Weapon Acquisition with Target Uncertainty

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This article is concerned with choosing a mix of weapons, subject to constraints, when the targets to be attacked are known imprecisely. It is shown that the correct method for optimizing the mix of weapons involves a pair of nested optimization problems (two-stage optimization). Two methods for optimizing the expected utility of a mix are discussed. The first involves a simultaneous attack model, in which it is implicitly assumed that all weapons are used at once. The second involves a sequential attack model, in which targets appear in random order and are attacked one at a time. Particular attention is given to the question of the appropriate mix of general-purpose and special-purpose weapons.

1. INTRODUCTION

The history of warfare shows that war is highly unpredictable in almost every facet (see, e.g., [2,6]). On the other hand, the complexity of modern warfare is so great that one must plan well in advance, even if there is considerable uncertainty. For example, most modern weapons have production lead times that exceed 1 year. This means that the first year (and subsequent years for many kinds of weapons) of the war will be fought with the ordnance on hand. An important question is then: how should one build up a supply of ordnance when the targets that will be attacked are unknown? This problem occurs at a number of different levels in the command structure. The highest level involves procurement decisions; that is, how should the Navy acquire ordnance when the targets are unknown? Once the ordnance is acquired, the same question arises in regard to different regions; that is, what kinds of weapons should be stocked at supply bases in the Western Pacific, Atlantic, and so on? After a given region is supplied, the problem arises one more time when an aircraft carrier is loaded for deployment. One solution to these problems would be to purchase such vast amounts of ordnance that any situation can be covered. Fiscal constraints prohibit such a solution. In addition, space constraints at supply bases and especially on the carrier make such a solution infeasible. That is, one must choose a mix of ordnance subject to some very serious space constraints.

This kind of question is not limited to ordnance. Johnson and Loane [4], for example, studied the problem of force composition with uncertainty in the missions that the forces are needed for. A similar problem is described in Mamer and Smith [7,8]. In this problem, one must select a repair kit consisting of a set of tools and subject to a budget constraint with uncertainty in the kind of repair that will be performed.

In this article, particular attention is paid to the problem of selecting general-purpose

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and special-purpose weapons.* Without providing a rigorous definition, a general-purpose (GP) weapon is one that can be used against a variety of targets under a variety of environmental conditions and defensive countermeasures. A special-purpose (SP) weapon is one that can be used against only a few targets, requires special environmental conditions, and may be susceptible to defensive countermeasures but has a higher probability of destroying the target than the SP weapon when the correct circumstances prevail. A typical example of a GP weapon is an unguided, ''dumb'' bomb that follows a simple ballistic trajectory after release while an example of an SP weapon is a laser- or television-guided bomb. The fundamental question then involves the proportion of SP and GP weapons in a stockpile of size N, given that the targets to be attacked are uncertain.

In order to help fix ideas, the following canonical problem is used throughout the article as a computational example (a more realistic problem is treated in [9]). In this example, there are three target types, with T_i denoting the number of targets of type i and $T = (T_1,T_2,T_3)$. There are four weapon types. Subscript 0 indicates the GP weapon and subscript $i \neq 0$ represents the SP weapon for a target of type i. The term N_i denotes the number of weapons of type i and $\bar{N} = (N_0,N_1,N_2,N_3)$. The total number of weapons is $N = \Sigma_{i=0}^{3} N_i$. The effectiveness of the various weapons against targets is summarized in a matrix that shows the probability that a single weapon to type i will destroy a target of type j. For the canonical problem, this matrix is

$$\begin{array}{ll} \text{Targets} & \quad\;\; T_1 \quad T_2 \quad T_3 \\ \text{Weapons}\begin{cases} N_0 \\ N_1 \\ N_2 \\ N_3 \end{cases} & \hspace{-1.5em}\begin{pmatrix} p_0 & p_0 & p_0 \\ p_1 & 0 & 0 \\ 0 & p_1 & 0 \\ 0 & 0 & p_1 \end{pmatrix}. \end{array} \tag{1}$$

Here $p_1 > p_0$ (so that the SP weapons are more effective). The final element in the canonical problem is the uncertainty in the distribution of targets. It is modeled in the following simple fashion. Let $t = (t_1,t_2,t_3)$ be the vector representing the proportions of various kinds of targets, so that $t_i = T_i/\Sigma\, T_i$. Assume that t can take exactly four values,

$$\begin{aligned} t &= (\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}) && \text{with probability } p, \\ t &= (\tfrac{28}{30},\tfrac{1}{30},\tfrac{1}{30}) && \text{with probability } (1-p)/3, \\ t &= (\tfrac{1}{30},\tfrac{28}{30},\tfrac{1}{30}) && \text{with probability } (1-p)/3, \\ t &= (\tfrac{1}{30},\tfrac{1}{30},\tfrac{28}{30}) && \text{with probability } (1-p)/3. \end{aligned} \tag{2}$$

If p were 1, then there would be no uncertainty in the distribution of targets. It is shown in Section 2 that if there is no uncertainty in the distribution of targets, the usual kinds of models lead to an optimal mix (optimal to be specified) involving only SP weapons.

When there is uncertainty in the distribution of targets, a mix of ordnance that

*The words *ordnance* and *weapon* will be used interchangeably, although this use is not completely correct. That is, weapons are built up from ordnance components. There is no trouble with the interchangeable use here, since the problems studied are relatively simple. In the application of the method to detailed scenarios [9], special care is given to these issues.

involves only SP weapons also involves a high level of risk: there may be a considerable probability that a target cannot be attacked because there are no weapons available for it. In this sense, GP weapons provide a kind of "insurance" when there are high levels of uncertainty. The main analytical problem is then to be able to develop models that capture this idea and can then be used to determine optimal acquisition policies.

In the next section, various modeling issues are discussed. These include single- and multiple-shot probabilities of kill, an introduction of the concept of risk aversion, and a summary of some of the current approaches to this problem. It is argued that the optimal mix should be chosen to optimize an expected utility. In Section 3, a simultaneous attack model is used to determine the optimal mix. Section 4 contains a description of a sequential attack model that captures the order of battle more realistically than the simultaneous attack model. Section 5 contains some concluding comments, as well as indications of directions for additional work.

2. MODELING ISSUES

This section contains a discussion of a variety of issues essential to the formulation and solution of the problem.

Single-Shot Probability of Kill

The fundamental operational variable is $\hat{P}_{ij}^k$ defined as follows:

$$\hat{P}_{ij}^k = \Pr\{\text{a single weapon of type } i \text{ destroys a target of type } j\}. \tag{3}$$

For some kinds of weapons, dumb bombs for example, $\hat{P}_{ij}^k$ is known with reasonable accuracy from historical data. For newer, more sophisticated weapons, $\hat{P}_{ij}^k$ itself may be uncertain. The methods introduced here can be used to deal with uncertainty in $\hat{P}_{ij}^k$ as well.

