

# The Chromatic Number of Random Regular Graphs

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**Abstract.** Given any integer  $d \geq 3$ , let  $k$  be the smallest integer such that  $d < 2k \log k$ . We prove that with high probability the chromatic number of a random  $d$ -regular graph is  $k$ ,  $k + 1$ , or  $k + 2$ .

## 1 Introduction

In [10], Luczak proved that for every real  $d > 0$  there exists an integer  $k = k(d)$  such that w.h.p.<sup>1</sup>  $\chi(\mathcal{G}(n, d/n))$  is either  $k$  or  $k + 1$ . Recently, these two possible values were determined by the first author and Naor [4].

Significantly less is known for random  $d$ -regular graphs  $\mathcal{G}_{n,d}$ . In [6], Frieze and Luczak extended the results of [9] for  $\chi(\mathcal{G}(n, p))$  to random  $d$ -regular graphs, proving that for all integers  $d > d_0$ , w.h.p.

$$\left| \chi(\mathcal{G}_{n,d}) - \frac{d}{2 \log d} \right| = \Theta \left( \frac{d \log \log d}{(\log d)^2} \right).$$

Here we determine  $\chi(\mathcal{G}_{n,d})$  up to three possible values for all integers. Moreover, for roughly half of all integers we determine  $\chi(\mathcal{G}_{n,d})$  up to two possible values. We first replicate the argument in [10] to prove

**Theorem 1.** *For every integer  $d$ , there exists an integer  $k = k(d)$  such that w.h.p. the chromatic number of  $\mathcal{G}_{n,d}$  is either  $k$  or  $k + 1$ .*

We then use the second moment method to prove the following.

**Theorem 2.** *For every integer  $d$ , w.h.p.  $\chi(\mathcal{G}_{n,d})$  is either  $k$ ,  $k + 1$ , or  $k + 2$ , where  $k$  is the smallest integer such that  $d < 2k \log k$ . If, furthermore,  $d > (2k - 1) \log k$ , then w.h.p.  $\chi(\mathcal{G}_{n,d})$  is either  $k + 1$  or  $k + 2$ .*

The table below gives the possible values of  $\chi(\mathcal{G}_{n,d})$  for some values of  $d$ .

$d$	4	5	6	7, 8, 9	10	100	1,000,000
$\chi(\mathcal{G}_{n,d})$	3, 4	3, 4, 5	4, 5	4, 5, 6	5, 6	18, 19, 20	46523, 46524

<sup>1</sup> Given a sequence of events  $\mathcal{E}_n$ , we say that  $\mathcal{E}$  holds *with positive probability* (w.p.p.) if  $\liminf_{n \rightarrow \infty} \Pr[\mathcal{E}_n] > 0$ , and *with high probability* (w.h.p.) if  $\liminf_{n \rightarrow \infty} \Pr[\mathcal{E}_n] = 1$ .

### 1.1 Preliminaries and Outline of the Proof

Rather than proving our results for  $\mathcal{G}_{n,d}$  directly, it will be convenient to work with random  $d$ -regular multigraphs, in the sense of the configuration model [5]; that is, multigraphs  $\mathcal{C}_{n,d}$  generated by selecting a uniformly random configuration (matching) on  $dn$  “vertex copies.” It is well-known that for any fixed integer  $d$ , a random such multigraph is simple w.p.p. As a result, to prove Theorem 1 we simply establish its assertion for  $\mathcal{C}_{n,d}$ .

To prove Theorem 2 we use the second moment method to show

**Theorem 3.** *If  $d < 2k \log k$ , then w.p.p.  $\chi(\mathcal{C}_{n,d}) \leq k + 1$ .*

*Proof of Theorem 2.* For integer  $k$  let  $u_k = (2k - 1) \log k$  and  $c_k = 2k \log k$ . Observe that  $c_{k-1} < u_k < c_k$ . Thus, if  $k$  is the smallest integer such that  $d < c_k$ , then either i)  $u_k < d < c_k$  or ii)  $u_{k-1} < c_{k-1} < d \leq u_k < c_k$ .

