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The complexity of G-free colourability

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Abstract

The problem of determining if a graph is 2-colourable (i.e., bipartite) has long been known to have a simple polynomial time algorithm. Being 2-colourable is equivalent to having a bipartition of the vertex set where each cell is K_2-free. We extend this notion to determining if there exists a bipartition where each cell is G-free for some *fixed* graph G. One might expect that for some graphs other than $K_2, \overline{K_2}$ there also exist polynomial time algorithms. Rather surprisingly we show that for *any* graph G on more than two vertices the problem is NP-complete.

1. Introduction

A *vertex k-colouring* of a graph is an assignment of one of k colours to each vertex such that adjacent vertices receive different colours. Such colourings have been studied extensively and form one of the oldest and deepest areas of graph theory. In this course of study many generalisations of the colouring concept have been suggested. The following two notions, introduced in [13], appear to be useful in expressing such generalisations in a uniform fashion:

Let a property on graphs be a subset of the set of all graphs. Given a property π, a nonnegative integer k and a graph H, a π *k-colouring* of H is a function col from the vertex set of H to $\{1,\dots,k\}$ such that the subgraph induced by each colour class has (belongs to) property π. The π *chromatic number* of H, $\chi_\pi(H)$, is the least k for which H has a π k-colouring.

Using the notion of π-colourings we see that colouring problems have been studied for a wide range of π. In standard colouring π is an independent set. In [9] π is having no path of length greater than some fixed m, while in [15] a similar bound is imposed on the size of any clique. The case where π is a forest has been studied in [10]. In [5] π is perfect and the corresponding chromatic number provides a new measure of a graph's imperfection.

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The fundamental property for which we would like to examine π-colourings is being G-free, i.e., having no induced subgraph isomorphic to G, for some fixed graph G. Note that all the properties mentioned above can be expressed as the intersection of some G-free properties. In fact, any hereditary property can be expressed as the, perhaps infinite, intersection of G-free properties. In this paper we examine the complexity of deciding whether a graph has a G-free k-colouring, captured by the following problem:

Definition 1. G-free k-Colourability.
Instance: A graph H.
Question: Is there a G-free k-colouring of H?

The complexity of G-free k-colourability has been studied for various cases of G and k in [2,3]. The following comprises all the cases for which the problem has been shown to be NP-complete:

- $G = P_4$ and $k \geqslant 3$.
- G is the disjoint union of two graphs and $k \geqslant 3$.
- G is 2-connected and $k \geqslant 2$.

We focus on the $k = 2$ case. The machinery we develop for this case makes the extension of our hardness result for $k \geqslant 3$ easy (Appendix A). When $k = 2$ and G has two vertices G-free k-colourability is the problem of deciding if the input graph (or its complement) is bipartite. The simple algorithm for solving this problem is based on the fact that a graph is bipartite iff it has no odd length cycles. On the other hand, by the last of the above results, we should not expect to find a polynomial time algorithm when G is 2-connected. We show that the structure of G is irrelevant as the problem is NP-complete for any graph other than $K_2, \overline{K_2}$ for all $k \geqslant 2$.

We prove this rather surprising result by using an approach very different from that used to prove NP-completeness when $k = 2$ and G is 2-connected. In that proof a reduction of Hypergraph 2-Colorability [11], for a special class of hypergraphs, to G-free 2-colourability is used by applying the Něsětril–Rödl construction [14] to the input hypergraph and G. The result is a graph whose G-free chromatic number is equal to the chromatic number of the hypergraph. For this, non-trivial equality to hold along with the hypergraph's special structure, the 2-connectivity of G is essential. Our reduction depends only on $|V_G| > 2$ and for every graph G it assumes the existence of a special graph, called a G-gadget. This allows for a uniform reduction for all graphs. To conclude the proof we reduce the construction of G-gadgets to the provision of uniquely G-free colourable graphs.