Multiple-Shot Probability of Kill

Imagine that m_{ij} weapons to type i are used to attack one target of type j. Assuming independence of shots, the probability that the m_{ij} weapons destroy the target is

$$1 - \Pr\{\text{none of the weapons destroys the target}\} = 1 - (1 - \hat{P}_{ij}^k)^{m_{ij}}. \tag{4}$$

Now suppose that there are T_j targets of type j and a total of n_{ij} weapons of type i to be applied to targets of type j. It is easily shown that the uniform allocation of weapons to targets (so that $m_{ij} = n_{ij}/T_j$) maximizes the expected number of j-kind targets destroyed. With this uniform allocation, the probability of destroying a j-type target is

$$p_{kj}(n_{ij},T_j) = 1 - \prod_i (1 - \hat{P}_{ij}^k)^{n_{ij}/T_j}. \tag{5}$$

Setting $P_{ij}^k = -\log(1 - \hat{P}_{ij}^k)$, Equation (5) can be rewritten as

$$p_{kj}(n_{ij},T_j) = 1 - \exp\left(-\frac{1}{T_j}\sum_i n_{ij}P_{ij}^k\right). \tag{6}$$

The expected number of targets destroyed, EK, is then

$$\begin{aligned}\mathrm{EK} &= \sum_j T_j p_{kj}(n_{ij}, T_j) \\ &= \sum_j T_j\left[1 - \exp\left(-\frac{1}{T_j}\sum_i n_{ij}P_{ij}^k\right)\right].\end{aligned} \tag{7a}$$

Equations (3)–(7) are the starting points for a number of ordnance acquisition programs in current use. For example, the nonnuclear ordnance requirement (NNOR) is roughly determined as follows. For a given target type, one uses $\hat{P}_{ij}^k$ to find the weapon i^* with the most kills per dollar. The number of that weapon purchased for use against targets of type j is then $n_{i^*j} = T_j/\hat{P}_{i^*j}^k$.

The Naval Weapons Center (NWC) at China Lake, California, advocates an approach in which the exponential term in (7) is expanded to order $1/T_j$. This gives

$$\mathrm{EK} \approx \sum_{i,j} n_{ij}P_{ij}^k \tag{7b}$$

and leads to a linear programming problem for the optimal mix of weapons.

Finally, the Air Force uses a functional similar to (7) in its computer programs which determine ordnance mixes [1]. It is worthwhile to note that optimization problems designed around (7a) will end up with solutions that have only SP weapons in them. To see this, consider the problem

$$\max_{\{n_{ij}\}} \sum_j T_j\left[1 - \exp\left(-\frac{1}{T_j}\sum_i n_{ij}P_{ij}^k\right)\right]$$

$$\text{subject to} \quad \sum_i\sum_j n_{ij} = N, \qquad n_{ij} \ge 0. \tag{8}$$

This problem is solved by standard Lagrange multiplier methods. The Lagrangian is

$$\begin{aligned}\mathfrak{L} = &\sum_j T_j\left[1 - \exp\left(-\frac{1}{T_j}\sum_i n_{ij}P_{ij}^k\right)\right] \\ &- \lambda\left(\sum_i\sum_j n_{ij} - N\right) + \sum_i\sum_j u_{ij}n_{ij}.\end{aligned} \tag{9}$$

In this equation, the multipliers u_{ij} are such that $u_{ij} \ge 0$ with $u_{ij} > 0$ if and only if $n_{ij} \equiv 0$. Setting $\partial\mathfrak{L}/\partial n_{ij} = 0$ gives

$$\lambda = u_{ij} + P_{ij}^k \exp\left(-\frac{1}{T_j}\sum_i n_{ij}P_{ij}^k\right), \qquad i,j = 1,2, \ldots \tag{10}$$

Observe that for fixed j, the exponential is constant over i. Thus,

$$\text{set } c_j = \exp(-1/T_j \textstyle\sum_i n_{ij}P_{ij}^k)$$

to obtain

$$\lambda = u_{ij} + P_{ij}^k c_j, \qquad i,j = 1,2, \ldots \tag{11}$$

Now order the effectiveness of the weapons against the jth kind of target so that $P_{1j}^k > P_{2j}^k > P_{3j}^k, \ldots$ It is easily shown that $n_{1j} > 0$, that is, that the most effective weapon is used against target type j. To do this, simply consider the case of $N = 1$; then proceed by induction from (8), using the derivative of the functional to estimate improvements as one goes from N to $N + 1$ weapons. Then, for any other i, Equation (11) implies that ($n_{1j} > 0$ implies $u_{1j} = 0$)

$$\lambda = P_{1j}^k c_j = u_{ij} + P_{ij}^k c_j, \tag{12}$$

so that $u_{ij} = (P_{1j}^k - P_{ij}^k)c_j > 0$ and thus $n_{ij} = 0$. We have thus shown that the solution to the optimization problem (8) involves only SP weapons. In particular, this means that one cannot take uncertainty in targets into account by solving (8) for each target vector and then performing some "suitable average" over target vectors. Such a procedure will produce weapon mixes that are devoid of GP weapons and thus very risky.

Another serious problem associated with (8) is that (8) implies a very unrealistic operational picture. That is, the operational view associated with (8) is that one purchases n_{ij} weapons of type i for use against targets of type j and puts them aside until a target of type j appears. In reality, one purchases weapons of type i and uses them as targets appear. A good model should capture this fact. That is, targets appear and weapons are used sequentially, not simultaneously, as is assumed in the formulation leading to (8).

Two-Stage Optimization Problems

The proper approach to the general problem of weapon acquisition and allocation, and one which takes the uncertainty in the target distribution into account, is the following. Assume that there is a function $f(\{n_{ij}\},T)$ that describes the effectiveness of the set of weapons $\{n_{ij}\}$ against target vector T. It is reasonable to assume that $f(\{n_{ij}\},T)$ is a sum of such functions, one for each target type:

$$f(\{n_{ij}\},T) = \sum_j f_j(\{n_{ij}\},T_j). \tag{13}$$

Now consider a mix of weapons $\{N_i\}$, $i = 0,1,2, \ldots$, with $\Sigma N_i = N$. The *value* of this mix of weapons against a given target vector is defined by

$$V(\{N_i\}|T) = \max_{\{n_{ij}\}} \sum_j f_j(\{n_{ij}\},T_j) \tag{14}$$

such that

$$\sum_j n_{ij} = N_i, \qquad n_{ij} \geq 0.$$

When the distribution of targets is uncertain, the value of the mix $\{N_i\}$ is determined by averaging (14) over the distribution on T:

$$V(\{N_i\}) = E_T[V(\{N_i\}|T)]. \tag{15}$$

Equation (15) represents the first step in the two-stage optimization problem.

The optimal mix of a given size N is then determined by

$$V = \max_{\{N_i\}} V(\{N_i\}) \tag{16}$$

such that

$$\sum_i N_i = N, \qquad N_i \geq 0.$$

Rewriting (16) by using (14) and (15), one clearly sees the two-range nature of the optimization problem:

$$V = \max_{\{N_i\}} E_T\left[\max_{\{n_{ij}\}} \sum_j f_j(\{n_{ij}\},T_j),\ \sum_j n_{ij} = N_i,\ n_{ij} \geq 0\right],$$
$$\sum_i N_i = N, \qquad N_i \geq 0. \tag{17}$$

It will be seen that (17) provides the proper approach to incorporating uncertainty into the weapons allocation problem. Before proceeding, a few comments about (17) are appropriate. For fixed $\{N_i\}$, the inner optimization problem provides a measure of the value of that given mix of weapons. In particular, if the target vector has a discrete distribution, then $V(\{N_i\}|T)$, as defined in (14), allows one to see how the given mix performs against each fixed target vector. The ability to do this is actually quite important when one tries to understand the results of the full two-stage optimization problem.

The difference between (8) and (17) is an example of the more general phenomenon in which

$$E_x\left[\max_u h(x,u)\right] \geq \max_u E_x[h(x,u)]$$

for an arbitrary function $h(x,u)$. Although this point may be "obvious," with the exception of [4], it seems to have gone unnoticed for problems associated with munitions allocation and acquisition.