A simple first moment argument (see e.g. [11]) implies that if  $d > u_k$  then w.h.p.  $\chi(\mathcal{C}_{n,d}) > k$ . Thus, if  $u_k < d < c_k$ , then w.h.p.  $\mathcal{C}_{n,d}$  is non- $k$ -colorable while w.p.p. it is  $(k + 1)$ -colorable. Therefore, by Theorem 1, w.h.p. the chromatic number of  $\mathcal{C}_{n,d}$  (and therefore  $\mathcal{G}_{n,d}$ ) is either  $k + 1$  or  $k + 2$ . In the second case, we cannot eliminate the possibility that  $\mathcal{G}_{n,d}$  is w.p.p.  $k$ -colorable, but we do know that it is w.h.p. non- $(k - 1)$ -colorable. Thus, similarly, it follows that  $\chi(\mathcal{G}_{n,d})$  is w.h.p.  $k, k + 1$  or  $k + 2$ . □

Throughout the rest of the paper, unless we explicitly say otherwise, we are referring to random multigraphs  $\mathcal{C}_{n,d}$ . We will say that a multigraph is  $k$ -colorable iff the underlying simple graph is  $k$ -colorable. Also, we will refer to multigraphs and configurations interchangeably using whichever form is most convenient.

## 2 2-Point Concentration

In [10], Luczak in fact established two-point concentration for  $\chi(\mathcal{G}(n, d/n))$  for all  $\epsilon > 0$  and  $d = O(n^{1/6-\epsilon})$ . Here, mimicking his proof, we establish two-point concentration for  $\chi(\mathcal{G}_{n,d})$  for all  $\epsilon > 0$  and  $d = O(n^{1/7-\epsilon})$ .

Our main technical tool is the following martingale-based concentration inequality for random variables defined on  $\mathcal{C}_{n,d}$  [12, Thm 2.19]. Given a configuration  $C$ , we define a *switching* in  $C$  to be the replacement of two pairs  $\{e_1, e_2\}, \{e_3, e_4\}$  by  $\{e_1, e_3\}, \{e_2, e_4\}$  or  $\{e_1, e_4\}, \{e_3, e_2\}$ .

**Theorem 4.** *Let  $X_n$  be a random variable defined on  $\mathcal{C}_{n,d}$  such that for any configurations  $C, C'$  that differ by a switching*

$$|X_n(C) - X_n(C')| \leq b ,$$

for some constant  $b > 0$ . Then for every  $t > 0$ ,

$$\Pr[X_n \leq \mathbf{E}[X_n] - t] < e^{-\frac{t^2}{dnb^2}} \quad \text{and} \quad \Pr[X_n \geq \mathbf{E}[X_n] + t] < e^{-\frac{t^2}{dnb^2}} .$$

Theorem 1 will follow from the following two lemmata. The proof of Lemma 1 is a straightforward union bound argument and is relegated to the full paper.

**Lemma 1.** *For any  $0 < \epsilon < 1/6$  and  $d < n^{1/6-\epsilon}$ , w.h.p. every subgraph induced by  $s \leq nd^{-3(1+2\epsilon)}$  vertices contains at most  $(3/2 - \epsilon)s$  edges.*

**Lemma 2.** *For a given function  $\omega(n)$ , let  $k = k(\omega, n, p)$  be the smallest  $k$  such that*

$$\Pr[\chi(\mathcal{C}_{n,d}) \leq k] \geq 1/\omega(n) .$$

*With probability greater than  $1 - 1/\omega(n)$ , all but  $8\sqrt{nd \log \omega(n)}$  vertices of  $\mathcal{C}_{n,d}$  can be properly colored using  $k$  colors.*

*Proof.* For a multigraph  $G$ , let  $Y_k(G)$  be the minimal size of a set of vertices  $S$  for which  $G - S$  is  $k$ -colorable. Clearly, for any  $k$  and  $G$ , switching two edges of  $G$  can affect  $Y_k(G)$  by at most 4, as a vertex cannot contribute more than itself to  $Y_k(G)$ . Thus, if  $\mu_k = \mathbf{E}[Y_k(\mathcal{C}_{n,d})]$ , Theorem 4 implies

$$\Pr[Y_k \leq \mu_k - \lambda\sqrt{n}] < e^{-\frac{\lambda^2}{16d}} \quad \text{and} \quad \Pr[Y_k \geq \mu_k + \lambda\sqrt{n}] < e^{-\frac{\lambda^2}{16d}} . \quad (1)$$

Define now  $u = u(n, p, \omega(n))$  to be the least integer for which  $\Pr[\chi(G) \leq u] \geq 1/\omega(n)$ . Choosing  $\lambda = \lambda(n)$  so as to satisfy  $e^{-\lambda^2/(16d)} = 1/\omega(n)$ , the first inequality in (1) yields

$$\Pr[Y_u \leq \mu_u - \lambda\sqrt{n}] < 1/\omega(n) \leq \Pr[\chi(G) \leq u] = \Pr[Y_u = 0] .$$

