ELSEVIER

Discrete Mathematics 179 (1998) 1–11

DISCRETE MATHEMATICS

The existence of uniquely $-G$ colourable graphs

D. Achlioptas[a], J.I. Brown[b,*], D.G. Corneil[a], M.S.O. Molloy[a]

[a] *Department of Computer Science, University of Toronto, Toronto, Canada*

[b] *Department of Mathematics, Statistics and Computing Science, Dalhousie University, Halifax, Nova Scotia, Canada B3H 3J5*

Received 17 November 1995; received in revised form 31 October 1996; accepted 11 November 1996

Abstract

Given graphs F and G and a nonnegative integer k, a function $\pi : V(F) \rightarrow \{1,\dots,k\}$ is a $-G$ *k-colouring* of F if no induced copy of G is monochromatic; F is $-G$ *k-chromatic* if F has a $-G$ k-colouring but no $-G$ $(k-1)$-colouring. Further, we say F is *uniquely* $-G$ *k-colourable* if F is $-G$ k-chromatic and, up to a permutation of colours, it has only one $-G$ k-colouring. Such notions are extensions of the well-known corresponding definitions from chromatic theory. It was conjectured that for all graphs G of order at least two and all positive integers k there exist uniquely $-G$ k-colourable graphs. We prove the conjecture and show that, in fact, in all cases infinitely many such graphs exist.

1. Introduction

There have been many generalizations of the notion of a vertex colouring of a graph. Some have attracted interest for their own sake, while others, for example, to hypergraphs, have yielded new results in chromatic theory. Most of the graph theoretic generalizations have revolved around colouring vertices so that the subgraph induced by each colour class has a given property P; properties of particular interest have included acyclicity, planarity and perfection [23,2,13,8,9,25,29]. Several authors [23,19,22,10,12] have proposed and investigated the generalized chromatic theory along these lines. In fact, in [14] it was shown that new results on hypergraph colourings related to criticality, unique colourability and complexity can be provided through generalized graph colourings.

One can view an 'ordinary' graph colouring in the following way. A function $\pi : V(F) \rightarrow \{1,\dots,k\}$ is a k-colouring of graph F if and only if no (induced) K_2 is monochromatic. For a fixed graph G of order at least 2, a function $\pi : V(F) \rightarrow \{1,\dots,k\}$ is called a $-G$ *k-colouring* of F if there is no induced copy of G that is

* Corresponding author. E-mail: brown@cs.dal.ca.

PII S0012-365X(97)00022-8

monochromatic; that is, the subgraphs of F induced by each colour class $\pi^{-1}(i)$ are G-free (i.e. do not contain an induced copy of G). F is $-G$ *k-colourable* if it has a $-G$ k-colouring, and F is $-G$ *k-chromatic*, written $\chi(F: -G) = k$, if k is the least positive integer for which F is $-G$ k-colourable. For example, the 5-cycle C_5 is $-K_2$ 3-chromatic but $-P_3$ 2-chromatic, where P_n and C_n denote the path and cycle on n points.

We remark that there is a natural complexity question related to $-G$ colourings:

$-G$ k-Colourability

Instance: A graph F.

Question: Is F $-G$ k-colourable?

This problem was introduced in [11], where it was shown to be NP-complete (and its infinite analogue undecidable) for various classes of graphs. Using the results of the present paper, the complexity was fully resolved in [1] by showing that the problem is NP-complete for all G of order greater than 2 and all k greater than 1.

One can view non $-G$ k-colourability from a Ramsey-like viewpoint (with respect to vertex partitions). Let $H \longrightarrow_k^v G$ denote the fact that for every function $\pi : V(H) \rightarrow \{1,\dots,k\}$ there is an induced subgraph G' of H with $G' \cong G$ and $V(G') \subseteq \pi^{-1}(i)$ for some i (i.e., G' is a monochromatic induced copy of G). Note that a graph F being $-G$ k-colourable is equivalent to $F \not\longrightarrow_k^v G$. Folkman [20] introduced such a notion for his study of Ramsey edge colourings of graphs, and showed that for all integers k and graphs G, there exist graphs F such that $F \not\longrightarrow_k^v G$. His work has spawned several papers on related existential and extremal problems [26–28,24,16–18].