2. Background

All the graphs considered are simple and loopless. The vertex and edge sets of a graph G are denoted by V_G and E_G, respectively. The *complement* of a graph G is denoted by $\overline{G}$. The term *subgraph* will always be taken to mean induced subgraph;

that is, H is a subgraph of G iff $V_H \subseteq V_G$ and $\{v_1, v_2\} \in E_H \iff \{v_1, v_2\} \in E_G$, $\forall\, v_1, v_2 \in V_H$. For a given set of vertices $U \subseteq V_G$, we let $G[U]$ denote the subgraph of G with vertex set U. We refer the subgraph of G *induced* by U by $G[U]$. The *neighbourhood* of a vertex v in a graph G is denoted by $\Gamma_G(v)$ where the subscript will be omitted when it is clear from the context.

The operation of *removing* a set of vertices $I \subseteq V_G$ from a graph G is denoted by $G - I$ and results in $G[V_G - I]$. The *disjoint union* of vertex disjoint graphs G and H is the graph with vertex set $V_G \cup V_H$ and edge set $E_G \cup E_H$.

A property π is a subset of the set of all graphs (closed under isomorphism) that contains K_0 and K_1. The property of being G-free, where $|V_G| \geqslant 2$, is the set of all graphs that contain no subgraph isomorphic to G. If π is a property, then the *complement* of π, $\pi^c = \{\overline{G}\colon G \in \pi\}$ is also a property. For example, if $\pi = G$-free then $\pi^c = \overline{G}$-free. Given a property π, a nonnegative integer k and a graph G, a π *k-colouring* of G is a function col: $V_G \to \{1, \ldots, k\}$ such that the subgraph induced by each colour class has property π. If W is a set of vertices, we take $\mathrm{col}(W) = \cup_{w \in W} \{\mathrm{col}(w)\}$. We say that G is *π k-colourable* if it has a π k-colouring. Note that including K_0 in every property guarantees that a π k-colouring is a π l-colouring for any $l \geqslant k$ and including K_1 guarantees that $\chi_\pi(G) \leqslant |V_G|$. We say that two colourings $\mathrm{col}_1, \mathrm{col}_2$ are *equivalent* if there exists a permutation σ of $\{1, \ldots, k\}$ such that $\mathrm{col}_1 = \sigma \circ \mathrm{col}_2$. A graph is *uniquely* π k-colourable if it has only one π k-colouring up to equivalence.

3. The reduction

3.1. Preliminaries

A fact following from the definition of a property and its complement is that

Fact 1. *A G-free k-colouring of a graph H is a $\overline{G}$-free k-colouring of $\overline{H}$.*

As mentioned above, when $|V_G| = 2$ the problem is solvable in polynomial time. We show that, if P$\neq$NP, this is the only case for which this is true:

Theorem 1. *If $|V_G| > 2$ then G-free 2-colourability is* NP-*complete.*

A first step in proving Theorem 1 is suggested by Fact 1. Since for any graph G at least one of G, $\overline{G}$ is connected, it suffices to examine the complexity of the problem for connected graphs. Thus in the following, G is assumed to be *connected* and on *more than two* vertices.

Our reduction treats all graphs in a uniform fashion. This uniformity comes from assuming for each graph G the existence of a special graph $\mathcal{G}$, which we call a G-gadget, defined as follows:

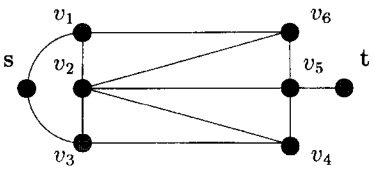

Fig. 1. An ad hoc P_3-gadget. (Observe that vertices v_2 and v_5 it must receive the same colour in any P_3–free 2–colouring, even if we remove vertices s and t)

Definition 2. A G-free 2-colourable graph $\mathscr{G}$ is called a G-gadget if $V_{\mathscr{G}}$ contains fixed vertices s, t such that in every G-free 2-colouring of $\mathscr{G}$:

- $\mathsf{col}(s) \neq \mathsf{col}(t)$.
- $\mathsf{col}(s) \notin \mathsf{col}(\Gamma(s))$ and $\mathsf{col}(t) \notin \mathsf{col}(\Gamma(t))$.