Clearly, if  $\Pr[Y_u \leq \mu_u - \lambda\sqrt{n}] < \Pr[Y_u = 0]$  then  $\mu_u < \lambda\sqrt{n}$ . Thus, the second inequality in (1) implies  $\Pr[Y \geq 2\lambda\sqrt{n}] < 1/\omega(n)$  and, by our choice,  $\lambda = 4\sqrt{d \log \omega(n)}$ .  $\square$

**Proof of Theorem 1.** The result is trivial for  $d = 1, 2$ . Given  $d \geq 3$ , let  $k = k(d, n) \geq 3$  be the smallest integer for which the probability that  $\mathcal{C}_{n,d}$  is  $k$ -colorable is at least  $1/\log \log n$ . By Lemma 2, w.h.p. there exists a set of vertices  $S$  such that all vertices outside  $S$  can be colored using  $k$  colors and  $|S| < 8\sqrt{nd \log \log \log n} < \sqrt{nd} \log n \equiv s_0$ . From  $S$ , we will construct an increasing sequence of sets of vertices  $\{U_i\}$  as follows.  $U_0 = S$ ; for  $i \geq 0$ ,  $U_{i+1} = U_i \cup \{w_1, w_2\}$ , where  $w_1, w_2 \notin U_i$  are adjacent and each of them has some neighbor in  $U_i$ . The construction ends, with  $U_t$ , when no such pair exists.

Observe that the neighborhood of  $U_t$  in the rest of the graph,  $N(U_t)$ , is always an independent set, since otherwise the construction would have gone on. We further claim that w.h.p. the graph induced by the vertices in  $U_t$  is  $k$ -colorable. Thus, using an additional color for  $N(U_t)$  yields a  $(k + 1)$ -coloring of the entire multigraph, concluding the proof.

We will prove that  $U_t$  is, in fact, 3-colorable by proving that  $|U_t| \leq s_0/\epsilon$ . This suffices since by Lemma 1 w.h.p. every subgraph  $H$  of  $b$  or fewer vertices has average degree less than 3 and hence contains a vertex  $v$  with  $\deg(v) \leq 2$ . Repeatedly invoking Lemma 1 yields an ordering of the vertices in  $H$  such that

each vertex is adjacent to no more than 2 of its successors. Thus, we can start with the last vertex in the ordering and proceed backwards; there will always be at least one available color for the current vertex. To prove  $|U_t| \leq 2s_0 \log n$  we observe that each pair of vertices entering  $U$  “brings in” with it at least 3 new edges. Therefore, for every  $j \geq 0$ ,  $U_j$  has at most  $s_0 + 2j$  vertices and at least  $3j$  edges. Thus, by Lemma 1, w.h.p.  $t < 3s_0/(4\epsilon)$ .  $\square$

### 3 Establishing Colorability in Two Moments

Let us say that a coloring  $\sigma$  is *nearly-balanced* if its color classes differ in size by at most 1, and let  $X$  be the number of nearly-balanced  $k$ -colorings of  $\mathcal{C}_{n,d}$ . Recall that  $c_k = 2k \log k$ . We will prove that for all  $k \geq 3$  and  $d < c_{k-1}$  there exist constants  $C_1, C_2 > 0$  such that for all sufficiently large  $n$  (when  $dn$  is even),

$$\mathbf{E}[X] > C_1 n^{-(k-1)/2} k^n \left(1 - \frac{1}{k}\right)^{dn/2}, \tag{2}$$

$$\mathbf{E}[X^2] < C_2 n^{-(k-1)} k^{2n} \left(1 - \frac{1}{k}\right)^{dn}. \tag{3}$$

By the Cauchy-Schwartz inequality (see e.g. [7, Remark 3.1]), we have  $\Pr[X > 0] > \mathbf{E}[X]^2 / \mathbf{E}[X^2] > C_1^2 / C_2 > 0$ , and thus Theorem 3.

To prove (2), (3) we will need to bound certain combinatorial sums up to constant factors. To achieve this we will use the following Laplace-type lemma, which generalizes a series of lemmas in [2–4]. Its proof is standard but somewhat tedious, and is relegated to the full paper.