Utility and Risk Aversion

The function $f_j(\{n_{ij}\},T_j)$ must still be specified. One natural choice is the number of j-type targets destroyed when there are n_{ij} weapons of type i allocated to the T_j targets of type j. A more general formalism, however, that allows one to capture the effects of uncertainty in the outcome of the attack involves the use of a utility function (see, e.g., [3], for a general discussion of utility theory). To do this, proceed as follows. Let K_j denote the number of targets of type j destroyed. Rather than working with the expectation of K_j, one works with the expectation of a function $u(K_j)$, where $u(\cdot)$ is a utility function that indicates preferences in outcomes. Discussions with naval officers indicate that $u(\cdot)$ should have the property of risk aversion. That is, if K_j^1 and K_j^2 are two choices for K_j occurring with probability p and $1 - p$, respectively, then

$$pu(K_j^1) + (1 - p)u(K_j^2) \leq u[pK_j^1 + (1 - p)K_j^2]. \tag{18}$$

If the equality holds, then the decision maker is called risk neutral; otherwise, the decision maker is called risk averse. The utility function chosen for this work is

$$u(K_j) = 1 - e^{-\rho_j K_j} \tag{19}$$

where ρ_j is a parameter. Clearly (18) is satisfied with a strict inequality if $u(K_j)$ is given by (19). The parameter ρ_j is determined as follows. One asks by how much must $pK_j^1 + (1 - p)K_j^2$ be reduced to achieve equality in (18). If ΔK_j is this amount (called the risk premium), then for (19), (18) becomes

$$p \exp(-\rho_j K_j^1) + (1 - p) \exp(-\rho_j K_j^2)$$
$$= \exp\{-\rho_j[pK_j^1 + (1 - p)K_j^2 - \Delta K_j]\}. \tag{20}$$

Equation (20) is an equation for ρ_j. Table 1 shows the value of ρ_j for a variety of choices of the other parameters. In particular, (20) provides a way to elicit the value of ρ_j from naval officers.

The utility function given in (19) has a number of useful properties. First observe that

$$\lim_{\rho_j \to 0} \frac{1}{\rho_j} u(K_j) = K_j \tag{21}$$

so that the expectation of $(1/\rho_j)u(K_j)$ is an implicit measure of the expected number of targets destroyed. Second, suppose that $K_j^1 = K$ and $K_j^2 = 0$. Then the function

$$\begin{aligned} \Delta u &= u(pK) - [pu(K) + (1 - p)u(0)] \\ &= (1 - p) - \exp(-\rho_j pK) + p \exp(-\rho_j K) \end{aligned} \tag{22}$$

is a monotonically increasing function of ρ_j, rising to $1 - p$ as $\rho_j \to \infty$. Thus, (21) and (22) show that increasing ρ_j corresponds to increasing risk aversion. Third, observe that ρ_j is nearly linear in both the K_j and ΔK_j. It will be seen that these properties lead to considerable robustness in the optimal mix, with respect to ρ_j. Fourth, observe that the risk aversion measured by ρ_j concerns outcomes against targets of type j only and does not provide information about risk aversion of the total (scenario-dependent) outcome. (This possibility is discussed in Section 5.)

With the utility function (19), one suitable choice for $f_j(\{n_{ij}\},T_j)$ is

$$f_j(\{n_{ij}\},T_j) = E_{K_j}[u(K_j)], \tag{23}$$

where E_{K_j} denotes the expectation over possible values of K_j. These possible values are $K_j = 0,1,2, \ldots ,T_j$ and are taken according to the binomial distribution

$$\Pr\{K_j = l\} = \binom{T_j}{l} p_{kj}^l (1 - p_{kj})^{T_j - l} \tag{24}$$

where p_{kj} is given by (6). Using (24) in (23) gives

$$\begin{aligned} f_j(\{n_{ij}\},T_j) &= \sum_{l=0}^{T_j} (1 - e^{-\rho_j l}) \binom{T_j}{l} p_{kj}^l (1 - p_{kj})^{T_j - l} \\ &= 1 - [1 - p_{kj}(1 - e^{-\rho_j})]^{T_j}. \end{aligned} \tag{25}$$

Table 1. Determination of ρ_j, the parameter in the utility function.

K_j^1	p	K_j^2	$1 - p$	ΔK_j	ρ_j
0	0.5	100	0.5	5	0.004
				10	0.008
				15	0.0127
				20	0.018
				25	0.024
0	0.5	1000	0.5	50	0.0004
				100	0.0008
				150	0.00127
				200	0.0018
				250	0.0024
				300	0.0033
				350	0.0046

The last equation follows by use of the moment-generating function for a binomial random variable. Observe from (25) that as $\rho_j \to \infty$, $f_j(\{n_{ij}\},T_j) \to 1 - (1 - p_{kj})^{T_j}$, which is the probability that at least one target is destroyed.

The two-stage optimization problem now becomes

subject to

$$V = \max_{\{N_i\}} E_T\left[\max_{\{n_{ij}\}} \sum_j \{1 - [1 - p_{kj}(1 - e^{-\rho_j})]^{T_j}\} \sum_j n_{ij} = N_i,\ n_{ij} \geq 0 \right),$$

where

$$p_{kj} = 1 - \exp\left(-\frac{1}{T_j} \sum_i n_{ij} P_{ij}^k \right) \tag{26}$$

and subject to

$$\sum_i N_i = N, \qquad N_i \geq 0.$$

The problem posed in (26) is solved in the next section.

Two Operational Considerations

The mix that solves (26) is one that optimizes the expected utility. Two operational considerations are worth discussing at this point. The first concerns an alternate way of viewing the problem. That is, according to (26), all weapons are used and, in a sense, all targets are attacked simultaneously. Thus, (26) can be called a simultaneous attack model (SIAM). In an alternate picture, targets appear one at a time (with probabilities determined by the vector t) and can be attacked if one still has a weapon that can be used against the target that most recently appeared. The attack process stops when a target appears and cannot be attacked. A justification for this approach is discussed in Section 4, where a simulation for the sequential viewpoint is described. With the sequential viewpoint, one can make at least $N_0 + \min_i N_i$ attacks and at most N attacks. In this case, the order in which targets appear is quite important. In particular, the attack process may stop with a considerable number of weapons remaining. This can be called a sequential attack model (SEAM). In this sense, the SIAM (26) may overestimate the value of SP weapons. The sequential view of the attack process will generally lead to results that favor more GP weapons than the expected utility model. The sequential view of the problem is especially well suited to a simulation approach. Such an approach is described in Section 4.

The second operational consideration concerns the meaning of the term *weapon*. In the naval context especially, a weapon involves a large number of components (e.g., an aircraft loaded with a certain combination of bombs; the bombs themselves are created by selecting a kind of fin to go with the basic bomb body). Thus, instead of working with weapons in (26), one should more realistically work with components. This can easily be done using the simulation approach as well [9].

3. RESULTS USING THE SIMULTANEOUS ATTACK MODEL

The optimization of expected utility by a SIAM is obtained by solving problem (26). In order to do that, one must first solve the inner optimization problem and then the outer optimization problem. The inner problem was solved using a nonlinear programming algorithm [10]. The outer problem is then solved by a gradient method. It is actually easier (and more instructive) to specify the mix of weapons and then simply study $V(\{N_i\})$. For example, for the canonical problem, consider the following mixes of size $N = 40$.

