$:

$$\begin{aligned}
g(\zeta) &\leq \frac{1}{\prod_{ij} \alpha_{ij}^{\alpha_{ij}}} \frac{d^{d/2} \prod_{ij} \alpha_{ij}^{(d/p)\alpha_{ij} \sum_{i\neq k, j \neq \ell} \alpha_{k\ell}}}{(d/p)^{\sum_{i<k, j\neq \ell} \beta_{ijk\ell}} \prod_{i<k, j\neq \ell} (\alpha_{ij}\alpha_{k\ell})^{(d/p)\alpha_{ij}\alpha_{k\ell}}} \\
&= \frac{1}{\prod_{ij} \alpha_{ij}^{\alpha_{ij}}} \frac{d^{d/2}}{(d/p)^{d/2}} \left(\frac{\prod_{ij} a_{ij}^{\alpha_{ij} \sum_{i\neq k, j\neq \ell} \alpha_{k\ell}}}{\prod_{i\neq k, j \neq \ell} \alpha_{ij}^{\alpha_{ij}\alpha_{k\ell}}} \right)^{d/p} \\
&= \frac{p^{d/2}}{\prod_{ij} \alpha_{ij}^{\alpha_{ij}}} \\
&\equiv g_{G(n,m)}(\{\alpha_{ij}\}) \ .
\end{aligned}$$

In [4, Thm 5], Achlioptas and Naor showed that for $d < c_{k-1}$ the function $g_{G(n,m)}$ is maximized when $\alpha_{ij} = 1/k^2$ for all i, j. In this case $p = (1 - 1/k)^2$, implying

$$g_{\max} \leq k^2 p^{d/2} = k^2 \left(1 - \frac{1}{k}\right)^d$$

and, therefore, that for some constant C_2 and sufficiently large n

$$\mathbf{E}[X^2] \leq C_2 \, n^{-(k-1)} \, k^{2n} \left(1 - \frac{1}{k}\right)^{dn} \quad .$$

7 Directions for Further Work

A Sharp Threshold for Regular Graphs. It has long been conjectured that for every $k > 2$, there exists a critical constant c_k such that a random graph $G(n, m = cn)$ is w.h.p. k-colorable if $c < c_k$ and w.h.p. non-k-colorable if $c > c_k$. It is reasonable to conjecture that the same is true for random regular graphs, i.e. that for all $k > 2$, there exists a critical integer d_k such that a random graph $\mathcal{G}_{n,d}$ is w.h.p. k-colorable if $d \leq d_k$ and w.h.p. non-k-colorable if $d > d_k$. If this is true, our results imply that for d in "good" intervals (u_k, c_k) w.h.p. the chromatic number of $\mathcal{G}_{n,d}$ is precisely $k + 1$, while for d in "bad" intervals (c_{k-1}, u_k) the chromatic number is w.h.p. either k or $k + 1$.

Improving the Second Moment Bound. Our proof establishes that if X, Y are the numbers of balanced k-colorings of $\mathcal{G}_{n,d}$ and $G(n, m = dn/2)$, respectively, then $\mathbf{E}[X]^2/\mathbf{E}[X^2] = \Theta(\mathbf{E}[Y]^2/\mathbf{E}[Y^2])$. Therefore, any improvement on the upper bound for $\mathbf{E}[Y^2]$ given in [4] would immediately give an improved positive-probability k-colorability result for $\mathcal{G}_{n,d}$.

In particular, Moore has conjectured that the function $g_{G(n,m)}$ is maximized by matrices with a certain form. If true, this immediately gives an improved lower bound, c_k^*, for k-colorability satisfying $c_{k-1}^* \to u_k - 1$. This would shrink the union of the "bad" intervals to a set of measure 0, with each such interval containing precisely one integer d for each $k \geq k_0$.

3-Colorability of Random Regular Graphs. It is easy to show that a random 6-regular graph is w.h.p. non-3-colorable. On the other hand, in [1] the authors showed that 4-regular graphs are w.p.p. 3-colorable. Based on considerations from statistical physics, Krząkała, Pagnani and Weigt [8] have conjectured that a random 5-regular graph is w.h.p. 3-colorable. The authors (unpublished) have shown that applying the second moment method to the number of balanced 3-colorings cannot establish this fact (even with positive probability).

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