It seems natural (in the light of chromatic theory) to consider those graphs that have the least number of $-G$ colourings. We define a graph F to be *uniquely $-G$ k-colourable* if it is $-G$ k-chromatic but, up to a permutation of colours, there is only one such $-G$ k-colouring. An example of a graph that is uniquely $-K_{1,3}$ 2-colourable is $K_{3,5}$. Various results on unique colourability with respect to other properties can be found in [7,10,13].

Uniquely k-colourable graphs have attracted considerable attention. Examples of such graphs are self-evident. Any connected bipartite graph is uniquely 2-colourable, and complete k-partite graphs are uniquely k-colourable. Thus the existence of such graphs for standard colourings is a non-issue.

However, for other types of graph colourings, the existence of uniquely colourable graphs is not certain. Grossman [21] has considered the unique Ramsey edge colourings, the problem of whether there exists for graph G a graph F such that F has exactly one 2-colouring (up to a permutation of colours) of its edges such that no partial subgraph isomorphic to G is monochromatic. This is the edge analogue of the colourings we are interested in here. Few results are known about the existence of such graphs, but in particular Grossman has shown that such graphs do not exist for stars ($K_{1,n}, n \geqslant 3$).

We show that this is not the case for $-G$ k-colourability. In [12], it was conjectured that $-G$ k-colourable graphs exist for all graphs G of order at least 2 and all positive integers k (see also [30]). What has been known so far is that the conjecture holds whenever G or its complement has a universal vertex [12] or is 2-connected [10,15,3].

In this paper we completely settle the conjecture, and in fact show (in sharp contrast to the unique Ramsey edge colouring case).

Theorem 1. *For any graph G of order at least 2 and any positive integer k, there exist infinitely many uniquely $-G$ k-colourable graphs.*

The proof will consist of three parts. The first part will classify all graphs of order at least 2 into three classes. The following sections will deal with the various classes.

2. Background

We begin with some background results and notation. In general, our hypergraph and graph notation will be standard (cf [4]).

The *complement* of a graph G is denoted by $\overline{G}$. A graph G is *k-connected* if it cannot be disconnected by the removal of fewer than k vertices. For a subset W of vertices of a graph G, we denote the induced subgraph of G on vertex set W by $\langle W\rangle$. We write $F = F_1 \uplus F_2$ if F is the disjoint union of graphs F_1 and F_2, and $F = F_1 + F_2$ if F is formed from $F_1 \uplus F_2$ by adding in all edges between F_1 and F_2 (this is often called the *join* of F_1 and F_2). A graph has a *universal vertex* if it is of the form $K_1 + F'$ for some graph F'.

The *order* of a graph or hypergraph is its number of vertices. A hypergraph is *r-uniform* if every (hyper)edge has size r. A *cycle* of length l in hypergraph H is an alternating sequence $v_0, e_0, \ldots, v_{l-1}, e_{l-1}, v_0$ ($l \geqslant 2$) of distinct vertices and edges such that $v_i \in e_{i-1} \cap e_i$ for $i = 0, \ldots, l-1 \pmod{l}$. The *girth* of H is the length of its smallest cycle (if H is acyclic, then we define the girth of H to be ∞).

A k-colouring of hypergraph H is a function $\pi : V(H) \to \{1, \ldots, k\}$ such that no edge of H is monochromatic. The definitions of k-colourability, chromatic number and unique colourability extend in the obvious way. Hypergraph colourings have been widely investigated. We relate hypergraph colourings and $-G$ colourings as follows (see [12,10]). Let G be a fixed graph of order n. For a graph F on vertex set V, form a hypergraph $\mathscr{H}^F_G$ on V whose edges are those subsets of vertices of F that induce a copy of G.

Proposition 1 (Brown [10]). *$\pi : V \to \{1, \ldots, k\}$ is a $-G$ k-colouring of F if and only if it is a k-colouring of hypergraph $\mathscr{H}^F_G$. In particular, $\chi(F\colon -G) = \chi(\mathscr{H}^F_G)$ and F is uniquely $-G$ k-colourable if and only if $\mathscr{H}^F_G$ is uniquely k-colourable.*

An important result on uniquely k-colourable hypergraphs that we shall need later is:

Theorem 2. *For all positive integers g, k and r with $r \geqslant 2$, there exist infinitely many uniquely k-colourable r-uniform hypergraphs of girth at least g.*

(This theorem was proved for $r = 2$ in [6] and for all $r \geqslant 3$ in [10,15].)

In fact, Theorem 2 was proved primarily to show the existence of uniquely $-G$ k-colourable graphs when G or $\overline{G}$ is 2-connected. We shall return to this in greater detail in a later section.