Mix number	$\tilde{N} = (N_0, N_1, N_2, N_3)$
1	(37,1,1,1)
2	(31,3,3,3)
3	(25,5,5,5)
4	(19,7,7,7)
5	(13,9,9,9)
6	(7,11,11,11)
7	(1,13,13,13)

By solving the inner problem for each mix and each value of p [the probability that the target mix is $t = (\frac{1}{3},\frac{1}{3},\frac{1}{3})$], one easily finds the range in which each mix dominates. Figure 1 shows the results of such calculations for $T = 30$ targets, $p_0 = 0.5$ (the

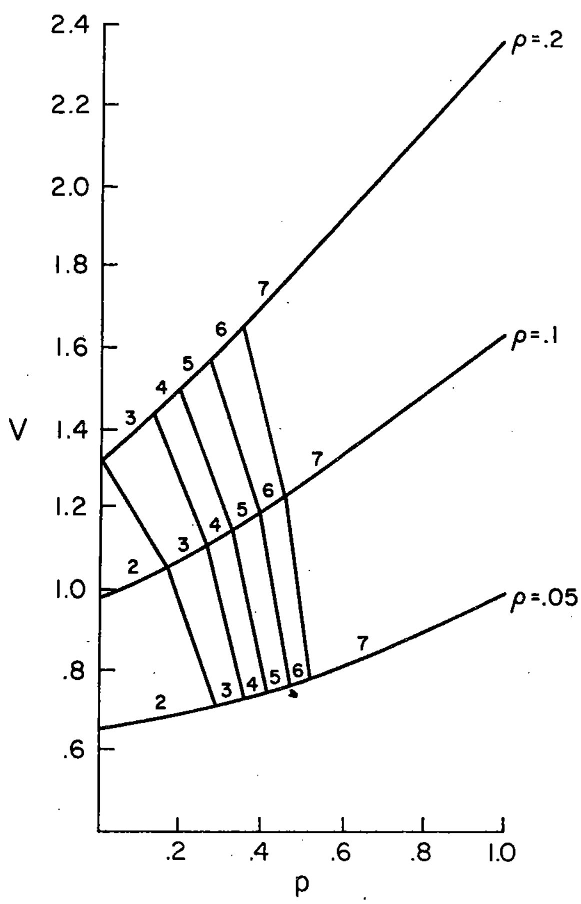

Figure 1. Results of the simultaneous attack model for the canonical problem. The value function V and dominating mix are shown as a function of p, the probability that the target vector is $t = (\frac{1}{3},\frac{1}{3},\frac{1}{3})$. Other parameter values: $T = 30$, $p_0 = 0.5$, $p_1 = 0.7$.

single-shot kill probability of the GP weapon) and $p_1 = 0.7$ (the single-shot kill probability of the SP weapon), and all $\rho_j = \rho$.

Figure 1 clearly shows the shift toward more general-purpose weapons as the uncertainty about the target vector increases. A mix of all SP weapons never dominates in this example.

A study of the results obtained in the inner optimization problem in (26) leads to the following approximate solution of the inner problem:

1. Apply SP weapons only to the target they were designed for.
2. Divide GP weapons uniformly among the targets.

Table 2 shows the results obtained using the approximate solution of the inner problem. The maximum error is 1.6% [corresponding to mix 3, $T = (28,1,1)$, and $\rho = 0.05$]. The great advantage of the approximate solution to the inner problem is that one no longer needs to solve an optimization problem. Instead, the inner problem is reduced to the evaluation of a functional.

The approximate method was used to solve the inner problem for $N = 1000$ weapons and 750 targets. The following weapon mixes were considered:

Mix number	$\bar{N} = (N_0,N_1,N_2,N_3)$
1	0,333,333,334
2	100,300,300,300
3	250,250,250,250
4	400,200,200,200
5	550,150,150,150
6	700,100,100,100
7	850,50,50,50

Figure 2 shows the dominating mix and value as a function of p for $p_0 = 0.5$, $p_1 = 0.7$, and $\rho = 0.0003$ [Figure 2(a)], $\rho = 0.003$ (Figure 2b). Figure 3 shows the dominating mix and value as a function of p for $p_0 = 0.5$, $p_1 = 0.8$, and $\rho = 0.0003$ [Figure 3(a)], $\rho = 0.003$ (Figure 3(b)]. These figures show results in accord with

Table 2. Exact and approximate solutions of the inner problem.[a]

	$V(\{N_i\})$					
	$\rho = 0.05$		$\rho = 0.1$		$\rho = 0.2$	
Mix	$T = (10,10,10)$	$T = (28,1,1)$	$T = (10,10,10)$	$T = (28,1,1)$	$T = (10,10,10)$	$T = (28,1,1)$
1	0.795	0.645	1.369	0.969	2.092	1.298
	0.796	0.645	1.372	0.969	2.095	1.275
2	0.835	0.654	1.429	0.987	2.158	1.311
	0.837	0.642	1.431	0.978	2.160	1.311
3	0.871	0.639	1.481	0.977	2.213	1.313
	0.872	0.629	1.483	0.966	2.216	1.308
4	0.903	0.621	1.526	0.960	2.261	1.305
	0.904	0.612	1.528	0.951	2.263	1.301
5	0.931	0.602	1.567	0.940	2.301	1.295
	0.933	0.595	1.568	0.933	2.303	1.291
6	0.957	0.580	1.602	0.918	2.336	1.283
	0.958	0.575	1.604	0.913	2.338	1.280
7	0.979	0.556	1.633	0.892	2.367	1.267
	0.981	0.554	1.635	0.889	2.368	1.266

[a]The upper entry in each row is the exact solution of the inner problem. The lower entry in each row is the approximate solution of the inner problem.

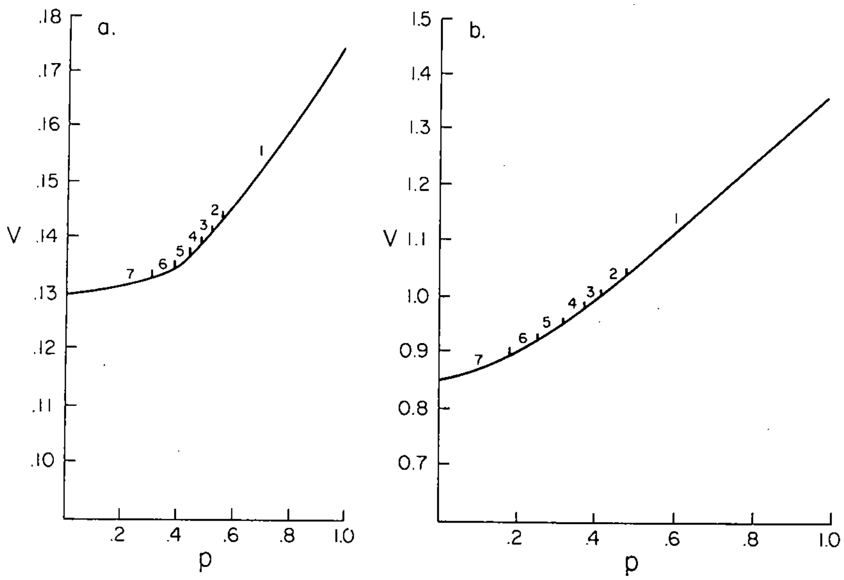

Figure 2. Solution of the canonical problem using the approximate solution of the inner problem. $p_0 = 0.5$, $p_1 = 0.7$, and (a) $\rho = 0.0003$ and (b) $\rho = 0.003$.

intuition. First, for p_1 fixed, as ρ increases, the points at which the dominating mix shifts move to the left (smaller values of p indicating more uncertainty). Second, for ρ fixed, as p_1 increases, the points at which the dominating mix shifts move to the left. [In fact, if $p_1 = 0.9$, then the mix (0,333,333,334) dominates all others for all values of p. It will be seen, however, that this is not the case if one takes the sequential view of the problem, as in the next section.]