Note that since we discuss 2-colourability the statement $\mathsf{col}(v) \notin \mathsf{col}(\Gamma(v))$ implies that $\Gamma(v)$ is monochromatic.

To prove Theorem 1 we will reduce Distinct NOT-ALL-EQUAL k-SAT (denoted Distinct NAE k-SAT and defined below) to G-free 2-colourability, where $k = |V_G| > 2$. We prove that Distinct NAE k-SAT is NP-complete by reducing NAE 3SAT [11] to it (Appendix B). An ad hoc P_3-gadget is shown in Fig. 1.

Definition 3. Distinct NOT-ALL-EQUAL k-SAT
Instance: Set U of variables, collection C of clauses over U such that for each clause $c \in C$, $|c| = k$ and all the literals in c are distinct.
Question: Is there a truth assignment for U such that each clause in C has at least one true literal and at least one false literal?
Note: We call such a truth assignment a NAE one.

3.2. *The construction*

Given an instance I of Distinct NAE k-SAT, (i.e., a set U of variables and a collection C of clauses) we will construct a graph $\mathscr{F}(I)$ as follows:

The vertex set of $\mathscr{F}(I)$ and most of its edges depend only on $|U|$ and $\ell = 2|C|$. We consider the graph defined by these two parameters as the *Skeleton* of $\mathscr{F}(I)$ and denote it by $\mathcal{S}$. The rest of $\mathscr{F}(I)$'s edges, the *Connections*, are a result of adding $|E_G|$ edges for each clause $c \in C$. We will describe the construction of $\mathscr{F}(I)$ in two separate parts, as suggested by its structure.
Skeleton: The skeleton $\mathscr{S}$ is the disjoint union of $|U|$ copies of a graph R_ℓ. To describe R_ℓ we define the *Join* operation between two graphs:

Definition 4. Given vertices v_1, v_2 of graphs H_1, H_2, respectively, we define the function $Join(H_1, v_1, H_2, v_2) = H$ such that:

- $V_H = (V_{H_1} - \{v_1\}) \cup (V_{H_2} - \{v_2\}) \cup \{v\}$, where v is a new vertex.
- $E_H = E_{H_1 - \{v_1\}} \cup E_{H_2 - \{v_2\}} \cup \{\{v, w\} : w \in \Gamma_{H_1}(v_1) \cup \Gamma_{H_2}(v_2)\}$.

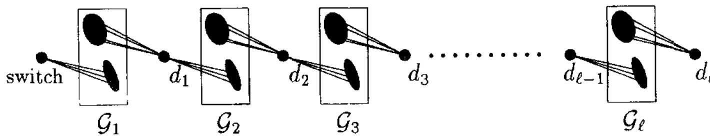

Fig. 2. R_ℓ

We say that vertex v is the result of identifying v_1 with v_2. Let H_1', H_2' be the subgraphs of H induced by $(V_{H_1} - v_1) \cup \{v\}$, $(V_{H_2} - v_2) \cup \{v\}$, respectively. It is easy to see that the former is isomorphic to H_1 and the latter to H_2.

In view of Definition 4, the graph R_ℓ can be seen as a "chain" of ℓ G-gadgets joined together by identifying the t vertex of each G-gadget with the s vertex of its successor (see Fig. 2). Formally, R_ℓ is defined by

$$R_{i+1} = Join(R_i, t_i, \mathscr{G}_{i+1}, s_{i+1}),$$

where $R_1 = \mathscr{G}_i = \mathscr{G}$ is some G-gadget and s_i, t_i are the s, t vertices, respectively, of the ith copy of $\mathcal{G}$ used ($\mathscr{G}_i$). The vertex resulting by identifying t_i with s_{i+1} is labelled as d_i and we will refer to all d_i vertices, collectively, as *outer* vertices. We also label t_ℓ as d_ℓ making it an outer vertex while vertex s_1 is labelled as "switch" and is *not* an outer vertex.

