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SEARCH THEORY IN NATURAL RESOURCE MODELING

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ABSTRACT. The role of search theory in the exploitation of natural resources is discussed in this paper. After a brief history and taxonomy of search problems, the mathematics of search is discussed. This includes underlying probability distributions, the differential equations of search, Bayesian use of search information and optimization problems in search theory. The theory is illustrated by applications in fisheries, pest control, animal foraging, and oil and mineral exploration.

1. Introduction. Before a natural resource can be exploited, one must know where it is. For some resources, forests, for example, this is not a big problem—it is easy to find the resource. For others, fish in the ocean, oil or mineral resources, and pests of agricultural crops, finding the resource is a major effort. It is for these resources that search theory can play a useful role, since search theory is essentially the applied mathematics of finding objects. The purpose of this paper is to review search theory and to show its role in natural resource modeling by way of examples in a variety of diverse applications. That is, we will show the underlying mathematical concepts in relative generality and then show how they are applied to very specific cases.

Search theory is basically concerned with information, particularly the gathering and updating of information and how one uses information in an optimal fashion. This leads one to approach problems from a Bayesian point of view—the Bayesian approach provides a natural and mathematically consistent way of dealing with and incorporating information.

The applications that we will discuss are a diverse group: fisheries, pest control, animal foraging, and oil and mineral exploration. The diversity of applications should help convince the reader of the ubiquitousness of the problems in which search theory is important. (There are, in fact, many applications that we will not discuss — some will be alluded to later on.)

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The next section contains a brief history of search theory with a "taxonomy" of mathematical problems. In the third section, we provide a review of search models and techniques. Certain areas of search theory are not discussed in detail, but pertinent literature citations are given. The fourth section contains specific examples that show how search theory is applied to diverse problems such as where fishermen should go fishing, how spraying decisions in agricultural pest control should be made, animal foraging, and assessing the quality of a region for mineral or oil reserves. Finally, the fifth section contains a summary and conclusions.

2. History and Taxonomy of Search Theory. It is obvious that men have been searching since man first walked the earth—and undoubtedly crude ideas about search theory have circulated since then. In modern times, search theory developed during World War II in response to German "Wolfpack" submarines operating off the eastern seaboard of North America. In response to this and other problems, the Navy approached P. Morse at M.I.T. for help. He organized the Anti-Submarine Warfare Operations Research Group (ASWORG) which contained, throughout the war, some of the best minds in the country, including W. Shockley (1942-44), G. E. Kimball (1942-45), J. Steinhardt (1942-45), W. Horvath (1943-45), C. Kittel (1943-45), B. O. Koopman (1944-45), and C. B. Allendorfer (1944-45). (A very enjoyable history of the group is found in Tidman [1984]. Morse's autobiography (Morse [1977]) is also quite interesting.)

The ASWORG was formed in April, 1942. By May, 1942 it had produced a memorandum "Preliminary Report on the Submarine Search Problem" which laid the foundations for further work in search theory. The group continued to work on search problems throughout World War II. At the end of the war, the group was renamed the Operations Evaluation Group (OEG) and published a compilation of its work on search problems, "Search and Screening" (OEG [1946]). Most of the material in this report was later published by B. O. Koopman — first in a series of papers (Koopman [1956a, b, 1957]) and later in a revised version of "Search and Screening" (Koopman [1980]).

The Navy continued to support research in search theory after the war. Many technical (mathematical) advances occurred in formulation and solution of search problems. Stone [1975] summarizes these advances (to many of which he contributed). Interest in search theory has been growing considerably since 1970 — particularly for non-military applications of search theory. In fact, in 1980 NATO sponsored an Advanced Research Institute on "Search Theory" with an expressed purpose of

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seeing what in the mathematics of search could be transferred to the non-military sector. Haley and Stone [1980] contains some of the proceedings of that ARI. A search problem has three components: 1) the object being sought (henceforth called the target, but a military application is not necessarily intended), 2) a searcher, and 3) a detection device. A taxonomy of search problems can be built around these three aspects. Other "taxonomies" of search are possible as well for example: 1) the length of search, and stopping rules, 2) the nature of information, and 3) the objective functional. We choose the former taxonomy because it leads naturally to the differential equations of search.

First, the target may be stationary or may move. In general, problems with stationary targets are somewhat easier to solve than problems with moving targets. Second, one can consider the level of detail in the description of the searcher. Early work in search theory concentrated on a description by means of "search effort" rather than specifying a search track. The difference between these two approaches usually arises in the description of the problem. That is, when search effort is used one usually thinks of dividing the entire search space into cells and thinks of putting a certain amount of search effort into a given cell. The probability of detecting the target, given that the target is in cell i and that search effort φ_i is applied to cell i, is then a function $\Phi(\varphi_i)$. (A typical example is $\Phi(\varphi_i) = 1 - e^{-b_i\varphi_i}$ where b_i is a parameter . The origin of this kind of formula will be discussed in the next section.) When a search track is considered, the search space is treated as a continuum and detection is characterized as follows. Let $X(t)$ denote the position of the target a time t, $Z(t)$ denote the position of the searcher at time t. Detection is characterized by a function $\Psi(x,t,z)$ defined by

$$(2.1)\quad \Psi(x,t,z)\Delta t = \Pr\{\text{detection in } (t, t+\Delta t) \mid X(t) = x, Z(t) = z\}.$$

Examples of such detection functions are presented in the next section.

The third aspect of taxonomy is the kind of search problem studied. The search problem can be a descriptive one in which a search track or search effort is specified and one asks for the time dependence of the probability density of the target. On the other hand, the search problem can be prescriptive, in which one seeks an optimal search track or an optimal search effort distribution. Examples of each kind of search problem will be discussed in Section 4, but this paper is mainly concerned with prescriptive problems. Mangel [1985a] discusses a number of descriptive search problems.

3. Review of search models. This section contains a description of modeling techniques of most interest in search theory. It contains subsections on 1) underlying probability models, 2) use of search information, 3) the differential equations of search theory, and 4) optimization problems of search theory.

This section is not comprehensive; instead, sufficient material is presented here so that the applications in the next section can be understood. In particular, some areas of search theory not discussed at all are a) search for a single target, moving or stationary, with the search stopping upon discovery (see e.g., Hellman [1970], Stone [1975], Lukka [1979], Washburn [1981], Mangel [1985b]), b) localization and tracking problems (e.g., Washburn [1981]), and c) transect theory (e.g., Seber [1982], Burnham, Anderson, and Laake [1980]).

3.1 Underlying probability models. To begin, imagine a large region a of area A that contains N objects. Assume that the region is searched at rate α so that the total area searched after time t is αt. Assume that the N objects are randomly distributed, that an object is discovered once or not at all, and that detections of different objects are independent events. If $X(t)$ is the number of objects detected in $(0,t)$, then (assuming $\alpha\, t < A$).

$$\Pr\{X(t)=n\} = \binom{N}{n}\left(\frac{\alpha t}{A}\right)^n\left(1-\frac{\alpha t}{A}\right)^{N-n}. \tag{3.1}$$

Equation (3.1) is a binomial distribution with mean $N\alpha t/A$ and variance $N(1-\alpha t/A)\alpha t/A$.

Suppose that N and A go to infinity in such a way that $N\alpha t/A \approx \lambda t$ where $0<\lambda<\infty$. Then one obtains the Poisson approximation to the binomial distribution

$$\Pr = e^{-\lambda t}\frac{(\lambda t)^n}{n!}. \tag{3.2}$$

It is easily seen that (3.2) can be derived from the assumption that

$$\begin{aligned}&\Pr\{\text{ another discovery in next dt}\} = \lambda dt + o(dt),\\ &\quad\Pr\{\text{ no discovery in next dt}\} = 1-\lambda dt + o(dt).\end{aligned} \tag{3.3}$$

The Poisson distribution has the property that the mean and variance of $X(t)$ are equal to λt. In a wide variety of resource problems, however, ranging from fish (e.g., Allen and Punsly [1984]) to pests (Bliss [1958]) to oil deposits (Uhler and Bradley[1970]) it is observed that the variance greatly exceeds the mean. One way to obtain a distribution with that property is the following. Assume that (3.2) holds locally in the region of interest, conditioned on λ. But now assume that λ varies globally and thus has a distribution associated with it. Choose the gamma density $f(\lambda)$ with parameters ν and α so that

$$f(\lambda) = \frac{\alpha^{\nu}\lambda^{\nu-1}e^{-\alpha\lambda}}{\Gamma(\nu)}. \tag{3.4}$$

If λ has the density (3.4), then the mean of λ is ν/α and the coefficient of variation (standard deviation divided by the mean) of λ is $1/\sqrt{\nu}$. Now (3.2) is viewed as a conditional distribution. The unconditional distribution is found by averaging over λ:

$$\begin{aligned} Pr\{x(t) = n\} &= \int_0^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} \frac{e^{-\alpha\lambda}\lambda^{\nu-1}\alpha^{\nu}}{\Gamma(\nu)} d\lambda \\ &= \frac{\Gamma(n+\nu)}{\Gamma(\nu)} \frac{t^n}{n!} \frac{\alpha^{\nu}}{(\alpha+t)^{n+\nu}} \end{aligned} \tag{3.5}$$

Setting $m(t) = (\nu/\alpha)t$ (so that $m(t)$ is the mean of $X(t)$), substituting and rearranging terms allows one to rewrite (3.5) as

$$\begin{aligned} Pr\{x(t) = n\} &\equiv p_n(m,\nu) \\ &= \frac{\Gamma(n+\nu)}{\Gamma(\nu)} \frac{1}{n!} \left(1 + \frac{m(t)}{\nu}\right)^{-\nu} \left(\frac{m(t)}{\nu + m(t)}\right)^{n}. \end{aligned} \tag{3.6}$$

which is a very common way of writing the negative binomial (NB) distribution (see Feller [1968] for a discussion of the NB distribution). Actually, the commonly used form for the NB distribution has ν replaced by k. In any case, when $X(t)$ has the distribution (3.6), the mean and variance of $X(t)$ are given by

$$\begin{aligned} E\{X(t)\} &= m(t), \\ \mathrm{Var}\{X(t)\} &= m(t) + \frac{m(t)^2}{\nu}. \end{aligned} \tag{3.7}$$

Thus, the parameter ν is often used to indicate the level of overdispersion (i.e., variance exceeding the mean). In an ecological context, some authors argue that ν may even be species specific (Taylor [1971]). As $\nu \to \infty$, the NB distribution approaches the Poisson; as $\nu \to 0$ it approaches the logarithmic distribution. The method of moments can be used to estimate m and ν. If μ and σ^2 are the observed mean and variance of $X(t)$, one estimates $m = \mu$, $\nu = \mu^2/(\sigma^2 - \mu)$. The maximum likelihood estimate for ν may be vastly superior to the moment estimate (see Kendall and Stuart [1979]), but the moment estimator is simple to implement.

It is worth noting that the same kind of analysis that leads to the NB distribution can be performed using the binomial distribution instead of the Poisson as the starting point. This is described in Mangel and Clark [1983] and Mangel [1985a].

All of these models are highly aggregated in that one begins by specifying a probability distribution in the number of detections or discoveries. A less aggregated approach is to start with a model for the motion of the targets, a detection function for the searcher, and then derive the probability distribution for the number of discoveries. In general, this means solving certain kinds of parabolic partial differential equations. Descriptons of this approach are found in Section 3.3.

3.2 *Use of search information.* For the kinds of problems described in the previous section, the search information will consist of something like "in an operating time of length T, n discoveries occurred." Such information can be incorporated in a consistent way if a Bayesian approach is used. For example, suppose that search corresponds to a Poisson process with parameter λ which is unknown. Assume further that before any search occurs, λ has a gamma density (3.4). (This is called the prior density in Bayesian analysis (DeGroot [1970], Berger [1980]). The gamma density is also called the conjugate prior for the Poisson, since if the prior on λ is a gamma, after sampling the updated (posterior) density is also a gamma. The use of such conjugate priors simplifies calculations considerably.)

After search, the posterior density of λ is com~~m~~puted by the use of Bayes' theorem

$$\begin{aligned} &\Pr\{\lambda \in (\overline{\lambda}, \overline{\lambda} + d\overline{\lambda}) \mid n \text{ discoveries in T}\} \\ (3.8) \qquad &= \frac{\Pr\{\lambda \in (\overline{\lambda}, \overline{\lambda} + d\overline{\lambda}), n \text{ discoveries in } T\}}{\Pr\{n \text{ discoveries in } T\}}. \end{aligned}$$

If $f(\lambda|n, T)$ is the posterior density, then

$$(3.9) \qquad f(\lambda|n,T) = \frac{\dfrac{e^{-\alpha\lambda}\lambda^{\nu-1}\alpha^{\nu}}{\Gamma(\nu)}\,\dfrac{e^{-\lambda T}(\lambda T)^{n}}{n!}}{\displaystyle\int_0^\infty \dfrac{e^{-\alpha\lambda}\lambda^{\nu-1}\alpha^{\nu}}{\Gamma(\nu)}\,\dfrac{e^{-\lambda T}(\lambda T)^{n}}{n!}\,d\lambda}.$$

Integrating and simplifying shows that

$$(3.10) \qquad f(\lambda|n,T) = \frac{e^{-(\alpha+T)\lambda}\lambda^{n+\nu-1}\alpha^{n+\nu}}{\Gamma(n+\nu)},$$

so that the posterior density on λ is also a gamma density with updated parameters $\nu+n$ and $\alpha+T$. These updated parameters incorporate the search information. One is able, in this case, to summarize the search information so neatly because the prior that was chosen (a gamma) integrates nicely against the model for the discovery process. Another useful class of prior densities are noninformative priors (Box and Tiao [1972], Martz and Waller [1982]) in which the search data change only the location of the numerator in (3.8), but not its shape. An example of a resource modeling problem in which a noninformative prior is used is found in Mangel and Beder [1985].