The following observation reduces our work in half.

Proposition 2 (Brown and Corneil[12]). *F is uniquely $-G$ k-colourable if and only if $\overline{F}$ is uniquely $-\overline{G}$ k-colourable.*

It is instructive and useful to sketch the two methods used previously to prove the existence of uniquely $-G$ k-colourable graphs in the cases where G or its complement has a universal vertex or is 2-connected. First, suppose that G is a graph of order at least 3 that has a universal vertex v, and let $G' = G - v$. By a result of Folkman [20], there are graphs F_k that are not $-G'$ k-colourable and whose clique number is equal to that of G' (and hence strictly smaller than that of G). If we substitute a copy of F_k for every vertex of a uniquely k-colourable graph, then the resulting graph is uniquely $-G$ k-colourable [12].

Consider the case where G is a 2-connected graph of order $n \geqslant 2$. Given an n-uniform hypergraph H of girth greater than n and a 2-connected graph G of order n, we construct a graph F by placing a copy of G down on every edge of H. More precisely, for each edge e of H we take any fixed bijection $\rho_e : e \to V(G)$ and for all $u,\ v \in V(H)$, uv is an edge if and only if there is an edge e of H containing u and v with $\rho_e(u)\rho_e(v)$ an edge of G. Let F be any such graph; we say that F is constructed by the *NR-construction* (this construction is due to Nešetřil and Rödl [26]). As noted in [26], the only 2-connected subgraphs of order at most n of F are those that are contained in an edge of H (it is here that the girth of H being more than n is used). It follows (using the previous notation) that $H = \mathscr{H}_G^F$. Hence if we choose H to be a uniquely k-colourable n-uniform hypergraph of girth greater than n, then from Proposition 1 any resulting F will be uniquely $-G$ k-colourable.

3. A classification of all graphs

In this section we show that any graph satisfies at least one of three properties. This theorem plays a key role in the later proof of the existence of uniquely $-G$ k-colourable graphs. The characterization may also be of use in other problems.

Theorem 3. *For any graph G, G or $\overline{G}$ satisfies at least one of the following properties:*

1. *it has a universal vertex,*
2. *there exists a set L of at most 2 leaves such that the graph formed by removing L is 2-connected and of order at least 3, or*
3. *it is P_4.*

Proof. To prove this result, we first need a definition and an observation. A vertex d of graph G is a *divisor* if $G - d = G_d^1 \uplus G_d^2$ where G_d^1 and G_d^2 both have order at least 2. The important observation concerning divisors is that if d is a divisor in G then $\overline{G-d} = \overline{G_d^1} + \overline{G_d^2}$ is 2-connected. This is clear since if G_d^1 and G_d^2 have orders n_1 and n_2, respectively, then $\overline{(G-d)}$ contains as a spanning subgraph the complete bipartite graph K_{n_1,n_2} which is 2-connected as $n_1,\ n_2 \geqslant 2$.

Let the order of G be n. We can assume that neither G nor $\overline{G}$ has a universal vertex, and that G (and $\overline{G}$) is not isomorphic to P_4. We may also assume that neither G nor $\overline{G}$ is 2-connected, for otherwise we are done. As at least one of G and $\overline{G}$ is connected, it is straightforward to verify that we have covered all (connected) graphs of order at most 4. Thus, we can also assume that G has order at least 5, and we need only show that the second case occurs, i.e. for one of G or $\overline{G}$, there is a set L of 1 or 2 leaves such that the graph formed by removing L is 2-connected and of order at least 3.

We will examine two cases depending on the existence of a divisor in G or $\overline{G}$:

- *G or $\overline{G}$ contains a divisor d.*
  Without loss of generality G has a divisor d. As neither G nor $\overline{G}$ has a universal vertex, the degree of d is in the set $\{1,\ldots,n-2\}$ for both graphs. From the observation of divisors, we know that $\overline{G-d}$ is 2-connected. If the degree of d in $\overline{G}$ is at least 2, then clearly $\overline{G}$ is also 2-connected, a contradiction. Thus, the degree of d in $\overline{G}$ is 1, that is, d is a leaf of $\overline{G}$. Taking $L = \{d\}$ we are done as $\overline{G} - L$ has order $n - 1 \geqslant 4$.