Before we proceed to the construction of the *Connections* we prove the following two lemmata. They demonstrate the role of $\mathcal{S}$ as a provider of vertices that will receive consistent colours in any of its G-free 2-colourings.

Lemma 1. *$\mathcal{S}$ is G-free 2-colourable.*

Proof. Since G is connected and $\mathscr{S}$ is the disjoint union of copies of R_ℓ it suffices to prove that R_ℓ is G-free 2-colourable. We will prove, inductively, that R_i is G-free 2-colourable for all $i \geqslant 1$. The base case, $i = 1$, holds by the definition of a G-gadget. We claim that if R_i is G-free 2-colourable, for $i = n$ then $R_{n+1} = Join(R_n, t_n, \mathscr{G}_{n+1}, s_{n+1})$ is also G-free 2-colourable. To see this, we G-free 2-colour R_n and $\mathscr{G}_{n+1}$ before we *Join* them so that $\mathrm{col}(t_n) = \mathrm{col}(s_{n+1})$. This is feasible by the inductive hypothesis and the base case, respectively. In R_{n+1} all the vertices retain their *"pre-Join"* colours while d_n takes the common *pre-Join* colour of t_n and s_{n+1}. The graph induced by $(V_{R_n} - \{t_n\}) \cup \{d_n\}$ is isomorphic to and identically coloured as R_n. Since the latter was G-free 2-coloured and G is connected, any monochromatic copy of G in R_{n+1} must contain d_n and at least one vertex $v \in \Gamma(d_n) \cap V_{\mathscr{G}_{n+1}}$. By the definition of a G-gadget and $\mathscr{G}_{n+1}$'s *pre-Join* colouring, $\mathrm{col}(d_n) \neq \mathrm{col}(v)$ for any such v and thus R_{n+1} is G-free 2-colourable. $\square$

Lemma 2. *In any G-free 2-colouring of R_ℓ:*

$$\mathrm{col}(d_i) = \mathrm{col}(switch) \text{ iff } i \bmod 2 = 0.$$

Proof. By the definition of a G-gadget and the construction of R_ℓ, in any G-free 2-colouring of the latter, $\mathsf{col}(d_i) \neq \mathsf{col}(d_{i+1})$, $\forall\, i \in [1,\ldots,\ell-1]$. Since, by the definition of a G-gadget, $\mathsf{col}(switch) \neq \mathsf{col}(d_1)$ the lemma follows. □

Connections: In the following by $switch_u$ and $d_{u,i}$ we denote the *switch* and d_i vertices, respectively, of the copy of R_ℓ associated with variable u ($R_{u,\ell}$) (Fig. 3). For a clause $c_j = \bigvee_{i=1}^{k} l_{j,i}$, let $\mathsf{I}(c_j) = \bigcup_{i=1}^{k} \mathsf{image}(l_{j,i})$, where

$$\mathsf{image}(l_{j,i}) = \begin{cases} d_{u,2j-1} & \text{if } l_{j,i} = \neg u \quad \text{for some} \quad u \in U, \\ d_{u,2j} & \text{if } l_{j,i} = u \quad \text{for some} \quad u \in U. \end{cases} \tag{1}$$

Since the literals in any clause c are distinct so are the vertices in $\mathsf{I}(c)$. For each clause $c \in C$ we connect the $k = |V_G|$ vertices in $\mathsf{I}(c)$ so as to induce a graph isomorphic to G. Taking $\ell = 2|C|$ guarantees that this is feasible for all $|C|$ clauses. We are now ready to prove the main theorem of this section.

Theorem 2. *If there exists a G-gadget then G-free 2-colourability is* NP-*complete.*

Proof. Since G is fixed the problem belongs to NP and $\mathcal{F}(I)$'s construction is polynomial. To prove hardness we will show that the answer for an instance I of Distinct NAE k-SAT is "Yes" iff $\mathcal{F}(I)$ is G-free 2-colourable.