For virtually all areas of search theory (one notable exception being line transect theory) search information is incorporated by means of the Bayesian kind of analysis just described. It should be pointed out, however, that in the dynamic framework of resource modeling even the simplest Bayesian methods usually run into the "curse of dimensionality" of dynamic programming. Various methods of circumventing this difficulty (conjugate priors, passive or semiactive use of information, etc.) will be used in the following sections. But it seems unlikely that any completely satisfactory solution of the dimensionality problem will be discovered, and much more research remains to be done.

3.3 *The differential equations of search theory.* In this section, certain differential equations that arise in search theory are described. They all pertain to the detection of a single target but it will be shown that they can be generalized for multiple targets. Consider first a stationary target. Assume that the probability of detecting this target in the next dt, given that it is at x and the searcher is at z, is $\Psi(x,z)dt$. Some examples of $\Psi(x,z)$ are the following. In the "cookie-cutter" model of detection, one assumes that;

$$(3.11) \qquad \Psi(x,z)dt = \begin{cases} 1, & \text{if } \|x-z\| \le R \\ 0, & \text{otherwise.} \end{cases}$$

For the case in which the target moves in the plane, the searcher in three space, and visual detection is assumed, Koopman[1980] argues that

$$\Psi(x,z) \approx \frac{kz_3}{\left((x_1-z_1)^2+(x_2-z_2)^2+(z_3)^2)\right)^{3/2}}, \tag{3.12}$$

where k is a constant and $z=(z_1,z_2,z_3), x=(x_1,x_2,0)$.

Suppose now that the searcher follows a track $Z(t)$. Let $u(x,t;Z(\tau))$ denote the probability that the target is not detected by time t given that its position is x and the search path is $Z(\tau), 0 < \tau < t$. Then assuming that detection up to t and past t are independent events gives

$$u\left(x,t+dt;Z(\tau)\right) = u\left(x,t;Z(\tau)\right)\Big(1-\Psi\left(x,Z(t)\right)dt\Big). \tag{3.13}$$

The difference equation (3.13) is equivalent to the following differential equation

$$\frac{du}{dt} = -\Psi\left(x,Z(t)\right)u, \tag{3.14}$$

with initial condition $u(x,0;Z(\tau)) = 1$ and solution

$$u(x,t;Z(\tau)) = \exp\left\{-\int_0^t \Psi\left(x,Z(\tau)\right)d\tau\right\}. \tag{3.15}$$

In the special case that $\Psi(x,Z(\tau)) = \Psi_0$, a constant, (3.15) becomes

$$u\left(x,t;Z(\tau)\right) = e^{-\Psi_0 t}. \tag{3.16}$$

Equation (3.15) is often called the random search formula (because of its original derivation by Koopman [1980]). It corresponds to memorylessness and lack of learning by the searcher, since $u(x,t+s;Z(\tau)) = u(x,t;Z(\tau))u(x,s;Z(\tau))$. The random search formula has, however, turned out to be exceedingly useful and a good description of many apparently non-random search processes (see, e.g., Washburn[1981]).

If $f_0(x)$ is the probability density for the location of the target, then the probability of detection by time t, $P_D(t)$, is

$$P_D(t) = 1-\int f_0(x)u\left(x,t;Z(\tau)\right)dx. \tag{3.17}$$

If there are N targets initially present and they are independently and identically distributed in the search region, then the probability of detecting k of them is

$$\text{(3.18)} \qquad \Pr\{k \text{ of } N \text{ detected}\} = \binom{N}{k} \left(P_D(t)\right)^k \left(1 - P_D(t)\right)^{N-k},$$

which ties back to the binomial distribution (3.1).

Suppose now that the target moves deterministically, i.e. that

$$\text{(3.19)} \qquad \frac{dx_i}{dt} = b_i(x) \qquad x_i(0) = x_{i0},$$

not subscripted

where $b(x)$ is a known velocity vector and $x_0 = (x_{1_0}, x_{2_0}, x_{3_0})$ is the unknown initial position of the target. The appropriate generalization of $u(x, t; Z(\tau))$ is

$$\text{(3.20)} \qquad \begin{aligned} u\left(x_0, t; Z(\tau)\right) = \Pr\{ &\text{ target is not detected in } (0,t) \mid \\ &\text{initial position is } x_0, \\ &\text{search path } Z(\tau), \\ &0 \le \tau \le t \text{ is followed}\}. \end{aligned}$$

The analog of (3.13) is

$$\text{(3.21)} \qquad \begin{aligned} &u\left(x_0, t + dt; Z(\tau)\right) \\ &= \left(1 - \Psi(x_0, Z)dt\right) u\left(x_0 + b(x_0)dt, t; Z(\tau)\right) \\ &= \left(1 - \Psi(x_0, Z)dt\right) \left[u\left(x_0, t; Z(\tau)\right) + \sum_i b_i(x_0) dt \frac{\partial u}{\partial x_{i0}} + o(dt) \right]. \end{aligned}$$

Equation (3.21) leads to the differential equation

$$\text{(3.22)} \qquad \frac{\partial u}{\partial t} = \sum b_i(x_0) \frac{\partial u}{\partial x_{i0}} - \Psi(x_0, Z)u,$$

with initial condition $u\left(x_0, \overset{0}{\cancel{t}}; Z(\tau)\right) = 1$. Equation (3.22) is a linear first order partial differential equation and is easily solved by the method of characteristics (Courant-Hilbert[1962]). It might be that the velocity function depends upon an additional parameter so that $b(x_0)$ is replaced by $b(x_0; \alpha)$ where α is unknown. An example would be a fish fleeing a net at a fixed speed but unknown direction. One model for $b(x_0;\alpha)$ is then

$$\begin{aligned} b_1(x_0;\alpha) &= v\cos\alpha \\ b_2(x_0;\alpha) &= v\sin\alpha \\ b_3(x_0;\alpha) &= 0, \end{aligned} \tag{3.23}$$

where v is the speed of the fish and α is the unknown direction. Such motion is often called conditionally deterministic. Clearly, it adds no conceptual difficulty to $u\left(x_0,t;Z(\tau)\right)$ and just a little bit of computational difficulty, in that one needs to condition on α, compute $u\left(x_0,t;Z(\tau)\mid\alpha\right)$ and then average over α.

As a third model for target motion, assume that the increment $dx = x(t+dt)-x(t)$ is a random variable itself. One common choice is that dx is normally distributed with mean vector $b(x)dt+o(dt)$ and covariance matrix $a(x)dt+o(dt)$. Symbolically, one writes that $x(t)$ satisfies the Itô equation (Schuss, [1980])

$$dx = b(x)dt + \sqrt{a(x)}dW, \tag{3.24}$$

where $W(t)$ is Brownian motion: $dW = W(t+dt)-W(t)$ is normally distributed with mean zero and variance dt and increments defined over disjoint intervals are independent random variables. The analog of (3.21) is now

$$\begin{aligned} &u\left(x_0,t;Z(\tau)\right) \\ &= \left(1-\Psi(x_0,Z)dt\right)E_{dx}\left\{u\left(x_0+dx,t-dt;Z(\tau)\right)\right\} \end{aligned} \tag{3.25}$$

where E_{dx} denotes the average over all dx. Taylor expanding, as in (3.21), and averaging leads to the differential equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\sum_{i,j} a_{ij}(x_0)\frac{\partial^2 u}{\partial x_{i_0}\partial x_{j_0}} \\ &\quad + \sum b_i(x_0)\frac{\partial u}{\partial x_{i0}} - \Psi(x_0,Z)u. \end{aligned} \tag{3.26}$$

This linear parabolic differential equation can often be approximately solved especially if $\|a_{ij}\|$ is in some sense small. If that is true, asymptotic methods can be applied (see Ludwig[1975], Schuss[1980] for general descriptions, and Mangel [1981,1985b]) for applications to search problems.

3.4 *Optimization problems of search theory.* In the next section, four detailed search problems will be discussed. Thus, in this section, we merely outline general kinds of optimal search problems and provide some simple, illustrative examples. The kinds of problems considered are: 1) maximizing the probability of detecting a single target in a fixed operating time, 2) minimizing the time to detect a single target, and 3) maximizing the total number of targets detected in a multi- target search problem.

For the first problem, imagine that the region of interest is divided into N "cells" that can be searched. Let

$$p_i = \Pr\{\text{ target is in cell } i\}, \tag{3.27}$$

with $\sum_{i=1}^{N} p_i = 1$. Assume that the random search formula (3.16) holds, so that

$$\begin{aligned}\Pr\{&\text{ detecting the target in search time } t_i \text{ in} \\ &\text{cell } i \mid \text{target is in cell } i\} = 1 - e^{-\beta_i t_i},\end{aligned} \tag{3.28}$$

where β_i is a constant. Suppose that the total search time is T and that this time is allocated over the N cells. The probability of detecting the target is $\sum_{i=1}^{N} p_i(1-e^{-\beta_i t_i})$ so that one has the optimal search problem

$$\begin{aligned} &\text{maximize } \sum_{i=1}^{T} p_i(1 - e^{-\beta_i t_i}) \\ &\qquad\text{such that } \sum_{i=1}^{N} t_i = T \\ &\qquad\text{all } t_i \geq 0. \end{aligned} \tag{3.29}$$

This is a simple nonlinear programming problem. It can be solved by applying the Kuhn-Tucker theorem (Avriel [1976]), the algorithm of Charnes and Cooper [1954], or the maximum principle. Observe that this problem involves a one-shot allocation of effort over the cells. An interesting modification, which we leave to the reader to formulate, is one in which there are two periods of length T. In such a case, the results of the search in the first period affect the allocation of effort in the second period. This effect is observed by updating the p_i after search. That is:

Pr{ target is in cell i |

it was not discovered in t_i hours of search}

$$= \frac{p_i(1-e^{-\beta_i t_i})}{p_i(1-e^{-\beta_i \text{helps}})+1-p_i} \quad (3.30)$$

$$= \frac{p_i(1-e^{-\beta_i t_i})}{1-p_i e^{-\beta_i t_i}}.$$

Next, consider the problem of minimizing the time to detect a single, randomly moving target. Observe that the probability of detecting this target by time T is

(3.31) $P_D(T) = 1 -$ [illegible] $- \int f_0(x_0)\overset{u}{\cancel{V}}(x_0, T; z(\tau))dx_0,$

where $f_0(x_0)$ is the probability density for the initial location of the target, and $u(x_0, T; Z(\tau))$ — the probability of nondetection in $(0, T)$ given that the initial position of the target is x_0 — satifies (3.26). The mean time to detection, $\overline{T}$, is then

$$\overline{T} = \int_0^\infty t P_D'(t)dt. \tag{3.32}$$

Suppose now that $Z(\tau)$ satisfies

$$\frac{dZ}{d\tau} = \overset{\sim\surd}{\cancel{V}}(\tau; Z), \tag{3.33}$$

where the velocity vector $\overset{\sim\surd}{\cancel{V}}(\tau; Z)$ is a control variable drawn from a control set V. Thus, one has the optimal control problem

$$minimize_{v \in V}\overline{T}, \text{ subject to } \frac{dZ}{d\tau} = v(\tau; Z). \tag{3.34}$$

This is a deceptively simple control problem — deceptively because buried in $\overline{T}$ is the solution of a parabolic partial differential equation. Methods for the solution of these kinds of problems are discussed by Lions [1971], Ahmed and Teo [1981], Teo and Wu [1984] for general problems, and Lukka [1979] for search problems in particular.

For the third problem, once again imagine a collection of N cells with many targets and assume that there are M search periods, each of length 1. Assume that there is a single searcher who searches one cell in each period and that the discoveries follow a Poisson distribution:

$$\Pr\{k \text{ discoveries} \mid \text{cell } i \text{ is searched}\} \tag{3.35}$$

$$= \frac{\lambda_i^k e^{-\lambda_i}}{k!}$$

Now assume that the λ_i are unknown and have a gamma prior distribution with parameters ν_i and α_i. Thus, search does two things: first it produces discoveries of the objects, and second it produces information which is summarized in the continual updating of the parameters ν_i and α_i (la (3.10)). Assume that the objective is to make as many discoveries as possible over the M search periods. Define $V_k(\nu, \alpha)$ by

$$\begin{aligned} V_k(\nu,\alpha) = {} & \text{maximum expected number of discoveries} \\ & \text{when } k \text{ periods remain and the parameters} \\ & \text{are } \nu = (\nu_1, \ldots, \nu_N), \text{ and } \alpha = (\alpha_1, \ldots, \alpha_N). \end{aligned} \tag{3.36}$$

We will now derive the dynamic programming equation (DPE) that $V_k(\nu, \alpha)$ satisfies. In order to do this, note that, conditioned on λ_i, the expected number of discoveries if cell i is searched is λ_i. When this is averaged over the gamma density, the expected number of discoveries, if cell i is searched, is ν_i/α_i. Thus,

$$V_1(\nu, \alpha) = \max_i \frac{\nu_i}{\alpha_i}. \tag{3.37}$$

The meaning of (3.37) is obvious: if only one period remains, search the cell where the expected number of discoveries is highest since any information obtained in this last period is valueless. For $k > 1$, the dynamic programming equation (DPE) for $V_k(\nu, \alpha)$ is .

$$V_k(\nu, \alpha) = \max_i \left\{ \frac{\nu_i}{\alpha_i} + E\{V_{k-1} \mid i\} \right\}. \tag{3.38}$$

In this expression, $E\{V_{k-1}|i\}$ is the expected value of V_{k-1}, given that cell i is searched. It is computed as follows:

$$E\{V_{k-1}|i\} = \sum_{n=0}^{\infty} \Pr\{n|i\} V_{k-1}(\nu + \delta_i n, \alpha + \delta_i), \tag{3.39}$$

where $\Pr\{n|i\}$ is the probability of n discoveries given that cell i is searched (see (3.4)-(3.5)):

$$\Pr\{n|i\} = \frac{\Gamma(n+\nu_i)}{\Gamma(\nu_i)n!} \frac{\alpha_i^{\nu_i}}{(\alpha_i+1)^{n+\nu_i}}, \tag{3.40}$$

and

$$\nu + \delta_i n = (\nu_1, \nu_2, \ldots, \nu_{i-1}, \nu_i + n, \nu_{i+1}, \ldots, \nu_N),$$
$$\alpha_i + \delta_i = (\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, \ldots, \alpha_N).$$

By recursively solving (3.38), one obtains the optimal sequence i_k^* of cells to search and the optimal value function V_k^*. The solution of (3.38) will be discussed in more detail in the next section.