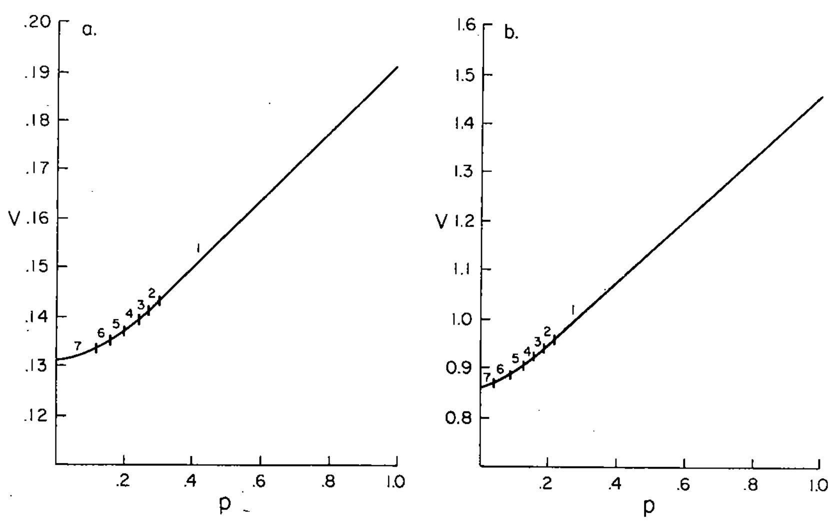

Figure 3. Solution of the canonical problem using the approximate solution of the inner problem. $p_0 = 0.5$, $p_1 = 0.8$, and (a) $\rho = 0.0003$ and (b) $\rho = 0.003$.

4. RESULTS USING THE SEQUENTIAL ATTACK MODEL

In this section, results for optimizing expected utility in the canonical problem are presented in which the sequential attack model was used. In order to emply the sequential attack model, a simulation was developed. The basic features of the simulation are the following:

1. The number of targets is essentially infinite. The proportions of different kinds of targets are uncertain. That is, the vector t is uncertain. For example, for the canonical problem, the distribution on t is given by (2).

2. Each iteration of the simulation begins with a specified mix of weapons $\tilde{N} = (N_0, N_1, \ldots)$. Targets appear one at a time, according to the probability distribution of $\tilde{t}$. When a target appears, the simulation looks at the current vector of weapons and chooses the weapon with the highest single-shot probability of kill against that target. That weapon is then used against the target and the vector of weapons is decremented by 1. When a target appears and there is no weapon that can be used against it, the current simulation iteration ends.

At first glance, this stopping rule for the simulation run appears arbitrary. To some extent, it is, but there are also good reasons for choosing it. First, one can think of all targets of sufficiently high threat value than when a target cannot be attacked, retreat from battle must occur. Second, a major objective of this work is to provide a tool for evaluating the effectiveness of different mixes of weapons. It is a serious failure of a mix if one is unable to attack a target that arises. Alternate rules for the simulation are discussed in Section 5.

3. At the end of each iteration of the simulation, the utility of that iteration is computed according to

$$u = \sum_j (1 - e^{-\rho_j k_j}), \tag{27}$$

where k_j is the number of j-type targets destroyed during the current iteration.

4. Since u given in (27) is a random variable, a large number of iterations of the simulation are performed for each vector N. The number of iterations is chosen so that

$$\frac{5.16[\langle u^2 - \langle u \rangle^2 \rangle]^{1/2}}{\langle u \rangle} \leq 0.01, \tag{28}$$

where $\langle \ \rangle$ denotes an average over the simulation runs. If u were normally distributed, then (28) indicates that the 99% confidence interval has a width that is 1% of the mean.

Table 3 shows the results of the simulation using the canonical problem with $p = 0.4$, $p_0 = 0.5$, and 30 or 40 weapons. It is instructive to compare the SEAM with the SIAM. For example, for $\rho = 0.05$, from Table 2 the expected utility of the mix (25,5,5,5) against $T = (10,10,10)$ is 0.871 and against $T = (28,1,1)$ is 0.639. For the mix (7,11,11,11), the values are 0.957 and 0.580, respectively. Setting $p = 0.4$ gives a total expected utility of 0.7318 for the mix (25,5,5,5) and 0.7308 for the mix (7,11,11,11). The corresponding values for the SEAM are 0.7560 and 0.7197, respectively. Observe that the utility for the SEAM case is higher than the SIAM case for the mix (25,5,5,5) but lower for the mix (7,11,11,11). This shows the additional value of GP weapons when targets are attacked sequentially. Note from Figure 1, for example, that for the SIAM and $p = 0.4$, the optimal mix of weapons is (19,7,7,7). For the same value of $p_1 = 0.7$, the optimal mix in the SEAM is (34,2,2,2), a

Table 3. Results using a simulation of the sequential model.

	$\langle u \rangle$			
	$\rho = 0.05$		$\rho = 0.01$	
Weapons mix	$p_1 = 0.9$	$p_1 = 0.7$	$p_1 = 0.9$	$p_1 = 0.7$
(30,0,0,0)	0.5878	0.5878	0.1422	0.1422
(24,2,2,2)	0.6418	0.6054	0.1539	0.1451
(18,4,4,4)	0.6636	0.6033	0.1577	0.1418
(12,6,6,6)	0.6763	0.5894	0.1599	0.1371
(6,8,8,8)	0.6806	0.5697	0.1600	0.1310
(0,10,10,10)	0.6613	0.5808	0.1542	0.1203
(40,0,0,0)	0.7333	0.7333	0.1866	0.1866
(34,2,2,2)	0.7923	0.7576	0.2006	0.1915
(25,5,5,5)	0.8274	0.7560	0.2065	0.1875
(16,8,8,8)	0.8400	0.7442	0.2081	0.1808
(7,11,11,11)	0.8274	0.7197	0.2028	0.1723
(0,13,13,14)	0.8030	0.6781	0.1945	0.1594

considerable shift toward more GP weapons. Even if the single-shot probability of kill of the SP weapons is increased to $p_1 = 0.9$, the results in Table 3 show that the optimal mix is still 40% GP weapons.

The shift toward more GP weapons in the SEAM is understood when one recognizes the importance of having weapons that can be used against any target. If a mix contains a large proportion of SP weapons, then it is likely that the attack process will end with many unused weapons. A larger proportion of GP weapons avoids this unpleasant situation.

Next consider the case in which $N = 1000$, $\rho = 0.0003$, and $p_1 = 0.9$. Recall that for the simultaneous attack model, the mix with only SP weapons dominated all others. Table 4 shows the values $V(\{N_i\}|t)$ for $t = (\frac{1}{3},\frac{1}{3},\frac{1}{3})$ and $(\frac{28}{30},\frac{1}{30},\frac{1}{30})$ as a function of the number of GP weapons. If $V_1 = V[\{N_i\}|(\frac{1}{3},\frac{1}{3},\frac{1}{3})]$ and $V_2 = V[\{N_i\}|(\frac{28}{30},\frac{1}{30},\frac{1}{30})]$, then the value of a given mix in the situation where $t = (\frac{1}{3},\frac{1}{3},\frac{1}{3})$ with probability p is simply

$$V(\{N_i\}) = pV_1 + (1 - p)V_2. \tag{29}$$

The values in Table 4 can thus be used to find $V(\{N_i\})$ for any value of p.