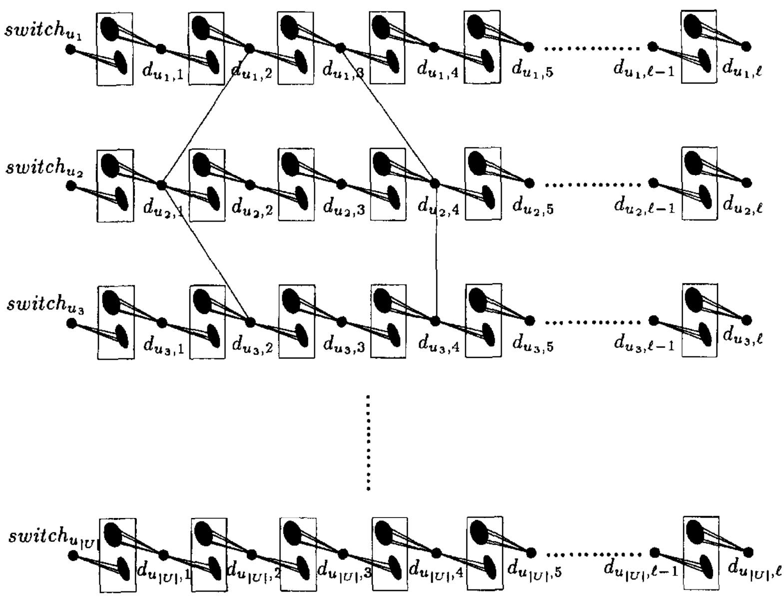

Fig. 3. An example of reducing NAE 3-SAT to P_3-free 2-colourability where $c_1 = (u_1 \vee \neg u_2 \vee u_3)$ and $c_2 = (\neg u_1 \vee u_2 \vee u_3)$. The P_3-gadget (denoted by a rectangle with two external vertices) could be the graph of Fig. 1.

- Given a G-free 2-colouring of $\mathcal{F}(I)$ we define a truth assignment for U where u is set to TRUE iff $\mathrm{col}(switch_u) = 1$. Note that for any truth assignment, inverting the truth value of every $u \in U$ does not affect the answer for I since there will be a true and a false literal in each clause iff this was true before the inversion. Thus, the choice of 1 is arbitrary.

We have to prove that under this truth assignment for U there is a true and a false literal in each clause. Since we are given a G-free 2-colouring of $\mathcal{F}(I)$ it must be that for any $c \in C$ the set $\mathrm{l}(c)$ is not monochromatic since, by the construction of the *Connections*, it induces a copy of G. Thus, by Lemma 2 and the construction of the truth assignment described above, in every clause c there will be a true and a false literal.

- Given a NAE truth assignment for U we define a G-free 2-colouring of $\mathcal{F}(I)$ by taking $\mathrm{col}(switch_u) = 1$ iff u is set to TRUE, and then extending this to a G-free 2-colouring for each $R_{u,\ell}$. The latter can be done as in Lemma 1 where we proved constructively that R_ℓ is G-free 2-colourable.

We have to prove that under this colouring of $\mathcal{F}(I)$ there is no monochromatic copy of G. The proof will be by contradiction, so let G^* be such a copy. Since each $R_{u,\ell}$ is G-free 2-coloured, V_{G^*} must contain vertices from at least two distinct copies of R_ℓ. We first claim that V_{G^*} can only contain outer vertices. If not, let $v \in V_{R_{u,\ell}} \cap V_{G^*}$ for some u while $v \neq d_{u,i}$, $\forall\ i \in \{1,\ldots,\ell\}$. We pick any vertex $v' \in V_{G^*} - V_{R_{u,\ell}}$; this vertex exists by our previous observation. Since G is connected there is a path joining v with v'. By the construction of R_ℓ this path includes some outer vertex. Examining any such vertex, we get a contradiction since in Lemma 1 we showed that for all $d_{u,j}$, $v \in (\Gamma(d_{u,j}) \cap V_{R_{u,\ell}})$ implies $\mathrm{col}(v) \neq \mathrm{col}(d_{u,j})$.