4. Applications of search theory in resource modeling.

We will now describe four diverse applications of search theory to resource modeling. These are 1) fisheries, 2) agricultural pest control, 3) animal foraging, and 4) oil and mineral exploration.

4.1 *Fisheries.* In many commercial fisheries a significant portion of the cost of fishing operations is taken up by search for concentrations of fish. Some fishermen are regularly more successful than average. While fishermen themselves may attribute such success to blind luck, it seems more likely that the most successful fishermen are those who are most adept at utilizing information from their environment. The successful fisherman knows where to look for fish at any time, and when to abandon an unsuccessful search. The unsuccessful fisherman fishes in the wrong places at the wrong times — perhaps using the wrong gear.

Attempting to understand the behavior of fishermen in an uncertain environment is important for fisheries regulators, especially when new types of regulations are being considered (Bockstael and Opaluch[1984], Hilborn[1985]). Regulations introduced without proper appreciation of fishermen's behavior are not likely to achieve the desired objectives of management, and thus tend to erode confidence in the managing authority.

Models of search in fisheries have been discussed by Shotten [1973], Swierzbinski[1981], Mangel [1982], Mangel and Clark [1983], Mangel and Plant [1985], Mangel and Beder [1985], and Clark [1985]. The key observation is that the fishing process actually produces two things: first the physical catch and, second, information about the stock level. Thus, one needs to study questions about the joint production (in economic language) of fish and information.

We will now consider a slight generalization of the third optimal search problem discussed in the previous section and show how it can be used to answer the question of where one should go fishing (Mangel and Clark, [1983]).

Consider a fleet of N vessels capable of searching for fish on several fishing grounds $A_1, \ldots, A_m$. Search for fish on each A_i is modeled as a Poisson process with mean encounter rate λ_i per vessel. It is assumed that k vessels in A_i search independently so that

$$\begin{aligned}(4.1)\qquad &\Pr\{k \text{ vessels encounter } n \text{ schools on } A_i \text{ in time } t|\lambda_i\}\\ &= p_{n,k,t}(\lambda_i) = \frac{(\lambda_i kt)^n}{n!}e^{-\lambda_i kt}, \quad n = 0, 1, 2, \ldots.\end{aligned}$$

This is expressed as a conditional probability in order to emphasize that we wish to consider the mean encounter rate λ_i to be uncertain. The fishermen do not know a priori what the mean encounter rate on A_i will be, but they combine their historical experience with current environmental observations to obtain a prior estimate of λ_i. Once fishing begins, they are able to update this estimate according to actual fishing success.

Let $f(\lambda_i)$ denote the prior distribution for λ_i. If n schools of fish are encountered by k vessels during time t, the posterior distribution becomes

$$(4.2)\qquad f(\lambda \mid n, k, t) = \frac{p_{n,k,t}(\lambda_i)f(\lambda_i)}{\int p_{n,k,t}\enclose{circle}{(\lambda i)}f\enclose{circle}{(\lambda i)}d\lambda;}.$$

If the prior density for λ_i is a gamma density (3.4) with parameters ν_i and λ_i, then the posterior density is also a gamma with parameters $\nu_i + n_i, \alpha_i + k_i t$. The updated (posterior) mean and variance are thus given by

$$(4.3)\qquad \overline{\lambda}_i' = \frac{\nu_i + n_i}{\alpha_i + k_i t}, \quad (\sigma_i')^2 = \frac{\nu_i + n_i}{(\alpha_i + k_i t)^2}.$$

(This model ignores the effect of removals of fish schools, which alters the value of λ_i as fishing proceeds. This extra complication will be discussed briefly below).

For simplicity of notation we now suppose there are only two fishing grounds, and let M denote the number of fishing trips that can be completed in a given season. During each trip, the fleet can be allocated with k_i vessels to ground $A_i, k_1 + k_2 = N$. With an objective of maximizing expected net revenue over the season, how should the fleet be allocated to A_1 and A_2 over time?

Before setting this problem up mathematically, it is worthwhile to think about what it involves. Why not, for example, simply send all N vessels to the ground with the highest expected catch rate (or highest expected net return, if costs differ between A_1 and A_2)? Indeed as (3.37) shows, if only one trip is possible, this is the optimal policy. But if several trips are possible, this policy may not be optimal, since it generates no new information about unsampled ground. Perhaps one or two vessels should be sent to the less attractive fishing ground, on the off chance that fish are unusually abundant there. If so, then the whole fleet can be switched for subsequent trips. Such "probing" for information is characteristic of optimal search strategies.

Let p_i denote the average dockside value of fish obtained from A_i and let c_i denote the cost of sending one vessel to A_i for one trip. Let $V_n(\nu, \alpha)$ denote the expected net revenue obtained from optimal allocation of N vessels, when n fishing trips remain in season, and where $\nu = (\nu_1, \nu_2)$ and $\alpha = (\alpha_1, \alpha_2)$ are the prior parameters at the beginning of these n remaining trips. (In effect, ν and α are the state variables of the problem, representing the state of information). If $n = 1$, the expected net revenue of value function is a generalization of (3.37) to include economic parameters:

(4.4)
$$\begin{aligned} V_1(\nu, \alpha) &= \max_{1 \le k \le N} \left[k \left(\frac{\nu_1}{\alpha_1} p_1 t - c_1 \right) + (N-k) \left(\frac{\nu_2}{\alpha_2} p_2 t - c_2 \right) \right] \\ &= \max_{1 \le k \le N} \left[\left(\frac{p_1 \nu_1}{\alpha_1} - \frac{p_2 \nu_2}{\alpha_2} \right) t - (c_1 - c_2) \right] k + N \left(\frac{p_2 \nu_2 t}{\alpha_2} - c_2 \right), \end{aligned}$$

and hence the optimal number of vessels on A_1 is

$$k^* = \begin{cases} N, & \text{if } \frac{p_1 \nu_1}{\alpha_1} t - c_1 > \frac{p_2 \nu_2}{\alpha_2} t - c_2 \\ 0, & \text{otherwise.} \end{cases} \tag{4.5}$$

The dynamic programming equation for $n > 1$ is then a generalization of (3.38):

$$\begin{aligned} V_{n+1}(\nu, \alpha) = \max_{1 \le k \le N} \Big[& k \frac{\nu_1}{\alpha_1} (p_1 t - c_1) + (N-k) \left(\frac{\nu_2}{\alpha_2} p_2 t - c_2 \right) \\ & + E\{V_n(v'_k, \alpha'_k)\} \Big], \end{aligned} \tag{4.6}$$

where the prior expectation is really a double expectation, over the prior distributions $f(\lambda_i)$ and over the sampling "experiments" (i.e., the generalization of (3.39)).

$$\begin{aligned}(4.7)\quad E\{\ldots\} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} V_n\left(\nu_1+n_1, \nu_2+n_2, \alpha_1+kt, \alpha_2+(N-k)t\right)\\ \Pr(n_1)\Pr(n_2).\end{aligned}$$

The probabilities $\Pr(n_i)$ are a generalization of (3.40):

$$(4.8)\quad \begin{aligned}\Pr(n) &= \int_0^{\infty} \Pr(n|\lambda, k, t)\gamma(\lambda; \nu, \alpha)d\lambda\\ &= \frac{(kt)^n}{n!}\frac{\alpha^n}{(\alpha+kt)^{n+\nu}}\frac{\Gamma(n+\nu)}{\Gamma(\nu)} \qquad (k \neq 0),\end{aligned}$$

$$(4.9)\quad \Pr(n) = \begin{cases}1, & n=0\\ 0, & n>0\end{cases} \quad (k=0).$$

Unlike (3.39), the expectation (4.7) involves a double series that must be summed numerically for each iteration of (4.6). Since the series converges slowly, this computation can be extremely time consuming. In Mangel and Clark [1983], only one iteration was attempted, and the idea of "semiadaptive" control was used to deal with $M > 2$ periods. With this approach, only information generated in the first fishing period is taken into consideration, and it is assumed that the allocation of vessels in periods $2, \ldots, M$ will be specified by the updated parameters after one trip. This is probably a good approximation to the optimum provided that sufficient sampling occurs on each ground in one period. (In real life it could hardly be expected that the values of λ_i would remain constant throughout the fishing season, unless the fish were of a particularly sedentary nature. Hence, a continual probing strategy would probably be best; no specific model of this situation has been developed, however.)

A single searcher. The semi adaptive calculation gives a poor approximation for the case of a lone searcher, since search on only one ground is considered. However, in this case we can make a simplification which allows for an additional period of search to be treated. For simplicity, assume $p_1 = p_2$ and $c_1 = c_2$. Then we can ignore costs and set $p_1 = p_2 = 1$, and (4.4) and (4.6) reduce to (3.37) and (3.38); the latter we rewrite more explicitly.

$$V_1(\nu,\alpha) = \max_{i=1,2} \frac{\nu_i}{\alpha_i} t, \tag{4.10}$$

$$\begin{aligned} V_2(\nu,\alpha) = \max\{ & \frac{\nu_1}{\alpha_1} t + t \sum_{n_1=0}^{\infty}, \max\left(\frac{\nu_1+n_1}{\alpha_1+t}, \frac{\nu_2}{\alpha_2}\right) \Pr(n_1), \\ & \frac{\nu_2}{\alpha_2} t + t \sum_{n_2=0}^{\infty} \max\left(\frac{\nu_1}{\alpha_1}, \frac{\nu_2+n_2}{\alpha_2+t}\right) \Pr(n_2)\}. \end{aligned} \tag{4.11}$$

The series in (4.11) can be reduced to a finite sum as follows (Clark and Mitchell [1984]):

$$\begin{aligned} & \sum_{n_1=0}^{\infty} \max\left(\frac{\nu_1+n_1}{\alpha_1+t}, \frac{\nu_2}{\alpha_2}\right) \Pr(n_1) \\ & \qquad = \sum_{n_1=0}^{m} \frac{\nu_2}{\alpha_2} \Pr(n_1) + \sum_{n_1=m+1}^{\infty} \frac{\nu_1+n_1}{\alpha_1+t} \Pr(n_1), \end{aligned}$$

where

$$m = \text{ integer } \left[\frac{\nu_2(\alpha_1+t)}{\alpha_2} - \nu_1\right]$$

if this is nonnegative, and $m = -1$ otherwise. Thus, the above sum becomes

$$\begin{aligned} & \sum_{n_1=0}^{\infty} \frac{\nu_1+n_1}{\alpha_1+t} \Pr(n_1) + \sum_{n_1=0}^{m} \left(\frac{\nu_2}{\alpha_2} - \frac{\nu_1+n_1}{\alpha_1+t}\right) \Pr(n_1) \\ & \qquad = \frac{\nu_1}{\alpha_1} + \sum_{n_1=0}^{m} \left(\frac{\nu_2}{\alpha_2} - \frac{\nu_1+n_1}{\alpha_1+t}\right) \Pr(n_1). \end{aligned} \tag{4.12}$$

Not only is the sum in Eq. (4.12) finite, but the number of terms also decreases, eventually to zero, as ν_1 increases; this further simplifies the subsequent iteration for V_3:

(4.13)

$$\begin{aligned} V_3(\nu,\alpha) = \max\Bigg\{ & \frac{\nu_1}{\alpha_1} t + \sum_{n_1=0}^{\infty} V_2(\nu_1+n_1, \nu_2, \alpha_1+t, \alpha_2) \Pr(n_1), \\ & \frac{\nu_2}{\alpha_2} t + \sum_{n_2=0}^{\infty} V_2(\nu_1, \nu_2+n_2, \alpha_1, \alpha_2+t) \Pr(n_2) \Bigg\}. \end{aligned}$$

The calculations leading to (4.12) make the evaluation of $V_2(.)$ in (4.13) an easy computation.

The computed results for the first three iterations are shown, in "feedback" form, in Figure 1. Here the prior mean and coefficient of variation for A_1 are fixed at $\overline{\lambda}_1 = 10.0$ and $CV_1 = 0.5$ (i.e., $\nu_1 = 4$, $\alpha_1 = 0.4$). The curves in the figure are "switching curves"—the optimal ground to sample on trip 1 is A_2 if $(\overline{\lambda}_2, CV_2)$ lies above or to the right of the switching curve, and A_1 otherwise. The three upper curves are shown corresponding to having one, two, or three trips remaining, respectively. (The fourth curve labeled SA will be explained later.)

