The Phase Transition in 1-in-k SAT and NAE 3-SAT

Dimitris Achlioptas[*] Arthur Chtcherba[†] Gabriel Istrate[‡] Cristopher Moore[§]

1 Introduction

Determining bounds for the random k-SAT threshold has been an active area of research in recent years [1, 3]. Yet, in spite of significant efforts, neither a tight analysis nor the structural properties of this threshold have been determined.

In this paper we study random instances of two other canonical variations of satisfiability, 1-in-k SAT and Not-All-Equal 3-SAT. Like random k-SAT, each generative model has one parameter $c = m/n$, the ratio of clauses to variables. Also similarly to random k-SAT, we focus on "threshold phenomena" occurring in these models and how they might relate to computational hardness.

For 1-in-k SAT, $k \geq 3$, we obtain the *exact* location of the threshold:

Theorem 1.1 *For all* $k \geq 3$, $c_{1,k} = 1/\binom{k}{2}$.

This is the first exact analysis of an NP-complete version of satisfiability. More importantly, the phenomenon underlying this sharp threshold is a variation of percolation (the emergence of a giant component) in random k-uniform hypergraphs. This allows us to prove the following result on the size of the "backbone" above the threshold. (A variable is in the backbone if every truth assignment satisfying a maximum number of clauses assigns the variable the same value [2]). Let $B(c) = \lim_{n\to\infty} \mathbf{E}(\#\text{ of variables in the backbone})/n$.

Theorem 1.2 *For all* $k \geq 3$, $\lim_{c\downarrow c_{1,k}} B(c) = 0$.

[*] Microsoft Research, Redmond, WA 98052, optas@microsoft.com

[†] Computer Science Department, University of New Mexico, Albuquerque NM 87131, artas@cs.unm.edu

[‡] Center for Nonlinear Sciences, Los Alamos National Laboratory, MS B258, Los Alamos, NM 87545, istrate@lanl.gov

[§] Computer Science Department and Department of Physics and Astronomy, University of New Mexico, Albuquerque NM 87131 and the Santa Fe Institute, 1399 Hyde Park Road, Santa Fe NM 87501, moore@santafe.edu. Corresponding author.

Theorem 1.2 has significant implications for the Statistical Mechanics approach to random satisfiability. In [6, 5] it was suggested that there exists a connection between the "order" of phase transitions and computational complexity. The canonical analogy offered is that the backbone of random 2-SAT exhibits a "second-order" (continuous) phase transition [2], while the backbone of random 3-SAT exhibits a "first-order" (discontinuous) phase transition.

Theorem 1.2 implies that random 1-in-k SAT has a second-order phase transition, i.e. $\lim_{c\downarrow c_{1,k}} B(c) = 0$. As we saw, this is also true for random 2-SAT. Since 1-in-k SAT is NP-complete, while 2-SAT is solvable in polynomial time this demonstrates that:

There is no direct connection between the order of phase transitions and computational complexity.

For NAE 3-SAT we prove that a (non-uniform) sharp threshold exists and provide upper and lower bounds for its location:

Theorem 1.3 $1.514 < c_3^{\mathrm{NAE}} < 2.215$.

Furthermore in Figure 1 we show the results of a numerical experiment. We generated random NAE 3-SAT expressions and used the Davis-Putnam algorithm to test whether or not they are satisfiable. The three curves are for formulas of 50, 100 and 200 variables. Each point represents the average of 100 trials. The transition seems to lie between 2 and 2.1. Note now that a NAE 3-SAT formula is equivalent to a 3-SAT formula with twice as many clauses, since the NAE clause (a, b, c) is equivalent to $(a \vee b \vee c) \wedge (\bar{a} \vee \bar{b} \vee \bar{c})$. It would be truly remarkable if in spite of the tremendous correlation between the clauses that result from translating random NAE 3-SAT instances to 3-SAT instances, we have:

$$2 \times c_3^{\mathrm{NAE}} = c_3^{\mathrm{SAT}} \approx 4.2\ .$$

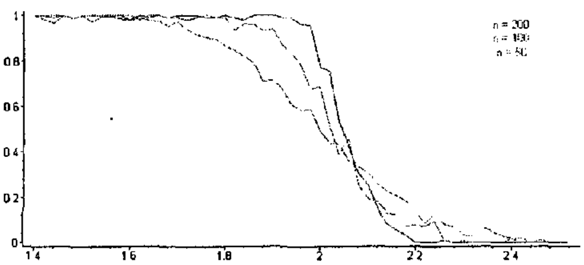

Figure 1: The probability that a random NAE 3-SAT formula is satisfiable.

2 1-in-k SAT

Satisfiability: We analyze the performance of the UNIT CLAUSE (UC) algorithm on random instances of 1-in-k SAT. UC satisfies a random unit clause if one exists, otherwise it picks an unset variable at random and assigns it 0/1 with equal probability.

Satisfying a literal x in a k-clause w creates $k-1$ unit clauses demanding that the remaining literals of w be false; dissatisfying x creates a 1-in-$(k-1)$ clause on the remaining literals of w. For each $2 \le i \le k$, we use differential equations to approximate the number of i-clauses that remain after t variables have been set. This allows us to show that the number of unit clauses generated per any step is asymptotically dominated by a Poisson random variable with mean $\lambda = \binom{k}{2}c$. As long as $\lambda < 1$, with constant probability UC will never create an empty clause, and thus succeeds in finding a satisfying assignment. Along with the existence of a non-uniform sharp threshold, proved separately, this implies $c_{1,k} \ge 1/\binom{k}{2}$.

Unsatisfiability: We prove that for any $\epsilon > 0$, if $c = (1+\epsilon)/\binom{k}{2}$ then with high probability the backbone of a random 1-in-k SAT formula has size at least αn, for some $\alpha = \alpha(\epsilon)$. This immediately implies $c_{1,k} \le 1/\binom{k}{2}$ since, in the presence of a linear size backbone, adding a single new clause causes unsatisfiability with constant probability.

To see that the size of the backbone is at least αn, consider the effect of satisfying a literal x. As seen above, this implies that $k-1$ other literals must be dissatisfied and, thus, the complements of these literals must be satisfied. Since every literal appears, on average, in $ck/2$ clauses we see that the expected number of literals "implied" by satisfying x is $c\binom{k}{2}$. Thus, for any $\epsilon > 0$, if $c = (1+\epsilon)/\binom{k}{2}$, we can construct a branching process to show that with probability bounded below by a constant, the set of literals implied by x contains a contradictory pair. Hence, with constant probability, x must always be dissatisfied, i.e. its underlying variable is in the backbone. Since this argument works for *every* literal x, the expected backbone size is linear. Applying a reasoning similar to the one used to prove Markov's inequality then yields the desired result.

3 NAE 3SAT

Satisfiability: We analyze the performance of an algorithm related to SHORT CLAUSE resolution, SC. As in SC, whenever there are no unit clauses, we choose a random 2-clause, and a random literal in it. Unlike SC, though, we set the underlying variable so as to minimize the number of unit clauses generated, rather than so as to satisfy the chosen literal. Again, the analysis is via differential equations.

Unsatisfiability: We apply the refinement of the first moment method introduced by Kirousis *et al.* [4]

References

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