We also claim that $\{m : d_{u,m} \in V_{G^*}\} \subseteq \{2r-1, 2r\}$, for some $r \geqslant 1$ (i.e., G^* can only "occur" between two successive columns of outer vertices, where the leftmost one is odd-numbered). To prove the claim we pick any vertex $d_{u,i} \in V_{G^*}$ and let $r = \lceil i/2 \rceil$. For any $d_{u'',k} \in V_{G^*}$ such that $k \notin \{2r-1, 2r\}$ we examine the path joining $d_{u,i}$ with $d_{u'',k}$. By the claim proved above, this path contains only outer vertices. Examining the first $d_{u',j}$ vertex on the path for which $j \notin \{2r-1, 2r\}$ we get a contradiction with $\mathcal{F}(I)$'s construction. This is because if $u = u'$, then $j = i \pm 1$ and by the construction of $\mathcal{S}$, $\mathrm{col}(d_{u,i}) \neq \mathrm{col}(d_{u',j})$. If $u \neq u'$ then we contradict the second part of $\mathcal{F}(I)$'s construction where we placed an edge between vertices $d_{u,i}$ and $d_{u',i'}$ only if $i' \in \{2r-1, 2r\}$. Thus, there must exist a $c \in C$ such that G^* is induced by $\mathrm{l}(c)$. By $\mathcal{F}(I)$'s G-free colouring and Lemma 2 such a monochromatic copy of G would contradict that we were given a NAE truth assignment. $\square$

4. Gadgets and unique colourability

The following lemma reduces the construction of G-gadgets to the provision of a special family of graphs:

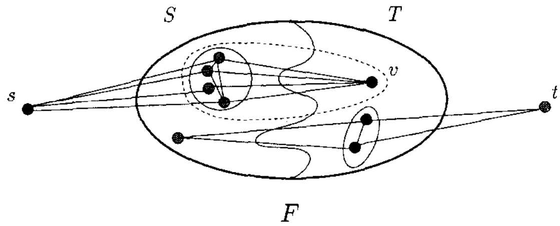

Fig. 4.

Lemma 3. *Given a uniquely G-free 2-colourable graph we can construct a G-gadget.*

Proof. Let F be a uniquely G-free 2-colourable graph. In its unique G-free 2-colouring, let $V_F = S \cup T$ where S, T are monochromatic (Fig. 4). We pick an arbitrary $v \in T$. For $S' = S \cup \{v\}$, $G[S']$ must contain a copy of G because otherwise S', $T - \{v\}$ defines another G-free 2-colouring of F. Based on this observation we add a new vertex s to V_F such that $\Gamma(s) = \Gamma(v) \cap S$. By the choice of $\Gamma(s)$, in every G-free 2-colouring of the resulting graph $\mathsf{col}(s) \neq \mathsf{col}(S)$ which implies, $\mathsf{col}(s) \neq \mathsf{col}(\Gamma(s))$. By picking an arbitrary vertex from S and following a similar argument we can add a vertex t such that $\mathsf{col}(t) \neq \mathsf{col}(\Gamma(t))$. Finally since $\mathsf{col}(S) \neq \mathsf{col}(T)$ we see that F along with s, t is a G-gadget. □

We conclude that in order to prove our main theorem, using Theorem 2 and Lemma 3, all we need is the existence of uniquely G-free 2-colourable graphs for all G on more than two vertices. Results on the existence of uniquely G-free k-colourable graphs were known for various cases of G and k and constructions were given in [4,6,12]. In [1] it was shown that for all $k \geqslant 1$ and for all G with more than two vertices uniquely G-free k-colourable graphs exist. This, apart from providing for G-gadgets, fully settled a conjecture of [2].