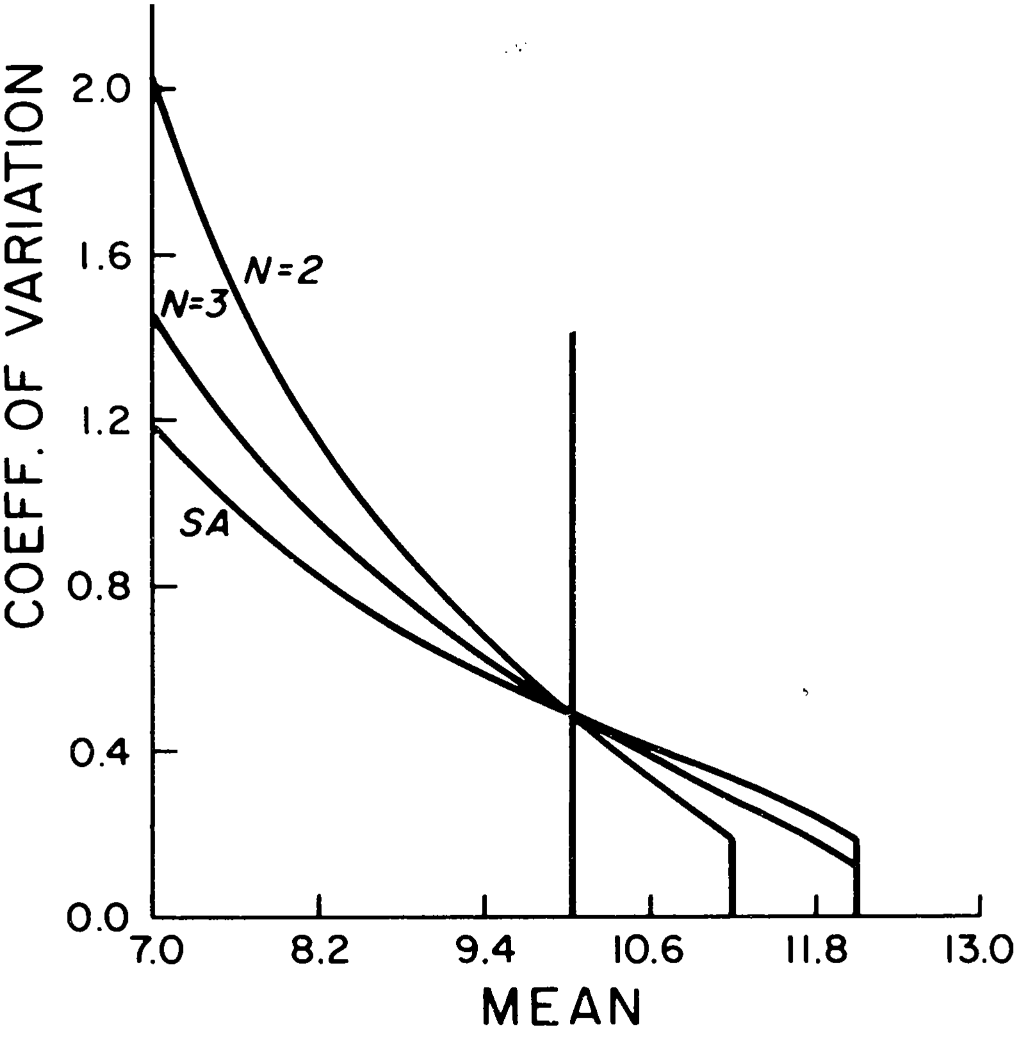

Figure 1. Indifference curves for the first of N remaining trips $N = 1, 2, 3$, plus ”semiactive–adaptive curve for $N = 3$.

Note first that all three switching curves pass through the "indifference point" $(\overline{\lambda}_2, CV_2) = (\overline{\lambda}_1, CV_1)$. For $N = 1$ the curve is a vertical line: A_2 is preferred to A_1 if and only if $\overline{\lambda}_2 > \overline{\lambda}_1$. For $N > 1$ the relative levels of uncertainty also affect the decision: A_2 will be preferred to A_1 even if $\overline{\lambda}_2 < \overline{\lambda}_1$, provided CV_2 is sufficiently larger than CV_1. The reason for this is clear: when the level of fish abundance is much more uncertain on A_2 than on A_1, the value of information obtained by sampling A_2 exceeds that for A_1 by a sufficient amount to compensate for the expected decrease in catch. This is the active aspect of active-adaptive control. (The switching curve for passive-adaptive control is simply the vertical line $\overline{\lambda}_2 = \overline{\lambda}_1$.)

The curve labeled SA corresponds to the semiactive policy, for $N = 3$. While this policy differs noticeably from the optimal policy, the actual expected payoff is only marginally inferior (Table 4.1). The columns in Table 4.1 are : nonadaptive (fish A_1 for all three trips); passive (fish A_1 on first trip, switch if $\overline{\lambda}_1' < \overline{\lambda}_2$, etc.); semiactive; and optimal.

Table 4.1. Total expected catch for various search policies: NA (non-adaptive), P (passive), SA (semiactive), and O (optimal). Parameter values: $\overline{\lambda}_1 = 10.0$, $CV_1 = 0.5$, $\overline{\lambda}_2 = 8.0$, $t = 5$, $N = 3$.

CV_2	NA	P	SA	O
0.2	150.0	159.4	159.2	159.4
0.6	150.0	160.9	159.2	161.3
1.0	150.0	161.7	159.2	163.8
1.4	150.0	164.0	172.4	173.1
1.8	150.0	165.1	180.3	180.7

Depletion and competition. The information obtained from searching can be extremely valuable to fishermen, and the question then arises as to the optimal production and sharing of such information. The sharing question cannot be addressed on the basis of the above simple model, however, since limited sharing only becomes relevant when fish concentrations are depletable. Modification of the simple search model to allow for depletion is discussed by Mangel and Clark [1983], to which the reader is referred for details. The basic probabilities in (4.1) now become

$$(4.14) \quad p_{n,k,t}(\lambda) = \begin{cases} \frac{1}{n}(\lambda/\delta)_n(1 - e^{-k\delta t})^n e^{-kt(\lambda - n\delta)}, & 0 \le n \le \lambda/\delta \\ 0, & n > \lambda/\delta, \end{cases}$$

where $(a)_n = (a-n+1)(a-n+2)\dots(a)$, for $a \geq n$. Here δ represents the amount by which the search rate λ is reduced by the removal of one school of fish (schools are assumed to be of equal size), so that λ/δ is the initial number of schools.

Mangel and Clark [1983] use this depletion model to analyze the competitive behavior of fishermen. It is of interest to ask whether competition would lead to an oversupply or an undersupply of search effort. In fact, both may occur! First, the competitive scramble to exploit a potentially productive fishing ground may generate excessive information as a side effect of the usual common-property over-exploitation phenomenon (Clark [1976]). On the other hand small patches may not be searched out at all, in cases where individual fishermen would be unable to protect their finds from intruders. While the two effects could counteract one another, it would only be by accident that an approximately optimal exploitation pattern would emerge. Mangel and Beder [1985] discuss other aspects (especially inference) of the model (4.14).

Regulation. The deterministic theory of fishery regulation has been discussed by Clark [1980], but the stochastic theory, which seems essential for predictive purposes (Hilborn [1985]), is as yet largely undeveloped. Mangel and Beder [1985] have discussed the problem of utilizing "on-line" search information to determine the optimal length of the fishing season. Mangel and Plant [1985] and Clark [1985] have modeled the effects of fishery quotas on fishermen's behavior.

Mangel and Plant [1985] note the following "duality" between regulation in terms of individual vessel effort and catch. The vessel's net seasonal profits are, in either case,

$$\pi = pH - c(E), \tag{4.15}$$

where H is the season's catch and E denotes effort. If E is controlled, then H becomes a random variable, while if H is controlled, then the effort E required to catch the quota H becomes random. Expected profits will generally differ for the two types of regulation.

Attempts by management agencies to introduce individual catch or effort quotas have usually met with strong resistance from the fishermen involved, although in some cases at least, the fishermen ultimately became strong supporters of a quota system. What fishermen may be most concerned about is that an individual quota will severely limit their chances of having a great year once in a while. In an unlucky season the fisherman may fail to fulfill his quota, while in a good season the quota constrains his catch. His expected catch is necessarily smaller than his quota.

The Seine Fishery for herring in the Bay of Fundy has operated under a vessel quota system since 1976. Figure 2 shows catches of the 47 seiners in this fishery in 1978; individual vessels had quotas ranging from 1200 to 1500 tons, depending on vessel size and past catch history. At least 23 of the 47 vessels failed to achieve their full quotas.

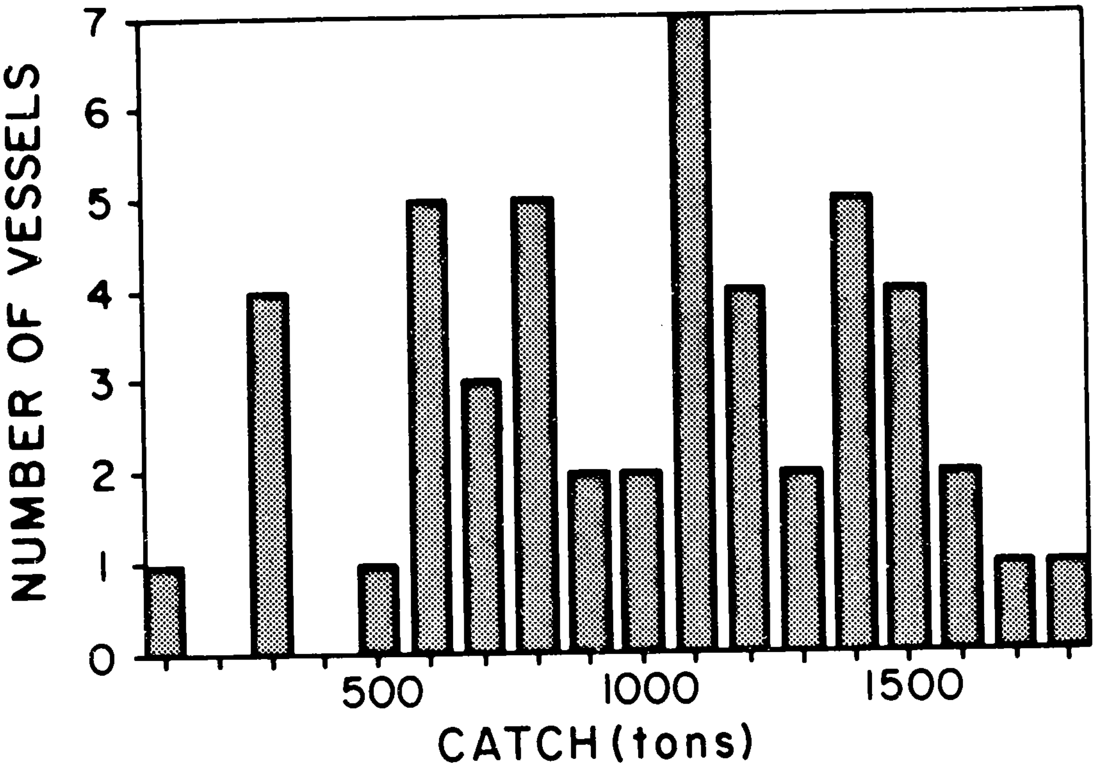

Figure 2. Individual vessel catches of herring in the Bay of Fundy, 1978. Vessel quotas ranged from 1200 to 1500 tons.

The following simple model (Clark[1985]) describes this situation. Let p_n denote the probability of finding n units of fish during a given time. We use the negative binomial distribution (3.6) with the mean $m(t) = m$, a constant over time, and aggregation parameter $\nu = k$.

Now suppose the fisherman has a quota of Q units of fish per week. Then the probability distribution for his actual catch is truncated at Q:

$$p_{n,Q} = \begin{cases} p_n(m,k)/C_Q, & \text{for } n \le Q \\ 0, & \text{for } n > Q, \end{cases} \tag{4.16}$$

where the normalization constant is

$$C_Q = \sum_{n=0}^{Q} p_n(m,k). \tag{4.17}$$

The expected catch under quota is

$$E\{n|Q\} = \sum_{n=0}^{Q} np_{n,Q} \tag{4.18}$$

and it is not hard to show that $E\{n|Q\} < E\{n\}$.

Given values of m, k, and Q, the expected reduction in catch can be computed easily from these formulas. The higher the variance in weekly catch rates, the greater will be the reduction in expected catches under quota. Also, a shorter quota period leads to a greater reduction in catch.

It follows that individual quota systems should be designed with a maximum of flexibility. Pooling and transfer of quotas should be allowed (but must be closely monitored). Short-period quota limits can be used to smooth out the delivery of fish to processors or marketers, but fishermen should be allowed to continue fishing until they have accumulated their seasonal quota. If properly formulated and controlled, an individual quota system has great potential both for controlling catches and for encouraging economic stability and efficiency in a common-property fishery. Approval and cooperation by the fishermen themselves is essential for the success of such a system.

4.2 *Agricultural pest control.* It has become apparent in recent years that heavy reliance on prophylatic spraying of insecticide as a means of controlling agricultural pests may exact a considerable toll from ecosystems. Thus, alternative control programs —called Integrated Pest Management (or IPM) —via cultural, biological, and chemical means have developed. Perkins [1982] presents an excellent discussion and history of the pesticide problem. A key element in the optimal management of agricultural pests and the concomitant reduction and/or avoidance of senseless use of pesticides — involves knowing how many pests are present. That is, by acquiring search information on pest infestation levels, a grower can make more informed decisions about how to control the pest. In this case, the search is equivalent to sampling the crop for the pest. Plant and Wilson [1985] and Plant and Mangel [1986] provide discussions of sequential sampling in agricultural pest control from the viewpoint of applied mathematicians. Mangel, Plant and Carey [1984] consider a search problem related to the detection and delimiting of infestations of Mediterranean fruit flies.