- *Neither G nor $\overline{G}$ contains a divisor.*
  We assume, without loss of generality, that G is connected. Let L be the set of leaves of G, and let $l = |L|$. Since G is connected and the removal of leaves does not affect connectivity, $F = G - L$ is connected.

  We claim that in fact F is 2-connected. If $|V(F)| \leqslant 2$ this is trivial by our assumptions that G is connected and has no universal vertex (i.e., F must be K_2). If $|V(F)| > 2$, suppose (to reach a contradiction) that F has a cut vertex d; that is, $F - d = F_1 \uplus F_2$, for some graphs F_1 and F_2. Without loss of generality, $|V(F_1)| \leqslant |V(F_2)|$. Let $L' \subseteq L$ be those leaves of G adjacent to any vertex of F_1. If $F_1 = \{v\}$, then v has degree 1 in F since F is connected, and as $v \notin L$, the degree of v in G is at least 2, that is, L' is not empty. Thus, $G - d$ is the disjoint union of the subgraphs induced by $F_1 \cup L'$ and $F_2 \cup (L - L')$. Note that both of these sets have cardinality at least 2, since if F_1 has order 1, then $|L'| \geqslant 1$ and $|V(F_2)| = |V(F)| - 1 \geqslant 2$, and if F_1 has order at least 2, then so does F_2. Thus, d is a divisor of G, a contradiction. It follows that F has no cut vertex; thus it is 2-connected.

  To conclude the proof we need to show that $l = |L| \leqslant 2$. Since G has order at least 5, if two leaves v_1, v_2 of G are adjacent to the same vertex d of G, then d is a divisor of G, a contradiction. Thus the edges incident to L form a matching M in G, so $|V(F)| = n - l \geqslant l$. This implies that $\overline{G}$ contains as a spanning subgraph

$(K_l + \overline{K_{n-l}}) - M$. It is straightforward to verify that if $l \geqslant 3$ then $(K_l + \overline{K_{n-l}}) - M$ is 2-connected, and hence $\overline{G}$ is 2-connected, a contradiction. Thus, $l = |L| \leqslant 2$ and we are done. □

4. The existence for all graphs but one

We now utilise the characterization of the previous section to prove the existence of uniquely $-G$ k-colourable graphs for all graphs G except one, P_4.

Theorem 4. *If k is any positive integer and G is any graph of order at least 2 other than P_4, then there exist infinitely many uniquely $-G$ k-colourable graphs.*

Proof. Let n denote the order of G. From Proposition 2, we can assume that $n \geqslant 3$ (as there are infinitely many uniquely k-colourable graphs). We can also assume from Proposition 2 and Theorem 3 that, without loss of generality either G has a universal vertex or if we remove a set L of at most 2 leaves of G, the remaining graph $G - L$ is a 2-connected graph of order at least 3.

First, let us consider the case where G has a universal vertex v. As mentioned in Section 2 (and proved in [12]), if F_k is a graph that is not $-(G-v)$ k-colourable and whose clique number is equal to that of $G-v$, then substituting a copy of F_k for every vertex of a uniquely k-colourable graph yields a uniquely $-G$ k-colourable graph. The existence of infinitely many uniquely k-colourable graphs completes the argument for G having a universal vertex.

Now assume that there is a set L of at most 2 leaves of G such that the remaining graph $G-L$ is a 2-connected graph of order at least 3. Let H be any n-uniform uniquely k-colourable hypergraph of girth at least $n+1$, as is known to exist by Theorem 2, and let its colour classes be $V_1, \ldots, V_k$. If we (independently) place on each edge of H a copy of G, the resulting graph F has the property that there is *at most* (up to a permutation of colours) one $-G$ k-colouring of F, namely the k-colouring of H, as in any other possible k-colouring, an edge of H is monochromatic, and hence a copy of G is monochromatic. The only problem is that F may have no $-G$ k-colouring at all! To prevent this, we should be more careful in placing the copies of G on each edge of H.