**Lemma 3.** *Let  $\ell, m$  be positive integers. Let  $\mathbf{y} \in \mathbb{Q}^m$ , and let  $M$  be a  $m \times \ell$  matrix of rank  $r$  with integer entries whose top row consists entirely of 1’s. Let  $s, t$  be nonnegative integers, and let  $\mathbf{v}_i, \mathbf{w}_j \in \mathbb{N}^\ell$  for  $1 \leq i \leq s$  and  $1 \leq j \leq t$ , where each  $\mathbf{v}_i$  and  $\mathbf{w}_j$  has at least one nonzero component, and where moreover  $\sum_{i=1}^s \mathbf{v}_i = \sum_{j=1}^t \mathbf{w}_j$ . Let  $f : \mathbb{R}^\ell \rightarrow \mathbb{R}$  be a positive twice-differentiable function. For  $n \in \mathbb{N}$ , define*

$$S_n = \sum_{\{\mathbf{z} \in \mathbb{N}^\ell : M \cdot \mathbf{z} = \mathbf{y}n\}} \frac{\prod_{i=1}^s (\mathbf{v}_i \cdot \mathbf{z})!}{\prod_{j=1}^t (\mathbf{w}_j \cdot \mathbf{z})!} f(\mathbf{z}/n)^n$$

and define  $g : \mathbb{R}^\ell \rightarrow \mathbb{R}$  as

$$g(\zeta) = \frac{\prod_{i=1}^s (\mathbf{v}_i \cdot \zeta)^{(\mathbf{v}_i \cdot \zeta)}}{\prod_{j=1}^t (\mathbf{w}_j \cdot \zeta)^{(\mathbf{w}_j \cdot \zeta)}} f(\zeta)$$

where  $0^0 \equiv 1$ . Now suppose that, conditioned on  $M \cdot \zeta = \mathbf{y}$ ,  $g$  is maximized at some  $\zeta^*$  with  $\zeta_i^* > 0$  for all  $i$ , and write  $g_{\max} = g(\zeta^*)$ . Furthermore, suppose that the matrix of second derivatives  $g'' = \partial^2 g / \partial \zeta_i \partial \zeta_j$  is nonsingular at  $\zeta^*$ .

Then there exist constants  $A, B > 0$ , such that for any sufficiently large  $n$  for which there exist integer solutions  $\mathbf{z}$  to  $M \cdot \mathbf{z} = \mathbf{y}n$ , we have

$$A \leq \frac{S_n}{n^{-(\ell+s-t-r)/2} g_{\max}^n} \leq B .$$

For simplicity, in the proofs of (2) and (3) below we will assume that  $n$  is a multiple of  $k$ , so that nearly-balanced colorings are in fact exactly balanced, with  $n/k$  vertices in each color class. The calculations for other values of  $n$  differ by at most a multiplicative constant.

### 4 The First Moment

Clearly, all (exactly) balanced  $k$ -partitions of the  $n$  vertices are equally likely to be proper  $k$ -colorings. Therefore,  $\mathbf{E}[X]$  is the number of balanced  $k$ -partitions,  $n!/(n/k)!^k$ , times the probability that a random  $d$ -regular configuration is properly colored by a fixed balanced  $k$ -partition.

To estimate this probability we will label the  $d$  copies of each vertex, thus giving us  $(dn - 1)!!$  distinct configurations, and count the number of such configurations that are properly colored by a fixed balanced  $k$ -partition. To generate such a configuration we first determine the number of edges between each pair of color classes. Suppose there are  $b_{ij}$  edges between vertices of colors  $i$  and  $j$  for each  $i \neq j$ . Then a properly colored configuration can be generated by i) choosing which  $b_{ij}$  of the  $dn/k$  copies in each color class  $i$  are matched with copies in each color class  $j \neq i$ , and then ii) choosing one of the  $b_{ij}!$  matchings for each unordered pair  $i < j$ . Therefore, the total number of properly colored configurations is

$$\prod_{i=1}^k \frac{(dn/k)!}{\prod_{j \neq i} b_{ij}!} \cdot \prod_{i < j} b_{ij}! = \frac{(dn/k)!^k}{\prod_{i < j} b_{ij}!} .$$

Summing over all choices of the  $\{b_{ij}\}$  that satisfy the constraints

$$\forall i : \sum_j b_{ij} = dn/k , \tag{4}$$

we get

$$\begin{aligned} \mathbf{E}[X] &= \frac{n!}{(n/k)!^k} \frac{1}{(dn - 1)!!} \sum_{\{b_{ij}\}} \frac{(dn/k)!^k}{\prod_{i < j} b_{ij}!} \\ &= 2^{dn/2} \frac{n!}{(n/k)!^k} \frac{(dn/k)!^k}{(dn)!} \sum_{\{b_{ij}\}} \frac{(dn/2)!}{\prod_{i < j} b_{ij}!} . \end{aligned}$$

By Stirling’s approximation  $\sqrt{2\pi n} (n/e)^n < n! < \sqrt{4\pi n} (n/e)^n$  we get

$$\mathbf{E}[X] > D_1 \frac{2^{dn/2}}{k^{(d-1)n}} \sum_{\{b_{ij}\}} \frac{(dn/2)!}{\prod_{i < j} b_{ij}!} , \tag{5}$$

where  $D_1 = 2^{-(k+1)/2} d^{(k-1)/2}$ .