5. Concluding remarks

Theorem 1 shows that, if P≠NP, the recognition of bipartite graphs and of graphs that we can partition into two cliques, are the only cases where G-free 2-colourability can be solved in polynomial time. This makes for a very sharp boundary between the tractable and the intractable cases. If we turn to the edge analogue of G-free 2-colourability, the few results that are known seem to establish a different picture. There, we wish to colour the edges of the input graph so that G does not appear as a monochromatic partial subgraph. In [7] it was shown that if $G = K_3$, the problem is NP-complete (*monochromatic triangle* [11]). On the other hand, if G is a *k-star* (a single vertex of degree k adjacent to k vertices of degree one, where $k \geqslant 3$) the problem is solvable in polynomial time as proved in [8]. An interesting observation is that uniquely (edge) G-free 2-colourable graphs exist when $G = K_3$ but *not* when G is a k-star, as proved in [12]. In

Theorem 2 and Lemma 3, in a sense, we reduced the complexity of G-free 2-colourability for the vertex case to the existence of uniquely G-free 2-colourable graphs. It might be worthwhile to investigate if this rather intriguing relationship carries over to the edge case.

Acknowledgements

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Appendix A. Three or more colours

To prove that G-free k-colourability is NP-complete when $k > 2$ we reduce Chromatic number [11] to it. A G^k-gadget is a G-free k-colourable graph with the same properties as a G-gadget *in every G-free k-colouring of it*. Thus, G-gadgets are merely G^2-gadgets. Given an instance I of *Chromatic number* (a graph) and a G^k-gadget $\mathscr{G}^k$ we get an instance I' of G-free k-colourability by "replacing" each edge of the graph with a copy of $\mathscr{G}^k$; i.e., by removing the edge and identifying its endpoints with the s, t vertices of the copy of $\mathscr{G}^k$. To prove that this, clearly, polynomial transformation is a reduction the following two observations suffice:

- Given a G-free k-colouring of I' we trivially have a colouring of I since the endpoints of each copy of $\mathscr{G}^k$ are assigned different colours.
- In the reverse direction, a k-colouring of the original graph gives rise to a G-free k-colouring of I' by G-free k-colouring the "interior" of each copy of $\mathscr{G}^k$. This is feasible since the "endpoints" s, t of each such copy are assigned different colours in the colouring of I. All the neighbours of a vertex $v \in V_I$ have a colour other than $\mathrm{col}(v)$ and G can be assumed to be connected (as before). This completes the argument.

In order to provide a G^k-gadget given a uniquely G-free k-colourable graph we merely mimic Lemma 3 by repeating the argument for s and t for all $k-1$ "other" colour classes.

Appendix B. Distinct NAE k-SAT

We prove that Distinct NAE k-SAT is NP-complete by reducing NAE 3SAT to it. The following elementary mechanism "extends" a clause of length ℓ to two clauses of length $\ell + 1$: *Given a clause $c_\ell = x_1 \vee x_2 \vee \ldots \vee x_\ell$ on the set of variables V_ℓ, we define two new clauses c_{ℓ_0}, c_{ℓ_1} on the set of variables $V_{\ell+1} = V_\ell \cup \{v_{\ell+1}\}$ (where $v_{\ell+1} \notin V_\ell$) as: $c_{\ell_0} = c_\ell \vee v_{\ell+1}, c_{\ell_1} = c_\ell \vee \neg v_{\ell+1}$.*

It is easy to see that there exists a truth assignment for V_ℓ under which c_ℓ has a *true* and a *false* literal *iff* there exists an assignment for $V_{\ell+1}$ under which the same is true for *both* c_{ℓ_0} and c_{ℓ_1}. Moreover, if all the literals in c_ℓ are distinct, the same will be true for both c_{ℓ_0} and c_{ℓ_1}. Starting from a clause with $l < k$ literals and applying the above mechanism $2^{k-l} - 1$ times, using new variables in each application, we get a set of 2^{k-l} clauses. By induction there is a truth assignment (for the original set of variables) under which the original clause has a *true* and a *false* literal *iff* there is a truth assignment (for the new set of variables) under which the same is true for *all* 2^{k-l} clauses.

Note that given an instance I of NAE 3SAT, we can safely remove all but one appearance of a literal in a clause. From this last remark and the above mechanism, a reduction of NAE 3SAT to Distinct NAE k-SAT is straightforward. Since k is not part of the input, such a reduction is polynomial.

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