Note that there is exactly one induced subgraph of G that is isomorphic to $G - L$, namely $G - L$ itself, since any induced subgraph G' of G of order at least 3 that contains a vertex v of L (a leaf of G) cannot be 2-connected. For each edge e of H, we place a copy of G down on e in such a way that the vertices of $G - L$ are not within any V_i; this can be done as no edge of e is contained within any V_i. Let F be any such resulting graph. By the properties of the NR-construction, the only copies of the 2-connected graph $G-L$ in F are those within an edge of H, and we have ensured that none of these are within any class V_i. It follows that indeed $V_1, \ldots, V_k$ yield a $-G$ k-colouring of F, as if no copy of $G - L$ is within a V_i, then clearly no copy of G is within a V_i. Thus, F is a uniquely $-G$ k-colourable graph. Since there are infinitely many such choices for hypergraph H [14], there are infinitely many uniquely $-G$ k-colourable graphs in this case as well. $\square$

5. The case $G = P_4$

The case $G = P_4$ is the one case that eludes the arguments of the previous section. For while P_4 indeed has a set L of two leaves such that $P_4 - L$ is 2-connected, the order of $P_4 - L$ is less than 3 and it does not have the property that it only has one copy of $P_4 - L$, as was needed for the proof of Theorem 4. It may be surprising that such difficulty lies in proving the existence of uniquely $-G$ k-colourable graphs for one particular graph, P_4. However, P_4 has often been an 'odd' graph, in that it is the smallest nontrivial self-complementary graph. The approach we take for P_4 is a probabilistic one. We refer the reader to [5] for a general reference on the probabilistic method.

Throughout this section, $k \geqslant 2$ is a fixed positive integer, n is an arbitrary large integer, and C and C' denote constants (depending on k, but not on n). In this section we only claim statements to be true for sufficiently large n, and all asymptotics are taken as n tends to infinity.

Theorem 5. *For all $k \geqslant 2$, there are infinitely many uniquely $-P_4$ k-colourable graphs.*

Proof. Let $V_1, \ldots, V_k$ be disjoint copies of $K_{n,n,\ldots,n}$, the complete n-partite graph with cells each of size n (we denote the lth cell of V_i by V_i^l). For any vertices v and w in different V_i's, we randomly (and independently) take the edge between vw with probability $\frac{1}{2}$. Thus our sample space Ω_n consists of $2^{\binom{k}{2}n^4}$ equally likely graphs. Clearly, any graph F in Ω_n is $-P_4$ k-colourable, as colouring each V_i with colour i yields a $-P_4$ colouring of F ($K_{n,n,\ldots,n}$ is P_4-free). We will show that most members of Ω_n are uniquely $-P_4$ k-colourable, by showing that with probability tending to one, there is no other $-P_4$ k-colouring (this shows, in particular, that $\chi(F{:}\,-P_4) = k$ as recolouring any vertex of a colour class of size at least 2 in a $-P_4$ $(k-1)$-colouring of F with a new colour yields a different $-P_4$ k-colouring).

Let

$$\alpha \in \left(0, \frac{1}{2k}\right), \quad \beta = 1 - \alpha(k-1), \quad \delta \in (0, \beta) \quad \text{and} \quad \gamma = \frac{\beta - \delta}{1 - \delta}$$

all be fixed rational numbers (they will be chosen precisely later). Note that if α is chosen small enough, β and δ can be chosen as close to 1 as we like.

Consider the following two events:

1. E_1^{α}: there are distinct $i, j \in \{1, \ldots, k\}$, distinct $l, l' \in \{1, \ldots, n\}$, $m \in \{1, \ldots, n\}$ and sets $A_1 \subset V_i^l$, $A_2 \subset V_i^{l'}$ and $A_3 \subset V_j^m$, each of cardinality αn such that the subgraph induced by $A_1 \cup A_2 \cup A_3$ is P_4-free.

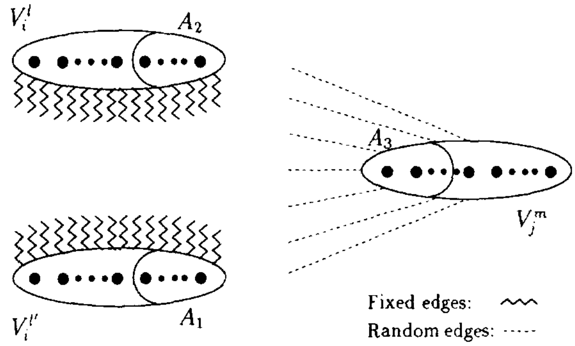

Fig. 1. E_1^α.

2. $E_2^{\alpha,\delta}$: there is a $j \in \{1,\dots,k\}$, a set $S \subset \{1,\dots,n\}$ of cardinality γn, sets $A_s \subset V_j^s$ $(s \in S)$, each of cardinality δn, and a vertex $x \notin V_j$ such that the subgraph induced by $\left(\bigcup_{s\in S} A_s\right) \cup \{x\}$ is P_4-free.

