UNIVERSITY OF CALIFORNIA

SANTA CRUZ

AN ANALYSIS OF THE DISCRETE AND CONTINUOUS GAMBLER'S RUIN PROBLEM

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by

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"An Analysis of the Discrete and Continuous Gambler's Ruin Problem"

I. Introduction:

For models of a broad spectrum of phenomena, it can be useful to incorporate a random element into an otherwise deterministic model. A population modeler might expect the population of interest to grow logistically, but also want to add random variation, or stochasticity, to this expectation. In such a case one would add a stochastic perturbation term to the model.

In some cases it is clear whether a modeler should choose a discrete or continuous disturbance term. The evolution of a blackjack player's holdings is a stochastic process that requires a discrete perturbation term. It should be discrete because for any initial bet, the change in holdings can only be either a net loss or gain of that bet amount, or zero for a draw. The Poisson increment, which I define rigorously in the **Background** section, is a discrete stochastic process that describes precisely this evolution of holdings. Now, if we change our point of view to that of a casino with many of such players, we may wish to change how we model disturbances. Although the casino's holdings will change discretely at each table, as the number of players increases, the casino's holdings will evolve more smoothly. Once we consider a sufficient number of players, the casino's holdings will appear to move continuously. Should we be in such a situation one might prefer to model the perturbations with Brownian motion, which may be thought of as the continuous-equivalent to the Poisson increment.

When modelers are presented with a choice between a continuous and discrete disturbance term, they should understand the implications of their choice. Most fundamentally, one should have a good understanding of how much the choice matters. In other words, are the results from a continuous model different from those from a discrete analogue. In some cases, should the difference prove to be insignificant, one may even choose the process with the most desirable mathematical properties. Thus, I am interested in the differences in results produced by discrete and continuous models. Since the Poisson increment and Brownian motion are basic discrete and continuous processes, respectively, I will limit my scope to them. Moreover, as a simple context will make for the clearest observation of their differences, I will focus on the gambler's ruin problem. The gambler's ruin problem is best defined as the following question: given an initial holdings value, what is the probability that a gambler bankrupts the casino before going broke?

My exploration of basic models of the gambler's ruin problem begins with background on the Poisson increment and Brownian motion. I follow the **Backround** section with **Analytical Results** sections, where I attempt to find analytical solutions to the gambler's ruin problem for both processes. I supplement the analytical results with their numerical analogues in the **Numerical Results** section, and finish with the **Conclusion**.

II. **Background**

The background section is included to familiarize the reader with the definitions and basic properties of Brownian motion and the Poisson increment.

Brownian motion, denoted $W(t)$ (for the mathematician Norbet Wiener), is defined by the following four axioms:

(i) $W(0) = 0$;

(ii) $W(t)$ is continuous;

(iii) The increments of $W(t)$, $dW := W(t + dt) - W(t)$, are independent;

(iv) $W(t)$ is normally distributed with mean 0 and variance t

(i.e. $W \sim N(0, t)$).

In Figure 1, I show a few sample paths, or realizations, of Brownian motion.

In order to become more familiar with the properties of Brownian motion, we will compute its first two moments. These results will also be of use in the **Analytical Results** section.

$$\begin{aligned} E\{dW\} &= E\{W(t+dt) - W(t)\} && (1) \\ &= E\{W(t+dt)\} - E\{W(t)\} = 0, && (2) \end{aligned}$$

with the last step following from the first axiom.

$$\begin{aligned} E\{dW^2\} &= E\{[W(t+dt) - W(t)]^2\} && (3) \\ E\{dW^2\} &= E\{W(t+dt)^2 - 2W(t+dt)W(t) + W(t)^2\} && (4) \\ E\{dW^2\} &= E\{W(t+dt)^2\} - 2E\{W(t+dt)W(t)\} + E\{W(t)^2\}. && (5) \end{aligned}$$

Since $W(t)$ has mean 0 and variance t, we can compute the first and third expectations in equation (5) to get

$$E\{dW^2\} = t + dt - 2E\{W(t+dt)W(t)\} + t. \qquad (6)$$

The next two steps follow from simultaneously adding and subtracting $W(t)^2$ and then factoring out $W(t)$ in the remaining expectation:

$$\begin{aligned} E\{dW^2\} &= 2t + dt - 2E\{W(t+dt)W(t) - W(t)^2 + W(t)^2\} && (7) \\ E\{dW^2\} &= 2t + dt - 2E\{[W(t+dt) - W(t)][W(t) - 0]\} - && (8) \\ &\quad 2E\{W(t)^2\} \\ E\{dW^2\} &= 2t + dt - 2t = dt. && (9) \end{aligned}$$

I use the fact that the increments of Brownian motion are independent to go from equation (8) to equation (9).