To bound the sum in (5) from below we use Lemma 3. Specifically,  $\mathbf{z}$  consists of the variables  $b_{ij}$  with  $i < j$ , so  $\ell = k(k-1)/2$ . For  $k \geq 3$ , the  $k$  constraints (4) are linearly independent, so representing them as  $M \cdot \mathbf{z} = \mathbf{y}n$  gives a matrix  $M$  of rank  $k$ . Moreover, they imply  $\sum_{i < j} b_{ij} = dn/2$ , so adding a row of 1's to the top of  $M$  and setting  $y_1 = d/2$  does not increase its rank. Integer solutions  $\mathbf{z}$  exist whenever  $n$  is a multiple of  $k$  and  $dn$  is even. We set  $s = 1$  and  $t = \ell$ ; the vector  $\mathbf{v}_1$  consists of 1's and the  $\mathbf{w}_j$  are the  $\ell$  basis vectors. Finally,  $f(\zeta) = 1$ . Thus,  $\ell + s - t - r = -(k-1)$  and

$$g(\zeta) = \frac{(d/2)^{d/2}}{\prod_{j=1}^{\ell} \zeta_k^{\zeta_j}} = \frac{1}{\prod_{j=1}^{\ell} (2\zeta_j/d)^{\zeta_j}} = e^{(d/2)H(2\zeta/d)} ,$$

where  $H$  is the entropy function  $H(\mathbf{x}) = -\sum_{j=1}^{\ell} x_j \log x_j$ .

Since  $g$  is convex it is maximized when  $\zeta_j^* = d/(2\ell)$  for all  $1 \leq j \leq \ell$ , and  $g''$  is nonsingular. Thus,  $g_{\max} = (k(k-1)/2)^{d/2}$  implying that for some  $A > 0$  and all sufficiently large  $n$

$$\begin{aligned} \mathbf{E}[X] &> D_1 \frac{2^{dn/2}}{k^{(d-1)n}} \times A n^{-(k-1)/2} \left( \frac{k(k-1)}{2} \right)^{dn/2} \\ &= D_1 A n^{-(k-1)/2} k^n \left( 1 - \frac{1}{k} \right)^{dn/2} . \end{aligned}$$

Setting  $C_1 = D_1 A$  completes the the proof.

## 5 The Second Moment

Recall that  $X$  is the sum over all balanced  $k$ -partitions of the indicators that each partition is a proper coloring. Therefore,  $\mathbf{E}[X^2]$  is the sum over all pairs of balanced  $k$ -partitions of the probability that both partitions properly color a random  $d$ -regular configuration. Given a pair of partitions  $\sigma, \tau$ , let us say that a vertex  $v$  is in class  $(i, j)$  if  $\sigma(v) = i$  and  $\tau(v) = j$ . Also, let  $a_{ij}$  denote the number of vertices in each class  $(i, j)$ . We call  $A = (a_{ij})$  the *overlap matrix* of the pair  $\sigma, \tau$ . Note that since both  $\sigma$  and  $\tau$  are balanced

$$\forall i : \sum_j a_{ij} = \sum_j a_{ji} = n/k . \tag{6}$$

We will show that for any fixed pair of  $k$ -partitions, the probability that they both properly color a random  $d$ -regular configuration depends only on their overlap matrix  $A$ . Denoting this probability by  $q(A)$ , since there are  $n!/\prod_{ij} a_{ij}!$  pairs of partitions giving rise to  $A$ , we have

$$\mathbf{E}[X^2] = \sum_A \frac{n!}{\prod_{ij} a_{ij}!} q(A) \tag{7}$$

where the sum is over matrices  $A$  satisfying (6).