We will illustrate how search theory can be used in agricultural pest control by considering the cotton-lygus bug (*Lygus hesperus* Knight) system. Recent work by Stefanou [1983] and Mangel, Stefanou, and Wilen [1985] shows that there are two periods during the development of the crop when lygus may do damage to the ultimate yield of cotton. (There is actually some disagreement about the possibility of damage that lygus may cause. Alternate viewpoints are provided by the work of Gutierrez, et. al. [1977,1979]. Regardless of the ultimate disposition of the lygus question, we believe that the example presented here is still useful for its illustrative purposes.)

If a grower did not sample his crop for pests, his decisions on spraying or other controls would be made using essentially whatever prior he has on the pest distribution. Scouting or searching for pests can provide additional information, which can be used to update the prior. Thus, in each period of crop development a grower has two decisions to make: first, whether to scout or not, and second, whether to apply some control or not. We will explicitly concentrate on the scouting decision here.

In order to formulate the search and spraying model, one needs to be able to characterize 1) the plant's dynamics, 2) the pest's dynamics, 3) the damage to the plant caused by the pest, and 4) the effect of spraying the pest. Thus introduce the following variables:

$$L_i = \text{ lygus population at the start of period } i$$

$$y_i = \begin{cases} 1, & \text{if spray was applied in period } i \\ 0, & \text{otherwise,} \end{cases} \tag{4.19}$$

$$s_i = \begin{cases} 1, & \text{if scouting was done in period } i \\ 0, & \text{otherwise.} \end{cases}$$

Let $D(L_1, L_2, y_1, y_2)$ denote the damage to the physical yield, measured at the end of the season, when the values of lygus and spray are (L_1, L_2) and (y_1, y_2) respectively. If p is the unit price for cotton, c_1 the cost of spraying and c_2 the cost of searching, then the total cost is

$$C_T = pD(L_1, L_2, y_1, y_2) + c_1(y_1 + y_2) + c_2(s_1 + s_2). \tag{4.20}$$

A detailed specification of the damage function which takes into account the dynamics of the cotton plant is found in Appendix 1 (also see Stefanou[1983] and Mangel, Stefanou, and Wilen [1985]). Pest control problems are different from most resource problems in that the costs are not assessed incrementally. That is, costs are measured in reduced yield at the end of the season by specifying one function rather than a sum of costs over the periods of interest. The total cost C_T can easily be extended to include a continuous cost of spraying by simply reinterpreting the meaning of y_i. The profit to a grower, π, is

$$\pi = pY_0 - C_T, \tag{4.21}$$

where Y_0 is the physical cotton yield in the absence of lygus.

Scouting provides information on the values of L_i. The method of scouting is by a sweep net and the unit of scouting is 50 sweeps of the net. Thus the data are represented as X = number of lygus observed per 50 sweeps. Each row corresponding to 50 sweeps will be called a "plot". We assume that the probability that a plot contains $L = \ell$ lygus is given by the Poisson distribution with parameter μ and that μ itself has a gamma distribution with parameters ν and α. Thus, the underlying model for the distribution of lygus is a negative binomial (equations (3.2) - (3.6)). For operational purposes, the results of scouting are divided into three categories: low ($X \leq 4$), medium ($5 \leq X \leq 8$), or high ($X > 9$) lygus counts per 50 sweeps. One needs, in addition, to specify the accuracy of the search. Using the discretized form of the lygus count, this can be done by introducing a matrix, σ, as follows:

$$\sigma = \begin{array}{c|ccc} X/L & Low & Medium & High \\ \hline Low & \sigma_{LL} & \sigma_{LM} & \sigma_{LH} \\ Medium & \sigma_{ML} & \sigma_{MM} & \sigma_{MH} \\ High & \sigma_{HL} & \sigma_{HM} & \sigma_{HH} \end{array} \tag{4.22}$$

where $\sigma_{ij} = \Pr\{X = i | L = j\}$ is a measure of the accuracy of the search process. If all $\sigma_{ij} = 1/3$, then search will produce no information and if the diagonal entries are equal to 1 the search is perfect. A convenient model is one in which the diagonal elements are equal to a parameter σ and the off-diagonal elements equal to $(1 - \sigma)/2$. To complete the formulation of the problem, one needs to specify the lygus dynamics. The following simple model is used

$$L_2 = Z(K, L_1) + (1 - Ky_1)L_1, \tag{4.23}$$

where L_1 is the lygus population in period i, K is the fraction of lygus killed by the spray and $Z(K, L_1)$ represents net immigration plus any increase in lygus due to the killing of predators.

We can now formulate the dynamic programming equations for the scouting and spraying decisions. Consider the second (latter) period first. Let the utility of profit be denoted by $U(\pi) = U_1\left(p(Y_0 - D) - c_1(y_1 + y_2)\right) - U_2\left(c_2(s_1 + s_2)\right)$. If the grower scouts in the second period, the expected utility of profit is

$$(4.24) \qquad V_S^{(2)} = E_{\mu|X_1}\left\{E_{L_1|X_1,\mu}\left\{E_{X_2|X_1,\mu}\left\{\max_{y_2} E_{L_2|X_2,X_1,\mu,y_1},\right.\right.\right.$$

$$\left.\left.\left.\left[U_1\left(p\left(Y_0 - D(L_1, L_2, y_1, y_2)\right) - c_1(y_1 + y_2)\right) - U_2\left(c_2(s_1 + s_2)\right)\right]\right\}\right\}\right\}.$$

This formidable set of expectations is the way of incorporating search information. Reading from the left on the right hand side of (4.24) one has: first, the posterior average over the updated distribution on μ, given the search information X_1; second, the average over possible values of L_1, given the search information; third, the a priori average over second period search information given first period search information (with the dynamics (4.23) implicitly built into this); and fourth, the average over the true second period population, given the search information. If no scouting is done in the first period, these expectations simplify to the expectations over the prior distribution.

If no search occurs in the second period, the expected utility of profit is

$$(4.25) \qquad V_{NS}^{(2)} = E_{\mu|X_1}\left\{E_{L_1|X_1,\mu} \max_{y_2} E_{L_2|X_1,\mu}, \left[U_1\left(p\left(Y_0\right.\right.\right.\right.$$

$$\left.\left.\left.\left. - D(L_1, L_2, y_1, y_2)\right) - c_1(y_1 + y_2)\right) - U_2(c_2 s_1)\right]\right\}.$$

That is, if no search is performed in the second period, the optimal spraying decision is made on the basis of first period information (posterior distributions) and lygus dynamics solely.

The optimal second period search decision is then made by determining the second period value, $V^{(2)}$:

$$(4.26) \qquad V^{(2)} = \max_{\{S,NS\}} \left(V_{NS}^{(2)}, V_S^{(2)}\right).$$

The solution of (4.26) determines whether or not the grower should scout in the second period. Note that this solution says nothing about whether or not the grower should spray, or the level of spraying. The level of spraying is determined by a set of optimization equations similar to (4.24) and (4.25) after the scounting information is obtained. For example, the grower might have a prior distribution which gives considerable weight to a high density of lygus. If scouting results showed few lygus, the posterior distribution would have less weight on a high density of lygus and spraying—which would have occurred without scouting—might not be done.

The first period spraying and scouting decisions are now determined by solving appropriate dynamic programming equations. In particular, if the grower does scout in the first period, the expected utility of profit is

$$V_S^{(1)} = E_\mu \left\{ \max_{y_1} E_{L_1|X_1,\mu} \left[\max \left(E_{X_2|X_1,\mu,y_1} V_S^{(2)}, \right. \right. \right. \\ \left. \left. \left. E_{X_2|X_1,\mu,y_1} V_{NS}^{(2)} \right) \right] \right\}. \tag{4.27}$$

If the grower does not scout in the first period, the expected utility of profit is

$$V_{NS}^{(1)} = E_\mu \left\{ \max_{y_1} E_{L_1|\mu} \left[\max \left(E_{X_2|\mu,y_1} V_S^{(2)}, \right. \right. \right. \\ \left. \left. \left. E_{X_2|\mu,y_1} V_{NS}^{(2)} \right) \right] \right\}. \tag{4.28}$$

The ultimate first period value and search decision are then made according to

$$V^{(1)} = \max_{\{S,NS\}} (V_S^{(1)}, V_{NS}^{(1)}). \tag{4.29}$$

Equations (4.24) - (4.29) provide a complete solution to the searching and spraying problem. They are easily programmed on a desktop micro computer (e.g., Stefanou [1983] provides results using Apple II+). In order to facilitate presentation of these results, the following additional information is needed:

1) In addition to the specification of lygus as simply Low, Medium, or High counts per 50 sweeps, the net growth parameter $Z(K, L_1)$ in dynamics (4.23) needs to be specified. Two choices will be used: $Z = 3$ lygus/50 sweeps (low Z), and $Z = 8$ lygus/50 sweeps (high Z). One could, of course, add an entire distribution to Z without any conceptual and very little computational difficulty.

2) The damage function D is actually quite complicated (Stefanou [1983] or Mangel, Stefanou, and Wilen [1985]) since it involves tracking the life history of the cotton plant from square production through the development of bolls and then relating bolls to yield. For purposes here, it is sufficient to index the cotton plant by its initial square load, Q (see Appendix 1 for further discussion).

3) The case of $\sigma_{ii} = \sigma$ and $(1 - \sigma)/2, \sigma_{ij} = 1 - \sigma/2$, $i \neq j$, in the scouting matrix (4.22) is used for the results presented here.

Tables 4.2 and 4.3 show the ultimate expected value of cotton production (dollars/acre) as a function of the four possible search strategies, the prior distribution on lygus, the initial square load, and the value of $Z(K, L_1)$. The bottom row in these tables compares the value of cotton production for the case of always searching versus never searching. In some sense, this is a good measure of the value of search information. For example, if a grower had to contract-out scouting services, the entries in the last row of these tables provide him with an idea of a "fair price" (to the grower, at least) for the scouting services.

Tables 4.4 and 4.5 show the value of always scouting versus the value of never scouting as a function of accuracy of the search information (σ), the prior distribution on lygus, the initial square load, and the value of $Z(K, L_1)$. As σ increases, the accuracy of scouting information also increases.

We stress that these results provide a way of assessing the value of search information and are not an endorsement of mindless spraying of pesticides. In fact, the reverse is true: our emphasis is on using search information in order to make better spraying decisions.

In conclusion, this example from pest control shows how search theory can play a major role in the optimal management of agricultural systems.

4.3 *Animal foraging.* Some species of animals live a solitary existence (except during breeding) while others are highly social. Many benefits have been suggested for group living (Pulliam and Caraco [1984]), but the question of interest here is whether foraging in groups can result in increased survival probabilities for individual foragers. A related question concerns the optimal size of foraging groups, and whether such optimal group sizes would in fact be observed in nature.

Let us take up the latter question first. Suppose that evolutionary fitness (e.g., survival probability) is a function $\varphi(n)$ of group size n. If $\varphi(n)$is monotonically decreasing, one would expect to see individual

Table 4.2. The Expected Value of Cotton Production Less Spray Cost Per Acre with Zero Scouting Cost, and Low Z.

Scouting Prior:	Low			Medium			High		
Strategies Q:	150	200	250	150	200	250	150	200	250
(Scout, Scout)	567.29	584.81	593.92	565.36	583.33	591.96	563.97	582.45	591.03
(Scout, Don't Scout)	567.29	584.81	593.92	565.36	583.33	591.96	563.97	582.45	591.03
(Don't Scout, Scout)	567.24	584.12	593.82	565.31	583.12	591.82	563.94	582.40	590.97
(Don't Scout, Don't Scout)	567.24	584.12	593.82	565.31	583.12	591.82	563.94	582.40	590.97
(Scout, Scout) minus (Don't Scout, Don't Scout)	.05	.69	.10	.05	.21	.14	.03	.05	.06

Notes:

Q =First Period Square Load
Price of Lint=\$.70/pound
Cost of Spray=\$8.00/acre
Effectiveness of Spray=.9
$\sigma = .9$

Table 4.3 The Expected Value of Cotton Production Less Spray Cost Per Acre (dollars/acre) with Zero Scouting Cost, and High Z.

Scouting Prior:	Low			Medium			High		
Strategies Q:	150	200	250	150	200	250	150	200	250
(Scout, Scout)	561.18	580.71	591.97	560.24	591.32	589.90	559.66	579.01	588.79
(Scout, Don't Scout)	561.15	580.71	591.97	560.23	591.32	589.90	559.66	579.01	588.79
(Don't Scout, Scout)	561.16	579.79	591.38	560.23	591.32	589.25	559.66	578.97	588.57
(Don't Scout, Don't Scout)	561.15	579.79	591.38	560.23	591.32	589.25	559.66	578.97	588.57
(Scout, Scout) minus (Don't Scout, Don't Scout)	.03	.92	.59	.01	.00	.65	.00	.40	.22

Notes:

Q =First Period Square Load
Price of Lint=\$.70/pound
Cost of Spray=\$8.00/acre
Effectiveness of Spray=.9

Effectiveness of Spray=.9

Table 4.4 The Value of Always Scouting vs. The Value of Never Scouting (cents/acre) for Varying Levels in the Accuracy of Scouting with Low Z.