I define the Poisson increment by

$$d\Pi = \begin{cases} \nu & w.p. \quad \frac{\lambda dt}{2} + o(dt) \\ 0 & \quad 1 - \lambda dt + o(dt) \\ -\nu & \quad \frac{\lambda dt}{2} + o(dt)) \end{cases}$$

Note that the probabilities $\frac{\lambda dt}{2} + o(dt)$ and $1 - \lambda dt + o(dt)$ are Taylor expansions of $\frac{1}{2}(1 - e^{-\lambda dt})$ and $e^{-\lambda dt}$, respectively. In Figure 2, I show a few realizations for $d\Pi$ with different values for ν and λ, with their product constant at $\nu^2\lambda = 1$.

Similarly, the first two moments of the Poisson increment are

$$E\{d\Pi\} = \frac{\lambda}{2}\nu + (1-\lambda)0 + \frac{\lambda}{2}(-\nu) = 0. \tag{10}$$

$$E\{d\Pi^2\} = \frac{\lambda}{2}\nu^2 + (1-\lambda)0 + \frac{\lambda}{2}\nu^2 = \lambda\nu^2. \tag{11}$$

One can see from comparing Figure 1 and Figure 2 that as λ grows (and ν shrinks), the Poisson paths become smoother and look more like the Brownian motion paths. By considering how the Poisson increment compares with the defining properties of Brownian motion, my intuition is that the Poisson increment converges to Brownian motion as $\lambda \to \infty$ and $\nu \to 0$ (with $\lambda\nu^2$ constant). First, $\Pi(0) = 0$ and independent increments are easily satisfied by any Poisson increment. As for continuity, I find it intuitive that as the jumps become small and frequent, Π becomes continuous. Lastly, normality should follow from the central limit theorem. Though it remains to be proved, the fact that the Poisson increment appears to converge to Brownian motion will prove useful. I will be able to analyze the differences in results for Poisson increments varying in closeness to Brownian motion.

III. **Analytical Results: Brownian Motion**

The fundamental quantity in the gambler's ruin problem is the probability of breaking the bank before going broke. Accordingly, in this section I focus on analytical solutions to the gambler's ruin problem using the two stochastic processes. If a gambler's holdings at time t are denoted by $X(t)$ and casino limit C, I define

$$u(x) = Pr\{\text{hit C before } 0|X(0) = x\} \tag{12}$$

I model a player's holdings as evolving due to a combination of a constant rate of decline (because, in the long run, the house always wins), m, and a stochastic perturbation. Using Brownian motion for the stochastic perturbation term, holdings evolve according to

$$dX = -mdt + \sigma dW, \tag{13}$$

where σ is a parameter that scales the intensity of the Brownian motion.

For an initial holdings value, the probability of bankrupting the casino before going broke is equal to the probability of the same outcome for the holdings in the next time period. The fact that holdings change randomly does mean that the value for holdings in the next time period is uncertain. However, once the stochastic perturbation term is specified, one can establish a probability distribution for future holdings. I show this idea graphically in Figure 3. This observation enables one to write $u(x)$ in terms of the probability distribution of future holdings:

$$u(x) = E_{dX}[u(x + dX)]. \tag{14}$$

Using equation (13), this becomes

$$u(x) = E_{dW}[u(x - mdt + \sigma dW)]. \tag{15}$$

I Taylor expand the right hand side of equation (15) around x:

$$u(x) = E_{dW}[u(x) + u_x(-mdt + dW) + \frac{1}{2}u_{xx}(-mdt + \sigma dW)^2 + o(dt)], \tag{16}$$

where u_x and u_{xx} denote the first and second order derivatives of u with respect to x, respectively. Taking expectations gives

$$\begin{aligned} u(x) &= u(x) - mdtu_x - \frac{1}{2}u_{xx}E_{dW}(m^2dt^2 - 2m\sigma dWdt + \sigma^2 dW^2) \qquad (17) \\ &\quad +o(dt) \\ u(x) &= u(x) - mdtu_x - \frac{1}{2}u_{xx}m^2dt^2 + \frac{1}{2}u_{xx}\sigma^2 dt + o(dt). \qquad (18) \end{aligned}$$

Subtracting $u(x)$ from both sides leaves

$$0 = -mdtu_x + \frac{1}{2}u_{xx}m^2dt^2 + \frac{1}{2}u_{xx}\sigma^2 dt + o(dt). \tag{19}$$

By dividing by dt and letting it go to 0, we obtain a second order differential equation:

$$0 = -mu_x + \frac{1}{2}u_{xx}\sigma^2., \tag{20}$$

with two boundary conditions:

$$(i) \qquad u(0) = 0 \tag{21}$$

$$(ii) \qquad u(C) = 1 \tag{22}$$

These boundary conditions simply state that a gambler who begins with zero holdings has already gone broke, where a gambler who starts with all of the casino's holdings has already bankrupted the casino. Using the integrating factor method, the general solution is

$$u(x) \quad = \quad \frac{k\sigma^2}{2m}(e^{\frac{2mx}{\sigma^2}} - 1) + u(0) \tag{23}$$

where k is an arbitrary constant. Using the first boundary condition, equation (23) simplifies to

$$u(x) \quad = \quad \frac{k\sigma^2}{2m}(e^{\frac{2mx}{\sigma^2}} - 1) \tag{24}$$