Fixing a pair of partitions  $\sigma$  and  $\tau$  with overlap matrix  $A$ , similarly to the first moment, we label the  $d$  copies of each vertex thus getting  $(dn - 1)!!$  distinct configurations. To generate configurations properly colored by both  $\sigma$  and  $\tau$  we first determine the number of edges between each pair of vertex classes. Let us say that there are  $b_{ijkl}$  edges connecting vertices in class  $(i, j)$  to vertices in class  $(k, \ell)$ . By definition,  $b_{ijkl} = b_{klij}$ , and if both colorings are proper,  $b_{ijkl} = 0$  unless  $i \neq k$  and  $j \neq \ell$ . Since the configuration is  $d$ -regular, we also have

$$\forall i, j : \sum_{k \neq i, \ell \neq j} b_{ijkl} = da_{ij} . \tag{8}$$

To generate a configuration consistent with  $A$  and  $\{b_{ijkl}\}$  we now i) choose for each class  $(i, j)$ , which  $b_{ijkl}$  of its  $da_{ij}$  copies are to be matched with copies in each class  $(k, \ell)$  with  $k \neq i$  and  $\ell \neq j$ , and then ii) choose one of the  $b_{ijkl}$ ! matchings for each unordered pair of classes  $i < k, j \neq \ell$ . Thus,

$$\begin{aligned} q(A) &= \frac{1}{(dn - 1)!!} \sum_{\{b_{ijkl}\}} \left( \prod_{ij} \frac{(da_{ij})!}{\prod_{k \neq i, \ell \neq j} b_{ijkl}!} \cdot \prod_{i < k, j \neq \ell} b_{ijkl}! \right) \\ &= 2^{dn/2} \frac{\prod_{ij} (da_{ij})!}{(dn)!} \sum_{\{b_{ijkl}\}} \frac{(dn/2)!}{\prod_{i < k, j \neq \ell} b_{ijkl}!} , \end{aligned} \tag{9}$$

where the sum is over the  $\{b_{ijkl}\}$  satisfying (8). Combining (9) with (7) gives

$$\mathbf{E}[X^2] = 2^{dn/2} \sum_{\{a_{ij}\}} \sum_{\{b_{ijkl}\}} \frac{n!}{\prod_{ij} a_{ij}!} \frac{\prod_{ij} (da_{ij})!}{(dn)!} \frac{(dn/2)!}{\prod_{i < k, j \neq \ell} b_{ijkl}!} . \tag{10}$$

To bound the sum in (10) from above we use Lemma 3. We let  $\mathbf{z}$  consist of the combined set of variables  $\{a_{ij}\} \cup \{b_{ijkl} : i < k, j \neq \ell\}$ , in which case its dimensionality  $\ell$  (not to be confused with the color  $\ell$ ) is  $k^2 + (k(k - 1))^2/2$ . We represent the combined system of constraints (6), (8) as  $M \cdot \mathbf{z} = \mathbf{y}n$ . The  $k^2$  constraints (8) are, clearly, linearly independent while the  $2k$  constraints (6) have rank  $2k - 1$ . Together these imply  $\sum_{ij} a_{ij} = 1$  and  $\sum_{i < k, j \neq \ell} b_{ijkl} = d/2$ , so adding a row of 1's to the top of  $M$  does not change its rank from  $r = k^2 + 2k - 1$ . Integer solutions  $\mathbf{z}$  exist whenever  $n$  is a multiple of  $k$  and  $dn$  is even. Finally,  $f(\zeta) = 2^{d/2}$ ,  $s = k^2 + 2$  and  $t = k^2 + 1 + (k(k - 1))^2/2$ , so  $\ell + s - t - r = -2(k - 1)$ .

Writing  $\alpha_{ij}$  and  $\beta_{ijkl}$  for the components of  $\zeta$  corresponding to  $a_{ij}/n$  and  $b_{ijkl}/n$ , respectively, we thus have

$$\begin{aligned} g(\zeta) &= 2^{d/2} \frac{1}{\prod_{ij} \alpha_{ij}^{\alpha_{ij}}} \frac{\prod_{ij} (d\alpha_{ij})^{d\alpha_{ij}}}{d^d} \frac{(d/2)^{d/2}}{\prod_{i < k, j \neq \ell} \beta_{ijkl}^{\beta_{ijkl}}} \\ &= \frac{1}{\prod_{ij} \alpha_{ij}^{\alpha_{ij}}} \frac{d^{d/2} \prod_{ij} \alpha_{ij}^{d\alpha_{ij}}}{\prod_{i < k, j \neq \ell} \beta_{ijkl}^{\beta_{ijkl}}} . \end{aligned} \tag{11}$$

In the next section we maximize  $g(\zeta)$  over  $\zeta \in \mathbb{R}^\ell$  satisfying  $M \cdot \zeta = \mathbf{y}$ . We note that  $g''$  is nonsingular at the maximizer we find below, but we relegate the proof of this fact to the full paper.