Accuracy of	Prior:	Low			Medium			High		
Information,σ	Q:	150	200	250	150	200	250	150	200	250
.33		0	0	0	0	0	0	0	0	0
.35		0	0	0	0	0	0	0	0	0
.50		0	18	0	1	1	0	0	0	0
.70		2	35	0	2	10	1	1	2	0
.80		3	52	3	5	15	4	1	3	2
.90		5	70	11	5	20	14	2	4	5
1.00		7	91	27	9	32	25	5	9	9

Notes:
Q =First Period Square Load
Price of Lint=\$.70/pound
Cost of Spray=\$8.00/acre
Effectiveness of Spray=.9

Table 4.5 The Value of Always Scouting vs. The Value of Never Scouting (cents/acre) for Varying Levels in the Accuracy of Scouting with High Z.

Accuracy of Information, σ	Prior:	Low			Medium			High		
	Q:	150	200	250	150	200	250	150	200	250
.33		0	0	0	0	0	0	0	0	0
.35		0	0	0	0	0	0	0	0	0
.50		0	15	6	0	4	7	0	1	2
.70		0	48	31	0	12	35	0	2	12
.80		0	64	45	0	16	49	0	3	17
.90		0	80	58	0	21	64	0	3	22
1.00		3	105	76	1	35	81	0	8	32

Notes: Q =First Period Square Load
Price of Lint=$.70/pound
Cost of Spray=$8.00/acre
Effectiveness of Spray=.9

foraging, but a game-theoretical analysis indicates that mutual "horning in" on food discoveries might be a competitive equilibrium strategy (see Clark and Mangel [1984]). Suppose, however, that $\varphi(n)$ is peaked, as in Figure 3.

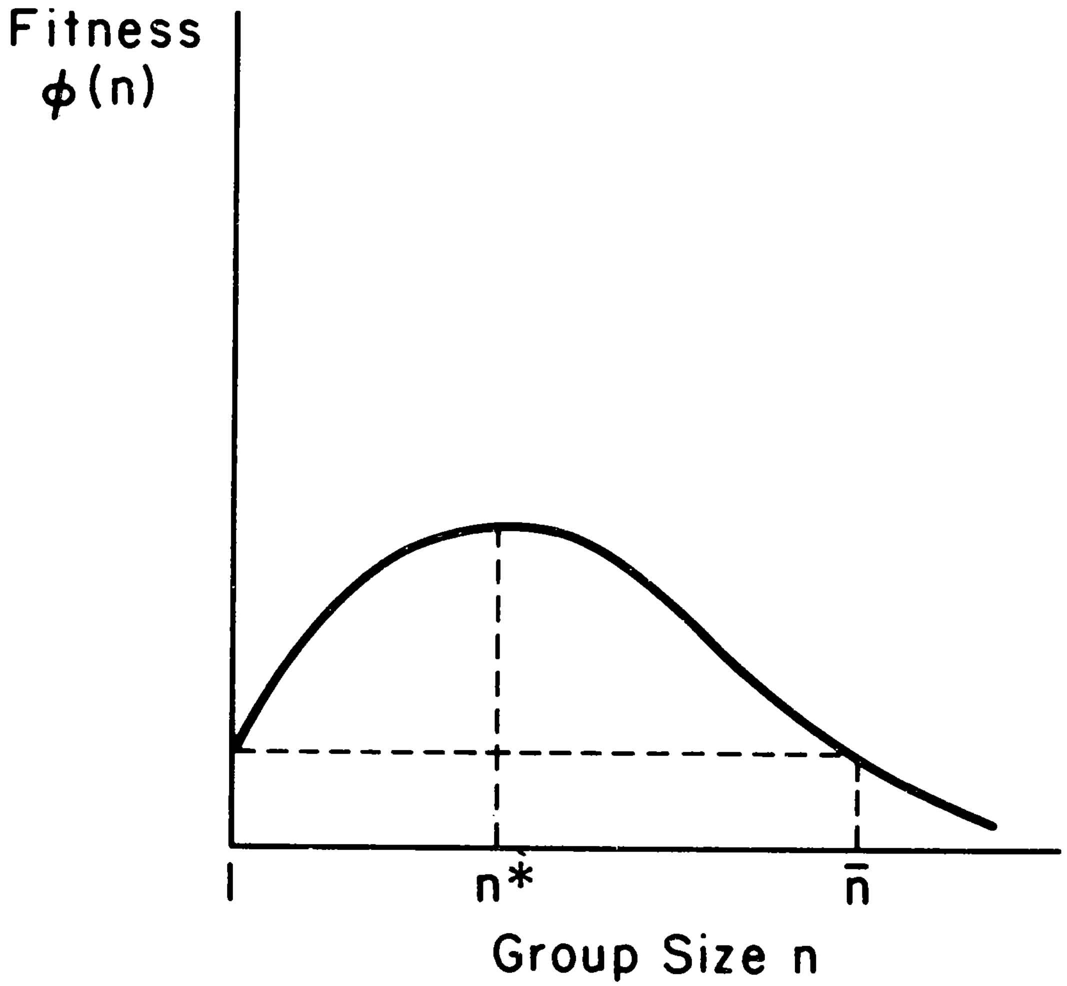

Figure 3. Peaked fitness curve $\varphi(n)$, with optimal group size n^* and equilibrium group size $\bar{n}$.

Then there exists an optimal group size $n = n^*$ at which individual fitness is maximized. But, would groups of size n^* be observed? If such a group did exist, it would attract animals from smaller groups. An individual, for example, would improve his lot by joining any group of size less than $\bar{n}$ where $\varphi(\bar{n}) = \varphi(1)$ (see Figure 3). Optimal sized groups thus appear to be unstable; observed groups should be larger, possibly much larger, than the optimum.

Because individual incentives usually conflict with group objectives, group members will often behave in apparently nonoptimal ways

(Schelling [1978]; Mangel and Clark [1986]). In the case of oil exploration, for example, the results of exploration in one area can drastically affect the value of nearby tracts. If these external benefits cannot be captured by individual drillers, total exploration effort may be severely suboptimal (Peterson [1975]). In fisheries, the level of search effort used may be either suboptimal or superoptimal, as noted in Section 4.1 above. The divergence in incentives has far-reaching implications in many areas, and much research on the topic remains to be done.

The next question is how would a peaked "curve" arise? If forage is patchy, would foragers improve their fitness by searching in groups? This question has a number of facets, and we will only discuss a few of them here (see Clark and Mangel [1986] for further details).

Suppose first that n foragers search independently and share equally each patch of food discovered. If λ is the individual search rate and τ the average time for an individual to consume a patch of average size B, then the average individual feeding rate for a group of size n is

$$\varphi(n) = \frac{B/n}{\tau/n + 1/n\lambda} = \frac{B}{\tau + 1/\lambda}, \tag{4.30}$$

—i.e., is independent of n. Groups of size n find and consume patches at n times the individual rate, so that the net individual feeding rate is unchanged.

In practice, for n much larger than 1, search would not be independent; the group search rate would be $n\lambda(n)$ with $n\lambda(n)$ increasing but $\lambda(n)$ decreasing. This gives

$$\varphi(n) = \frac{B}{\tau + 1/\lambda(n)}, \tag{4.31}$$

which is a decreasing function of group size n. (Also probably τ should be replaced by an increasing function $\tau(n)$.)

But now suppose that patches are ephemeral. Assume independent Poisson search and deterministic patch depletion given by

$$\begin{aligned} \frac{dX}{dt} &= -\alpha n X, \qquad t \geq 0 \\ X(0) &= B. \end{aligned} \tag{4.32}$$

If the group leaves the patch and resumes search after time T we obtain for the average feeding rate

$$\begin{aligned} \varphi(n,T) &= \frac{B(1 - e^{-\alpha n T})}{nT + 1/\lambda} \\ &= \psi(nT). \end{aligned} \tag{4.33}$$

Assume T is chosen to maximize this expression: If T is unrestricted we have

$$(4.34) \qquad T^* = T^*(n) = T_1^*/n,$$

where T_1^* maximizes the function $\psi(T)$. Hence, $\phi = \phi(n, T^*(n))$ is again constant in n.

For ephemeral patches, $T \overset{\text{or}}{\leq} T_{\max}$ = life of patch, so we obtain

$$(4.35) \qquad \phi(n) = \begin{cases} \phi(nT_{\max}), & \text{if } nT_{\max} < T^* \\ \phi(T^*), & \text{if } nT_{\max} \geq T^* , \end{cases}$$

and this is an increasing function of group size n, for $n \leq n_{max} = T^*/T_{\max}$. For example, if $\alpha = .001$/day, $\lambda = .01$/day, $T_{\max} = 1$ day, then $n_{\max} = 416$ and groups of size $n = n_{\max}$ increase their average food intake over individual searchers by a factor of 66.6.

Including search interference in the above model gives decreasing for $n > n_{\max}$, and thus is peaked, with peak $n^* \leq n_{\max}$. Our analysis therefore predicts that group foraging will be beneficial when forage is both patchy and ephemeral.

Variance of the Feeding Rate. So far we have equated fitness from foraging with the average feeding rate. But variance is also obviously important. As a simple model, let R denote the individual's minimum food requirement for a certain decision period, say, one day. Let $f(x, y)$ denote the probability density for food intake x, depending on a parameter y subject to the forager's control over some set Y. The probability of starvation (i.e., not meeting the minimum food requirement) for the period is then

$$(4.36) \qquad \Pr(x < R) = \int_{-\infty}^{R} f(x, y)dx = F(R, y),$$

where F is the cumulative distribution of x. The forager then attempts to minimize $F(R, y)$ over $y \in Y$.

As a very simple example, consider independent Poisson search by a group of n foragers, and ignore handling time τ. An individual forager has the choice of foraging alone or joining the group. If the individual forages in a group of size n, the search rate is $n\lambda t$, but the individual's share is $1/n$ of whatever is found. Thus, for Poisson search, the mean of the forage found in $(0, t)$ is λt and the variance is $\lambda t/n$. In this case, the function $f(x; n)$ is

$$(4.37) \qquad f(x; n) = \frac{(n\lambda t)^{nx} e^{-n\lambda t}}{(nx)!} \qquad nx = 0, 1, 2, \ldots .$$

The survival probability $1 - F(R; n)$ is shown in Figure 4 as a function of R for $n = 1, 2,$ and 5.

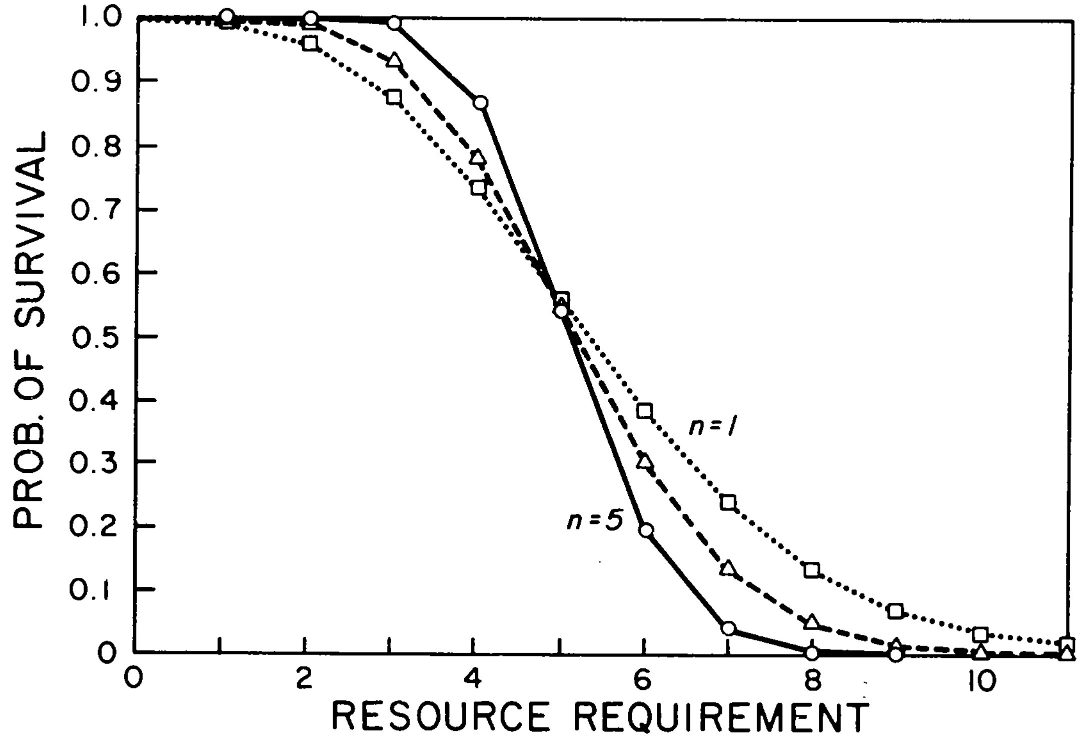

Figure 4. Survival probabilities for $n = 1, 2$, and 5, as functions of resource requirement, R; expected food discovery $\lambda t = 5$ units.

Note that for $R < \bar{x}$ we have fitness $1-F(R; n)$ as an increasing function of n, whereas for $R > \bar{x}$ fitness decreases with n. The forager should be risk averse, minimizing the variance in food intake, in the event that expected intake exceeds minimum requirement R; conversely, when R exceeds the expected intake, the forager should be risk prone (also see Carao [1981]).

Other Models. The relationship between the time profile of food intake and probability of survival is doubtlessly more complex than indicated by either of the above models. For example, the forager may be able to alter strategy over time, depending on its current food "deficit." If there are N decision periods, with starvation occurring if and only if total food intake is less than R, we can determine a feedback foraging strategy via dynamic programming (Clark and Mangel [1985]).