We claim now that if $F \in \Omega_n$ is not uniquely $-P_4$ k-colourable, then either E_1^α or $E_2^{\alpha,\delta}$ occurs. For suppose that there were a $-P_4$ k-colouring of $F \in \Omega_n$ whose colour classes were different from $V_1,\dots,V_k$. Then clearly in such a colouring there would be a colour class W' of cardinality at least n^2 that is different from each V_i; by choosing a subset of W', we can assume that there is a subset W of cardinality exactly n^2 such that the induced subgraph $\langle W\rangle$ is P_4-free. Assume that case E_1^α does not occur. If j is an index such that $|W \cap V_j|$ is maximum, then clearly $|W \cap V_j| \geqslant |W|/k = n^2/k$, so W intersects at least two $V_j^{j'}$'s in at least αn vertices (for otherwise, $|W \cap V_j| < n + (\alpha n)(n-1) < n^2/k$ for $n \geqslant 2k$, and we can assume the latter). Since E_1^α does not occur, it follows that for $i \in \{1,\dots,k\} - \{j\}$, and $i' \in \{1,\dots,n\}$, $|W \cap V_i^{i'}| < \alpha n$, and so $|W - V_j| < (k-1)(\alpha n)n = (k-1)\alpha n^2$, implying that $|W \cap V_j| \geqslant n^2 - (k-1)\alpha n^2 = \beta n^2$. If exactly l of the $V_j^{j'}$'s have $|W \cap V_j^{j'}| < \delta n$, then $l(\delta n) + (n-l)n \geqslant \beta n^2$ and it follows that at least γn of the $V_j^{j'}$'s intersect W in at least δn points, so by choosing subsets of these, we see that $E_2^{\alpha,\delta}$ holds (note that W contains a vertex x outside of V_j in this case as W is a set of cardinality n^2 that is not V_j). Thus it suffices to show that for some choices of α and δ, the probability of E_1^α or $E_2^{\alpha,\delta}$ occurring is o(1).

Let us handle the first case for any fixed α (see Fig. 1). We can choose i, j, A_1, A_2 and A_3 in

$$k(k-1)\binom{n}{2} n \binom{n}{\alpha n}^3,$$

ways. Moreover, as $\langle A_1 \cup A_2 \cup A_3\rangle$ is P_4-free, there are three cases for any vertex $v \in A_3$:

- for $p = 1,\ 2$, v is joined to all or none of A_p, or
- v is joined to at least one vertex but not all of A_1, and all of A_2, or
- v is joined to at least one vertex but not all of A_2, and all of A_1.

For fixed A_1, A_2 and A_3, the probability of the first case occurring is $(2 \cdot 2^{-\alpha n})^2$, and

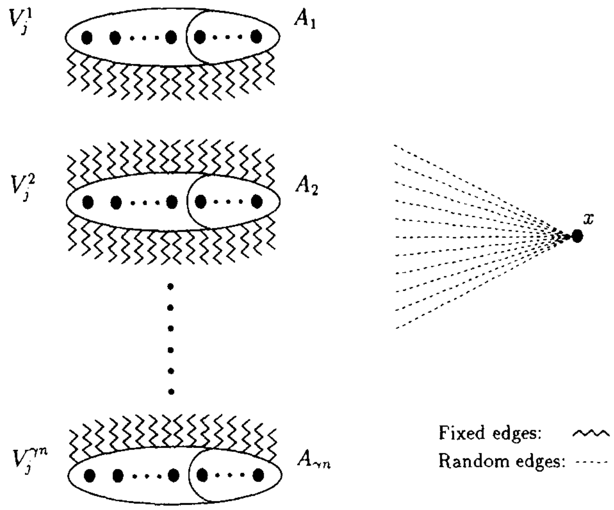

Fig. 2. $E_2^{\alpha,\delta}$ (with $S = \{1, \ldots, \gamma n\}$).

each of the other cases has probability $(1 - 2 \cdot 2^{-\alpha n})2^{-\alpha n}$. These are independent for each of the αn vertices $v \in A_3$, so we have

$$\begin{aligned}\mathrm{Prob}(E_1^{\alpha}) &\leqslant k(k-1)\binom{n}{2} n \binom{n}{\alpha n}^3 \left((2 \cdot 2^{-\alpha n})^2 + 2((1 - 2 \cdot 2^{-\alpha n})2^{-\alpha n})\right)^{\alpha n} \\ &< k^2 n^3 2^{3n} 2^{-(\alpha n)^2} (4 \cdot 2^{-\alpha n} + 2(1 - 2 \cdot 2^{-\alpha n}))^{\alpha n} \\ &< 4k^2 n^3 2^{-(\alpha n)^2 + 3n} 2^{\alpha n} \\ &= \mathrm{o}(1).\end{aligned}$$