## 6 A Tight Relaxation

Maximizing  $g(\zeta)$  over  $\zeta \in \mathbb{R}^\ell$  satisfying  $M \cdot \zeta = \mathbf{y}$  is greatly complicated by the constraints

$$\forall i, j: \sum_{k \neq i, \ell \neq j} \beta_{ijkl} = d\alpha_{ij} . \tag{12}$$

To overcome this issue we i) reformulate  $g(\zeta)$  and ii) relax the constraints, in a manner such that the maximum value remains unchanged while the optimization becomes much easier.

The relaxation amounts to replacing the  $k^2$  constraints (12) with their sum divided by 2, i.e., with the single constraint

$$\sum_{i < k, j \neq \ell} \beta_{ijkl} = d/2 . \tag{13}$$

But attempting to maximize (11) under this single constraint is, in fact, a bad idea since the new maximum is much greater. Instead, we maximize the following equivalent form of  $g(\zeta)$

$$g(\zeta) = \frac{1}{\prod_{ij} \alpha_{ij}^{\alpha_{ij}}} \frac{d^{d/2} \prod_{ij} \alpha_{ij}^{\sum_{k \neq i, \ell \neq j} \beta_{ijkl}}}{\prod_{i < k, j \neq \ell} \beta_{ijkl}^{\beta_{ijkl}}} , \tag{14}$$

derived by using (12) to substitute for the exponents  $d\alpha_{ij}$  in the numerator of (11). This turns out to be enough to drive the maximizer back to the subspace  $M \cdot \zeta = \mathbf{y}$ .

Specifically, let us hold  $\{\alpha_{ij}\}$  fixed and maximize  $g(\zeta)$  with respect to  $\{\beta_{ijkl}\}$  using the method of Lagrange multipliers. Since  $\log g$  is monotonically increasing in  $g$ , it is convenient to maximize  $\log g$  instead. If  $\lambda$  is the Lagrange multiplier corresponding to the constraint (13), we have for all  $i < k, j \neq \ell$ :

$$\begin{aligned} \lambda &= \frac{\partial}{\partial \beta_{ijkl}} \log g(\zeta) = \frac{\partial}{\partial \beta_{ijkl}} (\beta_{ijkl} \log(\alpha_{ij} \alpha_{k\ell}) - \beta_{ijkl} \log \beta_{ijkl}) \\ &= \log \alpha_{ij} + \log \alpha_{k\ell} - \log \beta_{ijkl} - 1 \end{aligned}$$

and so

$$\forall i < k, j \neq \ell: \beta_{ijkl} = C\alpha_{ij}\alpha_{k\ell}, \text{ where } C = e^{-\lambda-1} . \tag{15}$$

Clearly, such  $\beta_{ijkl}$  also satisfy the original constraints (12), and therefore the upper bound we obtain from this relaxation is in fact tight.

To solve for  $C$  we sum (15) and use (13), getting

$$\frac{2}{C} \sum_{i < k, j \neq \ell} \beta_{ijkl} = \frac{d}{C} = \sum_{i \neq k, j \neq \ell} \alpha_{ij}\alpha_{k\ell} = 1 - \frac{2}{k} + \sum_{ij} \alpha_{ij}^2 \equiv p .$$

Thus  $C = d/p$  and (15) becomes

$$\forall i < k, j \neq \ell: \beta_{ijkl} = \frac{d\alpha_{ij}\alpha_{k\ell}}{p} \tag{16}$$



Observe that  $p = p(\{a_{ij}\})$  is the probability that a single edge whose end-points are chosen uniformly at random is properly colored by both  $\sigma$  and  $\tau$ , if the overlap matrix is  $a_{ij} = \alpha_{ij}n$ . Moreover, the values for the  $b_{ijk\ell}$  are exactly what we would obtain, in expectation, if we chose from among the  $\binom{n}{2}$  edges with replacement, rejecting those improperly colored by  $\sigma$  or  $\tau$ , until we had  $dn/2$  edges – in other words, if our graph model was  $G(n, m)$  with replacement, rather than  $\mathcal{G}_{n,d}$ .