Foragers may face a higher risk of predation when searching for food than when under cover. Only when sufficiently hungry, is it worthwhile to start searching for food. The following jump-process model may be useful for analyzing such a situation:

$$dX = -\alpha dt + d\pi, \tag{4.38}$$

$$d\pi = \begin{cases} Y > 0, & \text{with probability } \lambda dt \\ 0, & \text{with probability } 1 - \lambda dt. \end{cases} \tag{4.39}$$

Here X represents the energy reserves of the forager, $\alpha > 0$ is the depletion rate of reserves, λ is a search-rate parameter, and Y the amount by which one item of food increases the reserves (Y could also be random). Starvation occurs if $X(t) < X_c$. The forager is able to choose Y and λ from some set of pairs S; in particular $\lambda = 0$ is an allowable choice —"hiding." We also assume

$$\Pr(\text{death by predation in time } dt) = \begin{cases} b dt, & \text{if } \lambda > 0 \\ 0, & \text{if } \lambda = 0. \end{cases} \tag{4.40}$$

The problem is to find the (Y, λ) policy in feedback form (Y, λ depending on X) so as to maximize overall survival probability.

For simplicity, assume Y is deterministic. Also set $X_c = 0$ without loss of generality. Define $p(x, s)$ as the maximum probability (i.e., using an optimal strategy) of survival to time $s > 0$, given $X(0) = x$. Then clearly

$$p(x, s) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \geq \alpha s, \end{cases}$$

and for $0 \leq x < \alpha s$ we can calculate $p(x, s)$ as follows. First, for $\alpha s - Y \leq x < \alpha s$ the forager survives if and only if it locates one food item before time s, without being killed by a predator. Hence,

$$\begin{aligned} p(x, s) = \int_0^s & \Pr\{\text{surviving to } t \text{ and not finding forage}\} \\ & \cdot \Pr\{\text{finding forage in } (t, t + dt)\} \\ & \cdot \Pr\{\text{surviving from } t + dt \text{ to } s \mid \text{one item of} \\ & \qquad \text{forage found}\} dt \\ & = \int_0^s e^{-bt} e^{-\lambda t} \lambda dt = \frac{\lambda}{\lambda + b}(1 - e^{-(b+\lambda)s}) \\ & \qquad \text{for } \alpha s - Y \leq x < \alpha s, \end{aligned} \tag{4.41}$$

since the third probability in the first integral equals 1 in this model. An inductive argument shows in general that

(4.42)

$$p(x,s) = \left(\frac{\lambda}{b+\lambda}\right)^n \left[1 - e^{-(b+\lambda)s}\left(1 + (b+\lambda)s + \dots + \frac{(b+\lambda)^{n-1}s^{n-1}}{(n-1)!}\right)\right] \text{ for } \alpha s - nY \le x < \alpha s - (n-1)Y.$$

For fixed time s, the survival probability decreases rapidly as a function of the "deficit" n. On the other hand, for fixed deficit n, the probability of survival increases with the length of time remaining, s, approaching the limit $\left(\lambda/(b+\lambda)\right)^n$ as $s \to \infty$.

The optimal foraging policy for this simple model is also very simple: forage whenever the deficit is positive. Clearly, a more realistic model would include infinite forager capacity and other factors; see Mangel and Clark [1986].

4.4 *Oil and mineral exploration.* Any form of exploration fundamentally involves a search process. Some applications of search theory to problems of oil and mineral exploration are found in the papers of Cozzolino [1972], Cozzolino and Falconer [1977], Gilbert [1976], Harris and Skinner [1982], Mangel [1983, 1985d], Menard and Sharman [1975], Rangarajan and Mehta [1980], Smiley [1979], Shusterich [1982], Uhler [1976, 1979], and Uhler and Bradley [1976]. In particular, the papers of Cozzolino and Falconer, Harris and Skinner, Menard and Sharman, and Rangarajan and Mehta apply versions of the random search formula (3.16) to problems of oil and mineral exploration and production. Uhler and Bradley show how the negative binomial distribution (3.5) and (3.6) can be used to provide a good model for the distribution of oil resources.

The random search formula (3.16) is memoryless since it is a simple exponential distribution. A number of authors have sought extentions of this formula to include depletion.

For example, Cozzolino [1972] uses two Poisson distributions to model search with depletion. Let N denote the initial number of objects in a certain search area A. The value of N is unknown, but a Poisson prior is assumed:

$$\Pr(N = n) = \frac{\lambda^n}{n!}e^{-\lambda}, \quad n = 0, 1, 2, \dots, \tag{4.43}$$

where $\lambda = E\{N\}$ is a known constant. Objects are located independently by Poisson search:

$$\Pr\ (\text{locate a given object in } dt) = \gamma dt. \tag{4.44}$$

Once located, an object is tagged or otherwise removed from the set of undiscovered objects.

The probability of locating n objects in time t (from $t = 0$), given N, is thus binomial:

$$\Pr(\text{locate } n \text{ objects in time } t|N)$$

$$= \binom{N}{n}\left(1 - e^{-\gamma t}\right)^{n} e^{-\gamma t(N-n)}, 0 \leq n \leq N.$$

A straightforward calculation then shows that the prior probability of locating n objects in time t is

$$\Pr(\text{locate } n \text{ objects in time } t)$$

$$= \frac{[\lambda(1 - e^{-\gamma t})]^{n}}{n!} \exp(-\lambda e^{-\gamma t}), \tag{4.45}$$

i.e., another Poisson distribution, with parameter $\lambda(1-e^{-\gamma t})$; as $t \to \infty$ this converges to the original distribution for N.

Finally, applying Bayes' rule, we obtain the posterior distribution for N:

$$\Pr(N = k|n \text{ items located in time } t)$$

$$= \frac{(\lambda e^{-\gamma t})^{k-n}}{(k-n)!} \exp(-\lambda e^{-\gamma t}), \quad k = n, n+1, \cdots$$

which can also be expressed as

$$\Pr(q \text{ items remain}|n \text{ items located in time } t)$$

$$= \frac{(\lambda e^{-\gamma t})^{q}}{q!} \exp(\lambda e^{-\gamma t}), \quad q = 0, 1, 2, \cdots \tag{4.46}$$

Thus the posterior distribution for the number of objects remaining to be discovered is a Poisson distribution with parameter $\lambda e^{-\gamma t}$, which also equals the expected number of items remaining. Note that this expectation decreases at the deterministic rate γ, and is completely independent of the actual number of objects located in time t. This is the sense in which the Poisson-Poisson search model is "completely memoryless." (Cozzolino's model also allows for variation in the size of the objects, but the memoryless property persists under this extension).

The memoryless property was also observed by Iwasa et al. [1981] in a foraging model. Specifically, assume Poisson search with parameter γ, no replacement, and let p_k be the prior probability that $N = k$. Let $r(n,t)$ denote the expected number of undiscovered objects after n objects have been located during an initial search period t. We then have:

Example 1. (Poisson prior)

$$p_k = \frac{\lambda^k}{k!}e^{-\lambda} \tag{4.47}$$

(mean = variance = λ).

$$r(n,t) = \lambda e^{-\gamma t}$$

Example 2. (Negative binomial prior)

$$p_k = \binom{\sigma + k - 1}{k}\left(\frac{1}{1+\alpha}\right)^{\sigma}\left(\frac{\alpha}{1+\alpha}\right)^{k} \tag{4.48}$$

(mean = $\sigma\alpha$, variance = $\sigma\alpha(1+\alpha)$)

$$r(n,t) = \frac{\sigma + n}{e^{\gamma t}\left(\frac{\alpha+1}{\alpha}\right) - 1}$$

Example 3. (Binomial prior)

$$p_k = \binom{M}{k} q^k (1-q)^{M-k} \tag{4.49}$$

(mean $= Mq$, variance $= Mq(1-q)$)

$$r(n,t) = \frac{M - n}{e^{\gamma t}\left(\frac{1-q}{q}\right) + 1}$$

A fourth example, the gamma prior, permits an arbitrary relationship between mean and variance, but does not appear to give a simple formula for $r(n,t)$ —see Mangel and Clark [1983], Equation (33). Mangel and Beder [1985] discuss extensions to non-Poisson search.

The number n of objects located in time t does not affect $r(n,t)$ for the Poisson case, but increases (resp. decreases) $r(n,t)$ for the negative binomial (resp. binomial) cases.

Next, let us discuss a stopping rule for exploration with these models. If r denotes the expected number of undiscovered objects at a given time, then the expected number discovered in the next dt is $r\gamma dt$. If

p denotes the value of one object and c the cost of search per unit time, then expected short-term benefits will exceed costs if and only if $pr\gamma > c$. In the Poisson case, no information is generated by searching, so the stopping rule is simply $pr\gamma = c$, or by (4.47)

$$t_s = \frac{1}{\gamma}\ell n\frac{p\gamma\lambda}{c}. \tag{4.50}$$

Although no consideration of alternative search areas was taken in deriving the stopping rule (4.50), this possibility can easily be included, simply by letting search cost c include the opportunity cost of search elsewhere. The larger this opportunity cost, the shorter the stopping time t_s, as one would expect.

The stopping rule becomes more complex if a non-Poisson prior is assumed, for in this case search provides both discoveries and information. The case of a gamma prior is discussed by Mangel and Clark [1983], but other priors seem not to have been analyzed.

Another stopping problem, and one that is especially well suited for Bayesian analysis, concerns the assessment of a region thought to contain an oil or mineral deposit. We will assume that this region is sampled by an exploratory search process (such as exploratory drilling) and let X_i denote the discovery from the i exploratory event. We assume that X_i follows a negative binomial distribution, so that

$$\Pr\{X_i = x\} = \frac{\Gamma(x+k)}{\Gamma(k)n!}\left(1+\frac{m}{k}\right)^{-k}\left(\frac{m}{m+k}\right)^{x}. \tag{4.51}$$

Here m is the mean of the distribution and k a parameter that represents the level of aggregation. As k decreases, the level of aggregation increases. The effect of this aggregation is easily shown by considering the likelihood that $X_i = 0$:

$$\Pr\{X_i = 0\} = \left(\frac{k}{k+m}\right)^{k}. \tag{4.52}$$

Figure 5 shows $\Pr\{X_i = 0\}$ as a function of k and m. When k is small, there can be a considerable chance of a zero observation, even if m is quite large.

The problem that we wish to consider here is one in which a decision must be made about the value of m. That is, one drills exploratory wells, measures X_i, and tries to determine whether m is greater than a critical value m_c or not (various extensions of this problem are considered in Mangel [1985c]). In particular, how many dry wells are needed before one can say that $m \leq m_c$ with sufficient confidence to conclude that the region is devoid of oil and minerals?

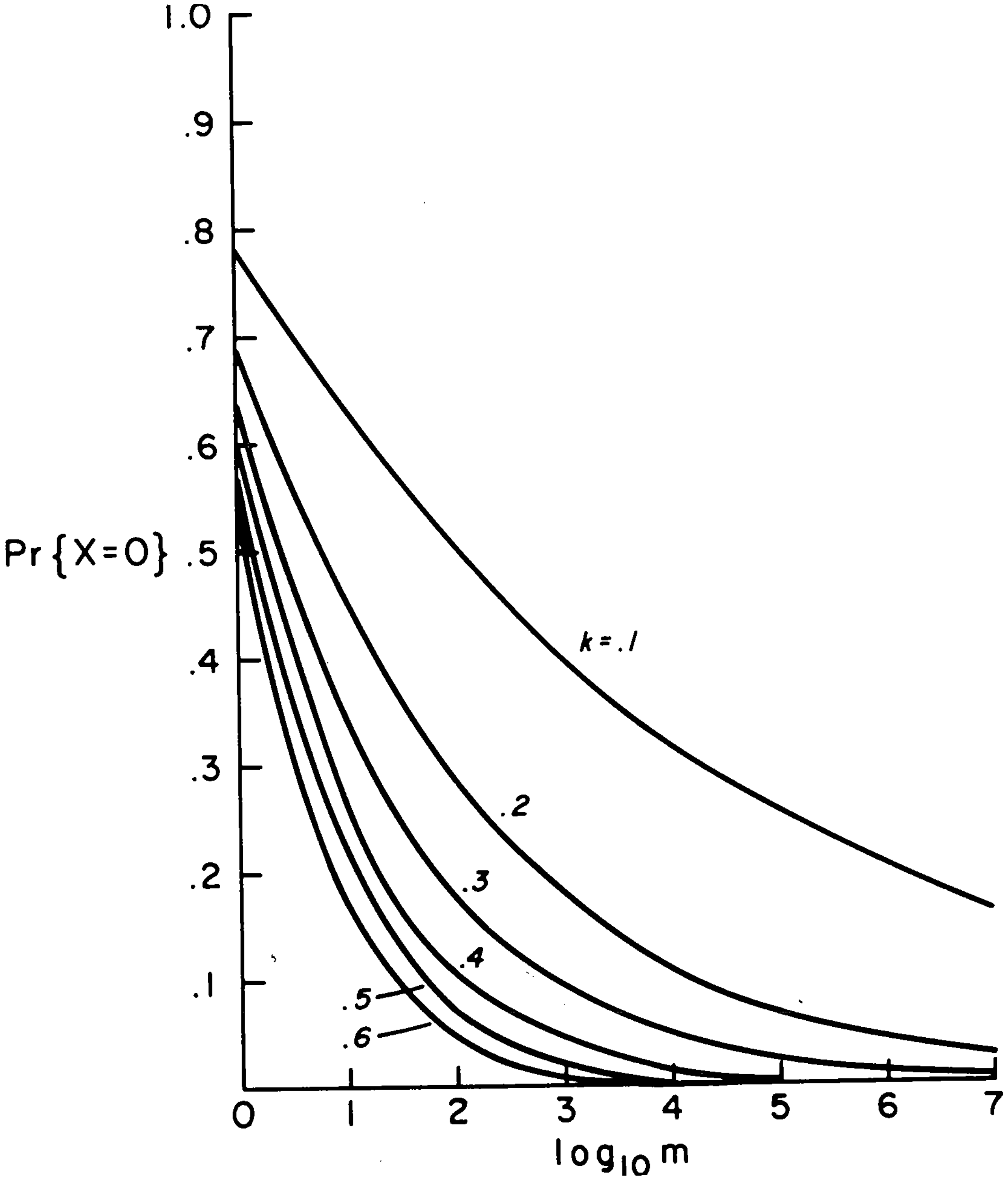

Figure 5. The likelihood of zero observation drawn from a NB distribution with parameters m and k.