We will now turn to the harder case of estimating the probability of $E_2^{\alpha,\delta}$. We will also see how we will choose δ (and α). Fig. 2 illustrates this case. Again as in the discussion for E_1^{α},

- the vertex x is joined to all or none of each A_s, $(s \in S)$, or
- for exactly one s, say s', x is joined to at least one vertex but not all of $A_{s'}$, and all of the vertices in the other A_s's.

Therefore,

$$\begin{aligned}\mathrm{Prob}(E_2^{\alpha,\delta}) &\leqslant k(k-1)n^2 \binom{n}{\gamma n} \binom{n}{\delta n}^{\gamma n} \left((2 \cdot 2^{-\delta n})^{\gamma n} + \gamma n (1 - 2 \cdot 2^{-\delta n})(2^{-\delta n})^{\gamma n - 1}\right) \\ &\leqslant C n^2 2^n \left(\delta^{-\delta n}(1-\delta)^{-(1-\delta)n}\right)^{\gamma n} \left(2^{\gamma n - \delta \gamma n^2} + \gamma n 2^{\delta n - \delta \gamma n^2}\right) \\ &\leqslant C' n^3 2^{2n} \delta^{-\delta \gamma n^2} (1-\delta)^{-(1-\delta)\gamma n^2} 2^{-\delta \gamma n^2}. \\ &= \mathrm{o}(1), \quad \text{for } .8 < \delta < 1.\end{aligned}$$

Note that if we choose $\alpha \leqslant 0.1/(k-1)$ $(\leqslant 1/2k)$, then β lies in $(0.9, 1)$, and hence we can choose $\delta \in (0.9, \beta)$, such that $\mathrm{Prob}(E_1^{\alpha}) + \mathrm{Prob}(E_2^{\alpha,\delta}) = \mathrm{o}(1)$. $\square$

We point out that the proof above shows much more than the existence of uniquely $-P_4$ k-colourable graphs. It proves that for large m there are $-P_4$ k-chromatic graphs on km vertices such that the colour classes in a $-P_4$ k-colouring of the graph each have cardinality m, and there are no other induced subgraphs of order m that are P_4-free.

6. Concluding remarks

We have shown that for any graph G of order at least 2 and any integer $k \geqslant 2$ there exist infinitely many uniquely $-G$ k-colourable graphs. This naturally gives rise to the following question: determine

$$u(G,k) \equiv \min\{n\text{: there is a uniquely } -G \ k\text{-colourable graph of order } n\}.$$

Clearly, $u(\overline{G},k) = u(G,k)$ and $u(K_2,k) = k$.

It seems extremely difficult to calculate or estimate $u(G,k)$. It is known [16,17] that $f(G,k)$, the minimum order of a $-G$ k-chromatic graph, for fixed k satisfies (for some constants C_1 and C_2)

$$C_1 n^2 \leqslant f(G,k) \leqslant C_2 n^2 \log^2 n,$$

where $n = |V(G)|$, and for fixed G (with neither G nor its complement complete),

$$C_3 k \log k \leqslant f(G,k) \leqslant C_4 k \log k,$$

(for some constants C_3 and C_4). These obviously imply some (weak) lower bounds on $u(G,k)$.

Even for small graphs G and $k = 2$, the problem is nontrivial. It is not hard to see that $u(K_2,2) = 2$, and $u(P_3,2) = 5$, as any graph of order at most 4 is either $-P_3$ 1-colourable or has at least 2 inequivalent $-P_3$ 2-colourings, and $K_{3,2}$ is uniquely $-K_{1,2}$ 2-colourable. In fact, $K_{2n-1,n}$ is uniquely $-K_{1,n}$ 2-colourable for all n, so $u(K_{1,n},2) \leqslant 3n-1$. We do not know if equality holds for $n \geqslant 3$.

Acknowledgements

The authors wish to thank the Natural Sciences and Engineering Research Council of Canada for financial assistance.