Table 4 leads to a number of interesting observations. First, unlike the SIAM, the mix with all SP weapons does not dominate all others in the SEAM. The reason for this has been mentioned before: the larger proportion of SP weapons in a mix, the more likely it is that the attack process will end with many unused weapons. Second,

Table 4. Values of a mix of 1000 weapons.

	Value when $t =$	
Weapons mix	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$(\frac{28}{30},\frac{1}{30},\frac{1}{30})$
(100,300,300,300)	0.25	0.11
(500,167,167,166)	0.20	0.13
(700,100,100,100)	0.18	0.14
(900,34,33,33)	0.16	0.15

observe from Table 4 that as the proportion of SP weapons increases, the difference between V_1 and V_2 increases. Thus, a larger proportion of SP weapons implies a larger variance in the ultimate value. Increasing risk aversion of the ultimate outcome will then imply a choice of mix that performs "about the same" over a wide range of scenarios. Thus, the proportion of GP weapons in a mix will increase with increasing risk aversion of the outcome. This suggests considering a new measure of value. Define σ^2 by

$$\sigma^2 = pV_1^2 + (1 - p)V_2^2 - (pV_1 + (1 - p)V_2)^2 \tag{30}$$

so that σ^2 is the variance of the overall value as a function of p. Then set

$$\hat{V}(\{N_i\}) = V(\{N_i\}) - \alpha\sigma, \tag{31}$$

where α is an adjustable parameter. As α increases, one assesses a greater penalty for variance in the outcomes. Any value of $\alpha > 0$ will imply a shift toward more GP weapons. Table 5 shows $\hat{V}(\{N_i\})$ for the four mixes in Table 4 as a function of p and α. From this table, one sees that the dominant mixes are 1 and 4 and that the region in which mix 4 dominates shifts to the right (i.e., greater values of p) as α increases. In the next section, a more complete discussion of alternate measures of value is presented.

The sequential model has one major disadvantage in that it requires a simulation. An approximate method for treating the SEAM will now be described, using the canonical problem as an example. Imagine a total of $N = \Sigma_{i=0}^{3} N_i$ weapons are available to be used. Consider all sequences of length N consisting of $\{M_1,M_2,M_3\}$ where $\Sigma\, M_i = N$ and M_i is the number of i-type targets attacked. The likelihood of a given sequence $\{M_1,M_2,M_3\}$ is determined by the multinomial distribution:

$$\Pr(\{M_1,M_2,M_3\}) = \frac{N!}{\prod_i M_i!} \prod_i t_i^{M_i}. \tag{32}$$

In addition, the M_i must satisfy certain constraints determined by the vector $\tilde{N} = (N_0,N_1,N_2,N_3)$. For the canonical problem, these constraints are

$$M_1 \le N_0 + N_1, \qquad M_2 \le N_0 + N_2, \qquad M_3 \le N_0 + N_3. \tag{33}$$

It is clear that not all sequences of length N will satisfy (33). It is possible that up to two of the three constraints are violated. The following approximation procedure was developed to deal with cases in which the constraints were violated. If the ith constraint is violated, then M_i is replaced by $\hat{M}_i$ where

$$\hat{M}_i = N_0 + N_i. \tag{34}$$

If the ith and jth constraints are violated, then M_i and M_j are replaced by $\hat{M}_i$ and $\hat{M}_j$, where

$$\hat{M}_i = \frac{M_i}{M_i + M_j} N_0 + N_i, \qquad \hat{M}_j = \frac{M_j}{M_i + M_j} N_0 + N_j. \tag{35}$$

Equations (34) and (35) are interpreted as follows. For (34), one has $M_i > N_0 + N_i$ so that the attack process had to stop at $\hat{M}_i = N_0 + N_i$. When two constraints are violated, all the GP weapons were used on targets i and j; (35) captures this idea.

Table 5. Values of $\hat{V}\{N_i\}$.[a]

	$\hat{V}(\{N_i\})$											
	$\alpha = 0$				0.5				1			
p	1	2	3	4	1	2	3	4	1	2	3	4
0.1	0.124	0.137	0.144	0.151	0.103	0.127	0.138	0.150	0.082	0.116	0.132	0.148
0.2	0.138	0.144	0.148	0.152	0.111	0.130	0.140	0.150	0.082	0.116	0.132	0.148
0.3	0.152	0.151	0.152	0.153	0.120	0.135	0.143	0.150	0.088	0.119	0.134	0.148
0.4	0.166	0.158	0.156	0.154	0.132	0.140	0.146	0.151	0.097	0.123	0.136	0.149
0.5	0.180	0.165	0.160	0.155	0.145	0.148	0.150	0.152	0.110	0.130	0.140	0.150
0.6	0.194	0.172	0.164	0.156	0.160	0.155	0.154	0.153	0.125	0.138	0.144	0.151
0.7	0.208	0.179	0.168	0.157	0.176	0.163	0.159	0.154	0.144	0.147	0.150	0.152
0.8	0.222	0.186	0.172	0.158	0.194	0.172	0.164	0.155	0.166	0.158	0.156	0.154
0.9	0.236	0.193	0.176	0.159	0.215	0.183	0.170	0.150	0.194	0.172	0.164	0.155

[a]Mix 1 is $N = (100,300,300,300)$, 2 is $N = (500,167,167,166)$, 3 is $N = (700,100,100,100)$, 4 is $N = (900,34,33,33)$.

The value $V(\{N_i\}|t)$ is calculated as follows. Let $v_i(M_i)$ denote the value associated with M_i attacks on the ith type of target. If $M_i \le N_i$, then

$$\begin{aligned} v_i(M_i) &= \sum_{k=0}^{M_i} (1 - e^{-\rho k})\Pr\{k \text{ of } M_i \text{ targets destroyed}\} \\ &= \sum_{k=0}^{M_i} (1 - e^{-\rho k})\binom{M_i}{k} p_1^k (1 - p_1)^{M_i - k} \\ &= 1 - [1 - p_1(1 - e^{-\rho})]^{M_i}. \end{aligned} \tag{36}$$

In what follows, it helps to set

$$S_1 = 1 - p_1(1 - e^{-\rho}), \qquad S_0 = 1 - p_0(1 - e^{-\rho}). \tag{37}$$

If $M_i > N_i$, but no constraint is violated, then

$$\begin{aligned} v_i(M_i) &= \sum_{k=0}^{N_i} (1 - e^{-\rho k})\binom{N_i}{k} p_1^k (1 - p_1)^{N_i - k} \\ &\quad + \sum_{k=N_i+1}^{M_i} (1 - e^{-\rho k})\binom{M_i}{k} p_0^k (1 - p_0)^{M_i - k} \\ &= 1 - S_1^{N_i} - S_0^{M_i} + S_0^{N_i}. \end{aligned} \tag{38}$$

If the ith constraint is violated, then M_i in (38) is replaced by $N_i + N_0$. If the ith and jth constraints are violated, then M_i in (38) is replaced by $N_i + N_0[M_i/(M_i + M_j)]$.

The value $V(\{N_i\}|t)$ is then

$$V(\{N_i\}|t) = E_M\left(\sum_i v_i(M_i)\right), \tag{39}$$

where E_M denotes the expectation over $M = (M_1, M_2, M_3)$, using the multinomial distribution (32).