We will adopt a Bayesian approach to this problem. (Bayesian approaches are especially well suited to problems of "negative information" — i.e., in which negative results are obtained upon search). Let

$\mathcal{L}$

$f_0(m)$ be the prior probability density for the value of the mean in the region of interest. The likelihood $P_n(m)$, that n independent samples are negative when the mean takes the value m is

$$P_n(m) \propto \left(\frac{k}{k+m}\right)^{nk}. \tag{4.53}$$

The posterior density for the mean given n zeros, $f(m \mid n)$ is then given by a normalized version of the product of $f_0(m)$ and $P_n(m)$:

$$f(m|n) = \frac{f_0(m)(k+m)^{-nk}}{\int f_0(m)(k+m)^{-nk} dm}. \tag{4.54}$$

The condition that $\Pr\{m < m_c\} \geq \alpha$, where α is a given confidence level, is then equivalent to

$$\int_0^{m_c} f_0(m)(k+m)^{-nk} dm \geq \alpha \int_0^{\infty} f_0(m)(k+m)^{-nk} dm. \tag{4.55}$$

One a prior density $f_0(m)$ is chosen, one can easily solve (4.55). One choice for $f_0(m)$ is the uniform prior

$$f_0(m) = 1 \qquad m \geq 0. \tag{4.56}$$

This prior is improper (it does not have a finite integral) but as long as $nk > 1$, the postrior density does integate. Using (4.56) in (4.55) and solving gives

$$n \geq \frac{1}{k}\left\{1 - \frac{\log(1-\alpha)}{\log\left(\frac{k+m_c}{k}\right)}\right\}. \tag{4.57}$$

A second choice for $f_0(m)$ is a non-informative prior (Box and Tiao [1972], Martz and Waller [1982]) which is one in which data change only the location but not the shape of the likelihood curve $f_0(m)P_n(m)$. In Appendix 2, we show that the noninformative prior for sampling the negative binomial distribution (4.51) is given by

$$f_0(m) \propto m^{-1/2}(k+m)^{-1/2}. \tag{4.58}$$

This prior is also not integrable, and, relative to the uniform prior, gives higher weights to small values of m. Using (4.58) in (4.55) gives

$$\int_0^{m_c} m^{-1/2}(k+m)^{-nk-1/2}dm \geq \alpha \int_0^{\infty} m^{-1/2}(k+m)^{-nk-1/2}dm. \tag{4.59}$$

The integral can be transformed to one over a finite domain by setting

$$\begin{aligned} m &= k\tan^2\theta \\ dm &= 2k\frac{\tan\theta}{\cos^2\theta}d\theta. \end{aligned} \tag{4.60}$$

The condition in (4.59) then becomes

$$\int_0^{\theta_c} (\cos\theta)^{2nk-1}d\theta \geq \alpha \int_0^{\pi/2} (\cos\theta)^{2nk-1}d\theta, \tag{4.61}$$

where $\theta_c = \arctan(\sqrt{m_c}/k)$.

Table 4.6 shows results obtained using (4.57) and (4.61). The uniform prior is "more cautious"—i.e., requires more negative samples — than the noninformative prior. The cause of this is clear: the noninformative prior leads to a posterior that is more heavily weighted towards small values of m.

In closing this section, it is worth noting that often one will have more information about the possible value of m — say, from independent geological or seismic sources. In such a case, one may wish to use an informative prior for $f_o(m)$. A good choice is then

$$f_0(m) \partial \mathrm{c}\,\mathrm{m}^{-\alpha}(k+m)^{-\beta}, \tag{4.62}$$

where α and β are constants adjusted to fit the available information. Use of priors similar to (4.62) is described in Mangel [1985c], as are extensions of this model.

Table 4.6 Deposit Size Estimation Based on Dry Wells

k	m_c	α	Number of Dry Wells Needed for $\Pr\{m \le m_c\} \ge \alpha$.	
			Uniform Prior	Noninformative Prior
.5	.25	.95	17	10
.5	.50	.95	11	7
.5	1.0	.95	7	4
.5	1.5	.95	6	4
.5	.5	.95	11	7
.5	.5	.97	12	8
.5	.5	.98	13	9
.5	.5	.99	15	11
.5	.5	.995	17	13
.2	.5	.95	17	9
.2	.5	.97	19	11
.2	.5	.99	23	14
.2	.5	.995	26	17

5. Discussion and conclusion. We have shown that search theory, as a mathematical topic, involves a diverse collection of mathematical ideas and techniques which includes differential equations, probability theory, and optimization theory. Search theory, in fact, provides a natural blending of those three areas of mathematics. As far as the applications are concerned, we discussed four fields of application: fisheries, pest control, foraging theory, and oil or mineral exploration. In each of these, the application of search theory leads to a new understanding of some aspect of the problem and also to new mathematics.

The interest and effort in natural resource modeling over the last fifteen years was concentrated in questions of exploitation. Search theory (and clearly one must find a resource before exploiting it) is essentially a question of exploration. There remains much to be done in coupling the theories of exploration and exploitation.

APPENDIX 1:
DAMAGE FUNCTION FOR THE COTTON-LYGUS SYSTEM

In this appendix, we describe the model for the cotton-lygus damage function (further details are given by Mangel, Stefanou and Wilen

[1985]). The cotton plant development is characterized by buds, called "squares" that develop into bolls, from which the ultimate yield is obtained. A discrete-time model is adopted with the following variables.

$$\begin{aligned} S(t) &= \text{number of medium and large squares per .001} \\ &\quad\ \text{acres at the start of week } t \\ L(t) &= \text{number of lygus adults plus nymphs per 4 sets} \\ &\quad\ \text{of 25 sweeps in week } t \\ B(t) &= \text{number of large bolls in week } t. \end{aligned} \tag{A.1.1}$$

(Actually, chronological time is not the truly appropriate variable; rather the degree-day is the appropriate measure with a conversion from chronological time to degree-days.) The lygus-cotton interaction is modeled by assuming the following dynamics for $S(t)$:

$$\begin{aligned} S(t) = \beta_1(t-1)S(t-1) &+ \beta_2(t-1)S(t-1)^2 \\ + \beta_3(t-1)L(t-1) &+ \beta_4(t-1)L(t-1)^2 \\ + \tilde{\nu}_S(t) \qquad & \text{for } t = 3, \ldots, 8, \end{aligned} \tag{A.1.2}$$

where the $\beta_i(t)$ are coefficients (determined by the data) and $\tilde{\nu}_S(t)$ is a noise term. The initial condition for (A.1.2) is $S(2) = Q$. The model (A.1.2) is one in which squares grow logistically in the absence of lygus, which exhibits a logistic "harvesting" effect.

In the data analyzed by Mangel, Stefanou, and Wilen [1985], no large bolls appeared before the fifth week. Assuming a two-week lag between squares and bolls, the following dynamics are used:

$$\begin{aligned} B(t) = \gamma_1(t-1)B(t-1) &+ \gamma_2(t-1)B(t-1)^2 \\ + \gamma_3(t-1)S(t-1) &+ \gamma_4(t-1)S(t-1)^2 \\ + \tilde{\nu}_B(t), \qquad & \text{for } t = 6, 7, \ldots, 12, \end{aligned} \tag{A.1.3}$$

with initial condition

$$B(5) = .013S(3) + .0002S(3)^2. \tag{A.1.4}$$

In (A.1.3), the $\gamma_i(t)$ are coefficients to be estimated and $\tilde{\nu}_B(t)$ is a noise term.

Mangel, Stefanou, and Wilen [1985] estimated the coefficients in (A.1.2-4) using ordinary least squares regression. When this is done, one finds that β_4 is not significantly different from zero at all and that β_3 is significantly different from zero at the 5% level only in weeks 4 and 8, thus leading to the two period problem discussed in the text. Figure 6 shows how well the estimated model followed the data used to generate it.

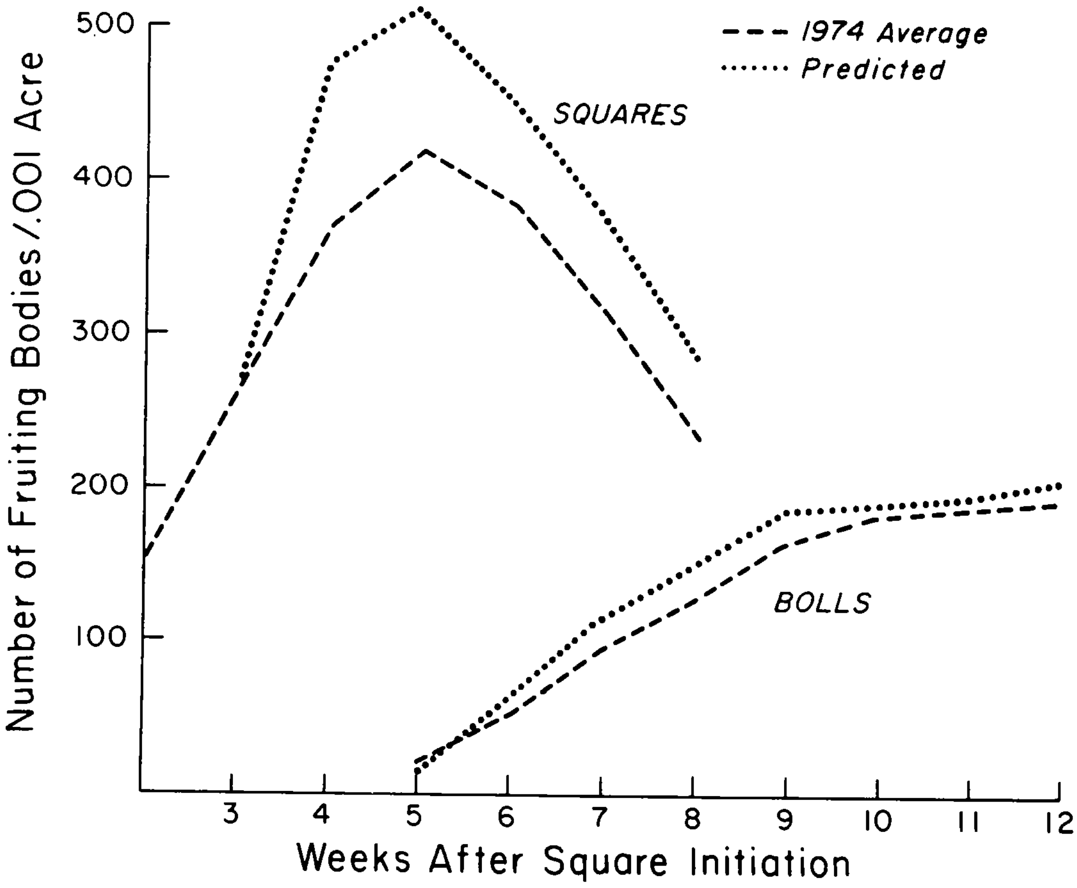

Figure 6. Comparison of the solution of equations (6) and the averages of the observed squares and bolls.

The yield of cotton (in pounds) is based on the assumption that 125,000 bolls yield 480 pounds. Thus, by solving (A.1.2-4) with $L(t) \equiv 0$, one obtains the value of Y_0 in (4.21) — the yield in the absence of lygus. Given values of $L(4)$ and $L(8)$, one can then compute the damage done by lygus in reducing physical yield.

APPENDIX 2:

NON INFORMATIVE PRIOR FOR THE MEAN OF A NEGATIVE BINOMIAL DISTRIBUTION

The approximate noninformative prior for the NB distribution is derived as described by Martz and Waller [1982], 224. Viewing (4.51) as the likelihood of m given x, the log-likelihood is

$$L(m|x) = -k\log(k+m) + x[\log m - \log(m+k)] + \ell(x,k), \tag{A.2.1}$$

where $\ell(x,k)$ contains terms independent of m. The derivatives of the log-likelihood are

$$\begin{aligned} \frac{\partial L}{\partial m} &= \frac{-k}{k+m} + \frac{x}{m} - \frac{x}{m+k} \\ \frac{\partial^2 L}{\partial m^2} &= \frac{k}{(k+m)^2} - \frac{x}{m^2} - \frac{x}{(m+k)^2}. \end{aligned} \tag{A.2.2}$$

Setting $\partial L/\partial m = 0$ shows that the maximum likelihood estimate is $\hat{m} = x$. Define

$$\begin{aligned} J(\hat{m}) &= \left.\frac{-\partial^2 L}{\partial m^2}\right|_{\hat{m}} \\ &= \frac{\hat{m}}{\hat{m}^2} - \frac{\hat{m}+k}{(k+\hat{m})^2} = \frac{k}{\hat{m}(k+\hat{m})}. \end{aligned} \tag{A.2.3}$$

The approximate non-informative prior is then

$$f_0(m) \propto J(m)^{1/2} = m^{-1/2}(k+m)^{-1/2}. \tag{A.2.4}$$

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