References

[1] D. Achlioptas, The complexity of G-free colourability, Discrete. Math., to appear.

[2] M.O. Albertson, R.E. Jamison, S.T. Hedetniemi, S.C. Locke, The subchromatic number of a graph, Discrete. Math. 74 (1989) 33–49.

[3] G. Benadé, I. Broere, J.I. Brown, A construction of uniquely C_4-free colourable graphs, Quaestiones Math. 13 (1990) 259–264.

[4] C. Berge, Graphs and Hypergraphs, North Holland, New York, 1979.
[5] B. Bollobás, Random Graphs, Academic Press, London, 1985.
[6] B. Bollobás, N. Sauer, Uniquely colourable graphs with large girth, Canad. J. Math. 28 (1970) 1340–1344.
[7] B. Bollobás, A.G. Thomason, Uniquely partitionable graphs, J. London Math. Soc. 16 (1977) 403–410.
[8] M. Borowiecki, P. Mihók, On (n,P)-partitionable graphs. Graphs, Hypergraphs and Applications, Teubner-Texte, Leipzig, 1985 pp. 15–18.
[9] I. Broere, C. Mynhardt, Generalized colorings of outerplanar and planar graphs, in: Y. Alavi et al., (Eds.), Graph Theory with Applications to Algorithms and Computer Science, Wiley, New York, 1985, pp. 151–161.
[10] J.I. Brown, A theory of generalized graph colourings, Ph. D. Thesis, Department of Mathematics, University of Toronto, 1987.
[11] J.I. Brown, The complexity of generalized graph coloring problems, Discrete. Appl. Math. 69 (1996) 245–253.
[12] J.I. Brown, D.G. Corneil, On generalized graph colorings, J. Graph Theory 11 (1987) 87–99.
[13] J.I. Brown, D.G. Corneil, Perfect colourings, Ars Combin. 30 (1990) 141–159.
[14] J.I. Brown, D.G. Corneil, Graph properties and hypergraph colourings, Discrete Math. 98 (1991) 81–93.
[15] J.I. Brown, D.G. Corneil, On uniquely $-G$ k-colourable graphs, Quaestiones Math. 15 (1992) 477–487.
[16] J.I. Brown, V. Rödl, A new construction of k-Folkman graphs, Ars Combin. 29 (1990) 265–269.
[17] J.I. Brown, V. Rödl, A Ramsey type problem concerning vertex colourings, J. Combin. Theory B 52 (1991) 45–52.
[18] J.I. Brown, V. Rödl, Folkman numbers for graphs of small order, Ars Combin. A 35 (1993) 11–27.
[19] E. J. Cockayne, Colour classes for r-graphs, Canad. Math. Bull. 15 (1972) 349–354.
[20] J. Folkman, Graphs with monochromatic complete subgraphs in every edge coloring, SIAM J. Appl. Math. 18 (1970) 19–24.
[21] J.W. Grossman, Graphs with unique Ramsey colourings, J. Graph Theory 7 (1983) 85–90.
[22] F. Harary, Conditional colorability of graphs. Graphs and Applications, in: F. Harary, J. Maybee (Eds), Proc. 1st Col. Symp. Graph Theory, Wiley, New York, 1985, pp. 127–136.
[23] S. Hedetniemi, On partitioning planar graphs, Canad. Math. Bull. 11 (1968) 203–211.
[24] P. Komjath, V. Rödl, Coloring of universal graphs, Discrete. Math. 63 (1986) 55–60.
[25] C.M. Mynhardt, I. Broere, Generalized colourings of graphs, in: Y. Alavi et al. (Eds), Graph Theory with Applications to Algorithms and Computer Science, Wiley, New York, 1985, pp. 583–594.
[26] J. Nešetřil, V. Rödl, Partitions of vertices, Comm. Math. Univ. Carolinae 17 (1976) 85–95.
[27] J. Nešetřil, V. Rödl, The Ramsey property for graphs with forbidden complete subgraphs, J. Combin. Theory. B 20 (1976) 243–249.
[28] J. Nešetřil, V. Rödl, On Ramsey graphs without cycles of short odd length, Comm. Math. Univ. Carolinae 20 (1979) 565–582.
[29] E. Sampathkumar, C.V. Venkatachalam, Chromatic partitions of a graph, Discrete. Math. 74 (1989) 227–239.
[30] D.B. West, Open Problems, SIAM Activity Group Discrete Math. Newsletter 5(2) (1995) 8–9.