Tables 6 and 7 contain a comparison of the results using the simulation and the

Table 6. Comparison of simulation and analytical methods for sequential attack model.

	$V(\{N_i\})$			
	$p_1 = 0.7$		$p_1 = 0.9$	
Weapons mix[a]	Simulation	Analytical	Simulation	Analytical
(30,0,0,0)	0.1422	0.1424 (0.1)[b]	0.1422	0.1424 (0.1)
(24,2,2,2)	0.1451	0.1463 (0.6)	0.1539	0.1557 (1.2)
(18,4,4,4)	0.1418	0.1430 (0.8)	0.1577	0.1593 (1.0)
(12,6,6,6)	0.1371	0.1401 (2.2)	0.1599	0.1618 (1.2)
(6,8,8,8)	0.1310	0.1336 (2.1)	0.1600	0.1624 (1.5)
(0,10,10,10)	0.1203	0.1112 (7.6)	0.1542	0.1447 (6.2)

[a]Other parameter values are $p_0 = 0.5$, $\rho = 0.01$, $p = 0.4$.
[b]Indicates

$$\text{Percent error} = \left|\frac{\text{simulation} - \text{analytical}}{\text{simulation}}\right| \times 100.$$

Table 7. Comparison of simulation and analytical methods for sequential attack model.

	$V(\{N_i\})$			
	$p_1 = 0.7$		$p_1 = 0.9$	
Weapons mix[a]	Simulation	Analytical	Simulation	Analytical
(40,0,0,0)	0.1866	0.1920 (2.9)[b]	0.1866	0.1920 (2.9)
(34,2,2,2)	0.1915	0.1928 (0.7)	0.2006	0.2023 (0.8)
(25,5,5,5)	0.1875	0.1887 (0.7)	0.2065	0.2092 (1.3)
(16,8,8,8)	0.1808	0.1829 (1.2)	0.2081	0.2131 (2.4)
(7,11,11,11)	0.1723	0.1754 (1.2)	0.2028	0.2142 (5.6)
(0,13,13,14)	0.1594	0.1450 (9.0)	0.1945	0.1885 (3.1)

[a]Other parameter values are $p_0 = 0.5$, $\rho = 0.01$, $p = 0.4$.
[b]Indicates

$$\text{Percent error} = \left|\frac{\text{simulation} - \text{analytical}}{\text{simulation}}\right| \times 100.$$

approximate method. The results shown in Tables 6 and 7 indicate that the analytical approximation is extremely accurate over a wide range of variables. For $N = 1000$ weapons, the analytical approximation to the simulation (i.e., Table 4) is not as accurate. The error is of the order of 20%, but all trends in Table 4 are followed in the analytical approximation.

5. SUMMARY AND DISCUSSION

The main purpose of this article is to introduce a new methodological concept for dealing with uncertainty in allocation and acquisition problems. Although the general concept (embodied in the observation $E_x\{\max_u h(x,u)\} \geq \max_u E_x\{h(x,u)\}$) is simple and well known, its importance for weapon acquisition has been virtually unnoticed. The key idea here is that one must compute the value of a specified mix of weapons by averaging over possible target scenarios and then compare the value associated with different mixes. If one does not take into account the variation over scenarios, then the acquisition process will lead to mixes of weapons that perform poorly against the average over scenarios. A second, but not as important, methodological concept introduced here is the use of an expected utility function of the number of targets destroyed, rather than simply an expected kill model. The use of a utility function has two advantages. First, it allows one to capture the natural risk aversion associated with military operations. Second, it provides a natural way of incorporating the uncertainty of the attack process into calculations.

Two methods for computing the value, $V(\{N_i\})$, of a set of weapons are described in this article. The first is a SIAM [Eq. (26)] in which one presumes all weapons are used. The optimal mix of weapons is determined by a two-stage optimization procedure in which one associates a certain set of weapons with a certain target type. The second method for computing the value of a mix of weapons is based on a SEAM in which targets appear randomly according to some distribution and one continues to attack targets until either all weapons are used up or a target appears and cannot be attacked (i.e., no GP weapons and SP weapons for that target remain). The SEAM was studied by using a simulation and an approximate analytical procedure. The SIAM overestimates the value of SP weapons, relative to the SEAM. This is caused by the assumption in the SIAM that all weapons can be used.

Thus far, the work presented here has generated the following "rules of thumb" regarding weapons mix decisions. First, GP weapons provide a kind of "insurance" against uncertainty in the sense of providing a capability for doing something in all situations. That is, one may not have an "optimal" capability if the situation is specified in enough detail, but one does have a "not too bad" capability across a variety of situations. second, even when the fraction of targets in a scenario is known, the exact order in which targets appear is not known. This introduces another kind of uncertainty that may affect the ultimate value of a mix of weapons. Third, SIAMs overestimate the value of SP weapons. Fourth, given a mix of weapons, the following heuristic is often a good approximation to the solution of the inner problem. Apply SP weapons to the target that they most effectively destroy. Apply GP weapons uniformly to all targets. Fifth, if one is risk averse to outcomes over scenarios, the optimal mix will involve a high percentage of GP weapons.

A number of modifications and additional aspects of the work are worth discussing.

Uncertain $\hat{P}_{ij}^k$ or Target Type

Throughout this article, it was assumed that the single-shot probability of kill, $\hat{P}_{ij}^k$, was known with certainty. This may not be true for at least two reasons. First, for many of the newer SP weapons, the value of $\hat{P}_{ij}^k$ in combat can only be inferred from the existing test data in noncombat situations. For this reason, there is considerable uncertainty in the actual value of $\hat{P}_{ij}^k$. Second, the value of the single-shot kill probability also depends on the countermeasures (CMs) taken to define the target. Some CMs are purely passive and affect all $\hat{P}_{ij}^k$ (e.g., hardening a site). Other CMs affect some weapons and not others. (For example, a SAM site will not affect the ballistic flight of an iron bomb but will affect the flight of a smart weapon that must be guided by someone inside the attacking aircraft.)

These considerations are easily taken into account. For example, in the canonical model, one could attach a distribution to p_1, say $p_1 = 0.9$ with probability π and $p_1 = 0.7$ with probability $1 - \pi$ (or any more complicated distribution as well). There is no conceptual difficulty (and very little computational difficulty) when the $\hat{P}_{ij}^k$ is modified in this way.

Similarly, to take into account different defensive CMs, one simply needs to extend the definition of the target vector. That is, one can think of a "target" as consisting of a physical target type with a certain countermeasure. Once again, there is no conceptual or computational difficulty with doing this.

Components

As mentioned at the end of Section 2, the operational forces do not purchase or load "weapons." Rather, they acquire and load components from which a variety of weapons can be built. For this reason, one should really work with a set $\{C_i\}$ of components rather than a set of weapons $\{N_i\}$ and thus compute the value of a set of components. That is, one needs to compute the value $V(\{C_i\})$ of a mix of components $\{C_i\}$. This involves no new conceptual difficulties and only a slight increase in computational complexity. The only significant change in computational complexity arises when one tries to maximize $V(\{C_i\})$ over $\{C_i\}$, subject to a constraint, since there are many more components than weapons. This is a tractable problem, however [9].

Attack Force Attrition

A factor not taken into account in either the SIAM or SEAM models is the attrition rate of the attacking forces. In particular, it is often argued that the attrition rate will be lower when SP weapons are used. It is possible to enlarge the problem studied to include attrition as a factor in either model. The difficulty, however, is that the attrition rates are highly uncertain for almost every situation. For this reason, it is unlikely that the basic concepts that emerged in this article will change considerably when attrition is included.