Substituting the values (16) in (14) and applying (13) yields the following upper bound on  $g(\zeta)$ :

$$\begin{aligned} g(\zeta) &\leq \frac{1}{\prod_{ij} \alpha_{ij}^{\alpha_{ij}}} \frac{d^{d/2} \prod_{ij} \alpha_{ij}^{(d/p)\alpha_{ij}} \sum_{i \neq k, j \neq \ell} \alpha_{k\ell}}{(d/p)^{\sum_{i < k, j \neq \ell} \beta_{ijk\ell}} \prod_{i < k, j \neq \ell} (\alpha_{ij} \alpha_{k\ell})^{(d/p)\alpha_{ij} \alpha_{k\ell}}} \\ &= \frac{1}{\prod_{ij} \alpha_{ij}^{\alpha_{ij}}} \frac{d^{d/2}}{(d/p)^{d/2}} \left( \frac{\prod_{ij} a_{ij}^{\alpha_{ij}} \sum_{i \neq k, j \neq \ell} \alpha_{k\ell}}{\prod_{i \neq k, j \neq \ell} \alpha_{ij}^{\alpha_{ij} \alpha_{k\ell}}} \right)^{d/p} \\ &= \frac{p^{d/2}}{\prod_{ij} \alpha_{ij}^{\alpha_{ij}}} \\ &\equiv g_{G(n,m)}(\{\alpha_{ij}\}) . \end{aligned}$$

In [4, Thm 5], Achlioptas and Naor showed that for  $d < c_{k-1}$  the function  $g_{G(n,m)}$  is maximized when  $\alpha_{ij} = 1/k^2$  for all  $i, j$ . In this case  $p = (1 - 1/k)^2$ , implying

$$g_{\max} \leq k^2 p^{d/2} = k^2 \left(1 - \frac{1}{k}\right)^d$$

and, therefore, that for some constant  $C_2$  and sufficiently large  $n$

$$\mathbf{E}[X^2] \leq C_2 n^{-(k-1)} k^{2n} \left(1 - \frac{1}{k}\right)^{dn} .$$

## 7 Directions for Further Work

*A Sharp Threshold for Regular Graphs.* It has long been conjectured that for every  $k > 2$ , there exists a critical constant  $c_k$  such that a random graph  $G(n, m = cn)$  is w.h.p.  $k$ -colorable if  $c < c_k$  and w.h.p. non- $k$ -colorable if  $c > c_k$ . It is reasonable to conjecture that the same is true for random regular graphs, i.e. that for all  $k > 2$ , there exists a critical integer  $d_k$  such that a random graph  $\mathcal{G}_{n,d}$  is w.h.p.  $k$ -colorable if  $d \leq d_k$  and w.h.p. non- $k$ -colorable if  $d > d_k$ . If this is true, our results imply that for  $d$  in “good” intervals  $(u_k, c_k)$  w.h.p. the chromatic number of  $\mathcal{G}_{n,d}$  is precisely  $k + 1$ , while for  $d$  in “bad” intervals  $(c_{k-1}, u_k)$  the chromatic number is w.h.p. either  $k$  or  $k + 1$ .

*Improving the Second Moment Bound.* Our proof establishes that if  $X, Y$  are the numbers of balanced  $k$ -colorings of  $\mathcal{G}_{n,d}$  and  $G(n, m = dn/2)$ , respectively,

then  $\mathbf{E}[X]^2/\mathbf{E}[X^2] = \Theta(\mathbf{E}[Y]^2/\mathbf{E}[Y^2])$ . Therefore, any improvement on the upper bound for  $\mathbf{E}[Y^2]$  given in [4] would immediately give an improved positive-probability  $k$ -colorability result for  $\mathcal{G}_{n,d}$ .

In particular, Moore has conjectured that the function  $g_{G(n,m)}$  is maximized by matrices with a certain form. If true, this immediately gives an improved lower bound,  $c_k^*$ , for  $k$ -colorability satisfying  $c_{k-1}^* \rightarrow u_k - 1$ . This would shrink the union of the “bad” intervals to a set of measure 0, with each such interval containing precisely one integer  $d$  for each  $k \geq k_0$ .

*3-Colorability of Random Regular Graphs.* It is easy to show that a random 6-regular graph is w.h.p. non-3-colorable. On the other hand, in [1] the authors showed that 4-regular graphs are w.p.p. 3-colorable. Based on considerations from statistical physics, Krzakała, Pagnani and Weigt [8] have conjectured that a random 5-regular graph is w.h.p. 3-colorable. The authors (unpublished) have shown that applying the second moment method to the number of balanced 3-colorings cannot establish this fact (even with positive probability).

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