Improvement of the Optimization

Most of the results in this article concerned $V(\{N_i\})$. Near optimal mixes of weapons were then determined by inspection of a set of $V(\{N_i\})$. In [9], a stochastic approximation scheme is used to obtain the optimal mix of components. That is, stochastic approximation methods [11] can be used to maximize $\langle V(\{N_i\})\rangle$ where the $\langle\ \rangle$ indicate an average over simulations.

Measuring the Level of Uncertainty

As the total number of weapons in the mix increases, one expects that the limit theorems of probability theory will take over and that there will be decreasing relative variation in the target vector that is attacked. This idea can be captured in the following way.

Consider the canonical problem in which the target vector is $t^1 = (t_1^1, t_2^1, t_3^1)$ with probability p and $t^2 = (t_1^2, t_2^2, t_3^2)$ with probability $1 - p$. Assume that N weapons are used and let M_i denote the number of i-type targets that appear. Then M_i has a multinomial distribution with

$$E\{M_i\} = \begin{cases} t_i^1 N & \text{with probability } p, \\ t_i^2 N & \text{with probability } 1 - p, \end{cases} \tag{40}$$

and

$$\operatorname{var}\{M_i\} = \begin{cases} t_i^1(1 - t_i^1)N & \text{with probability } p, \\ t_i^2(1 - t_i^2)N & \text{with probability } 1 - p. \end{cases} \tag{41}$$

Consider the total average (over scenarios) coefficient of variation in the target vectors that appear. This coefficient of variation is

$$\mathrm{CV} = \sum_i \left[p\left(\frac{1 - t_i^1}{t_i^1 N}\right)^{1/2} + (1 - p)\left(\frac{1 - t_i^2}{t_i^2 N}\right)^{1/2} \right]. \tag{42}$$

Equation (42) clearly shows that as N increases, the total CV decreases. In particular, one must continually readjust the value of p to keep the level of uncertainty the same. This observation is meaningless in a real-world context, where there is presumably one value of p. But it is important for situations in which one is comparing different mixes of weapons that have considerably different values of N.

Parameter Sensitivity

The key parameters that enter into the calculations needed to evaluate a mix of weapons are the target scenarios and their likelihoods, the value of the parameter in the utility function, and the values of the kill probabilities. An examination of Figures 1–3 shows that the dependence of the optimal mix on the likelihoods of the scenario gradually changes with the likelihood. There are no sudden jumps from a high proportion of GP weapons, say, to a low proportion of GP weapons. Similarly, examination in Figure 1 shows a gradual change of the optimal mix with respect to p. Some estimation of parameter sensitivity to the kill probabilities can be gleaned from Tables 3, 6, and 7. In those tables, one sees a shift toward more SP weapons as p increases. This increase can be quite dramatic, as the first column of Table 3 shows.

One could study parameter sensitivity by constructing "elasticities" of the form

$$\epsilon_p = \frac{1}{u}\frac{\Delta u}{\Delta p},$$

where p is any of the parameters, u is the utility, and Δu is the change in utility when p is changed by Δp. Comprehensive studies of such elasticities are presently underway.

Numerical experience with the models presented in this article shows that the objective functional is relatively flat around the optimum. This means that the results are relatively insensitive to small deviations in values of the mix.

Other Value Functions

The utility function $u(k) = 1 - e^{-\rho k}$ was introduced to capture the effect of uncertainty in the number of targets destroyed. The computations of the value function $V(\{N_i\}|t)$ then use this utility function. There is, however, another kind of uncertainty in this article that is dealt with just as a linear expectation. This is the uncertainty over scenarios (i.e., over the t vectors). Both the SIAM and SEAM compute the value as a simple linear expectation:

$$V(\{N_i\}) = E_t\{V(\{N_i\}|t)\}. \tag{43}$$

A more sophisticated way of dealing with the uncertainty across scenarios is to replace (44) by a utility-of-the-utility model. That is, replace (44) by

$$V(\{N_i\}) = E_t\{U[V(\{N_i\}|t)]\}, \tag{44}$$

where $U(\cdot)$ is a given "super" utility function. Equation (31) is an example of a simple, heuristic super utility function. One could even choose $U(V) = 1 - e^{-\hat{\rho}V}$, where $\hat{\rho}$ is a parameter. Perhaps the greatest difficulty with the utility-of-the-utility concept is that the parameters ($\hat{\rho}$ for example) do not have the simple intuitive interpretations that ρ has in $1 - e^{-\rho k}$. In this super utility function exhibits risk aversion, the effect will be a shift toward more GP weapons (see Table 4, for example).

Another possibility would be to forego the utility function entirely and work with a more complicated measure of value, such as the prospect theory of Kahneman and Tversky [5].

Other Simulation Rules

The rules for running and stopping the simulation described in Section 4 can be modified in a number of ways. Possible modifications of the running rules are (a) targets appear in clumps of random size rather than one at a time or (b) targets are attacked until they are destroyed. Preliminary investigation of this second modification indicates that it may have a nonnegligible effect on the outcome of the optimization procedure. For the preliminary investigations thus far, the effect has been a shift toward more SP weapons.

A natural modification of the stopping rule is to continue the iteration of the simulation until all weapons have been used and to keep track of the disutility associated with targets that cannot be attacked. Another stopping rule is to continue the iteration until the total number of targets that are not attacked reaches x, where x is a parameter supplied by the user. Studies using these rules are currently underway.

Supply Networks

As discussed in Section 1, there are a number of different levels at which the methodology introduced here can be used. For the problems associated with supplying a region or an individual aircraft carrier, one important complication is the existence of the supply network. For purposes of concreteness, consider the case of an aircraft carrier about to be loaded for a deployment. Although the primary source of weapons is the carrier's own magazine, there usually is a supply ship that travels with the carrier. Further away (in both times and space) are land-based intermediate supply depots (ISDs). Finally, there are supply bases in the continental United States (CONUS). Supply ships can be used to load weapons at the ISD and deliver them to a carrier task force. The ISD can then be resupplied from CONUS. Associated with each of the resupply actions is a delay. One can envision that the existence of a supply network will mitigate some of the negative effects of mixes composed mainly of SP weapons, as long as the pace of operations is slow enough that the carrier does not run out of weapons. In order to model the effects of the supply network, one needs to consider a dynamical version of the problem considered in this paper. The natural formulation is a discrete time one, in which the total value is the sum of values in each "period." The mix of weapons on the carrier will then increase due to supply actions and decrease due to attacks.

One of the most interesting questions associated with the use of a supply network involves the initial allocation of weapons. That is, how should one spread an initial allocation over CONUS and various ISDs (located in widely separated geographic areas) so as to optimize the ultimate value of a mix. Such questions are currently under investigation.

Generalization of Inventory Theory

One of the concepts used throughout this work was interchangeability. That is, more than one kind of weapon could be used to attack a given target. This can be interpreted in terms of classical inventory theory as follows. As targets appear, they present a "demand" for weapons. The demand can be served by more than one weapon from the stock of items. In this sense, one is faced with a generalization of classical inventory theory since this problem is one with multiple stocks, multiple demands, and interchangeability of demands and stocks. One can easily think of many situations in which such models apply (indeed, it is difficult to think of a consumer item, from automobiles to soda pop, in which multiple stocks are not common). The methodology introduced here can be used to address such generalizations of classical inventory theory.

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