We can solve for k by using the second boundary condition:

$$u(C) = \frac{k\sigma^2}{2m}(e^{\frac{2mC}{\sigma^2}} - 1) = 1, \tag{25}$$

which implies that

$$k = \frac{2m}{\sigma^2}\frac{1}{e^{\frac{2mC}{\sigma^2}} - 1}. \tag{26}$$

So the solution for equation (23) specific to our boundary conditions is

$$u(x) = \frac{e^{\frac{2mx}{\sigma^2}} - 1}{e^{\frac{2mC}{\sigma^2}} - 1}. \tag{27}$$

For [m,σ,C]=[0.01,1,100] the the probability of breaking the bank for an initial holdings of x is $\frac{e^{0.02x}-1}{e^2-1}$, which I show in Figure 4. As expected, a gambler who begins with a higher level of holdings is more likely to bankrupt the casino. However, due to the constant expected rate of decline, m, hitting the casino limit is still unlikely. A player will need to start with a holdings of about \$72 just to have a "fifty-fifty" chance of breaking the bank.

IV. Analytical Results: Poisson Increment

For the solution to the gambler's ruin problem using the $d\Pi$ process, the change in holdings follows

$$dX = -mdt + d\Pi. \tag{28}$$

I begin an attempt at a solution to $u(x)$ given equation (28) by, again, writing $u(x)$ in terms of the probability distribution of dX:

$$u(x) = E_{dX}[u(x + dX)] \tag{29}$$

$$u(x) = E_{d\Pi}[u(x - mdt + d\Pi)] \tag{30}$$

$$u(x) = (1 - \lambda dt)u(x - mdt) + \frac{\lambda dt}{2}[u(x - mdt + \nu) + u(x - mdt - \nu)] \tag{31}$$

I Taylor expand $u(x - mdt)$ around x to get

$$\begin{aligned} u(x) &= (1 - \lambda dt)[u(x) - mdtu_x + o(dt)] \\ &\quad + \frac{\lambda dt}{2}[u(x - mdt + \nu) + u(x - mdt - \nu)]. \end{aligned} \tag{32}$$

Subtracting $u(x)$ from both sides yields

$$\begin{aligned} 0 &= -\lambda dtu(x) + (1 - \lambda dt)[-mdtu_x + o(dt)] \\ &\quad + \frac{\lambda dt}{2}[u(x - mdt + \nu) + u(x - mdt - \nu)]. \end{aligned} \tag{33}$$

Now I Taylor expand $u(x - mdt + \nu)$ and $u(x - mdt - \nu)$ around $x + \nu$ and $x - \nu$, respectively. Using $|_{x+\nu}$ to denote evaluation at $x + \nu$ (and analogously for $x - \nu$), this is

$$\begin{aligned} 0 &= -\lambda dtu(x) + (1 - \lambda dt)[-mdtu_x + o(dt)] \\ &\quad + \frac{\lambda dt}{2}[u(x + \nu) + u_x|_{x+\nu}(-mdt) + \frac{1}{2}u_{xx}|_{x+\nu}(-mdt)^2 + o(dt) \\ &\quad + u(x - \nu) + u_x|_{x-\nu}(-mdt) + \frac{1}{2}u_{xx}|_{x-\nu}(-mdt)^2 + o(dt)]. \end{aligned} \tag{34}$$

Next I divide by dt and let it go to 0 to obtain the following differential difference equation:

$$0 = -\lambda u(x) - mu_x + \frac{\lambda}{2}[u(x + \nu) + u(x - \nu)]. \tag{35}$$

This equation will be difficult, if not impossible to solve for all values of ν. However, if ν is small I can Taylor expand the right hand side of equation (35) around x:

$$\begin{aligned} 0 &= -\lambda u(x) - mu_x + \frac{\lambda}{2}[u(x) + u_x\nu + \frac{1}{2}u_{xx}\nu^2 + o(\nu) \\ &\quad +u(x) - u_x\nu + \frac{1}{2}u_{xx}\nu^2 + o(\nu)] \end{aligned} \quad (36)$$

Combining like terms, equation (36) simplifies to

$$0 = -mu_x + \frac{\lambda}{2}u_{xx}\nu^2 + o(\nu) \quad (37)$$

When ν is small, equation (37) is approximately

$$0 = -mu_x + \frac{\lambda}{2}u_{xx}\nu^2 \quad (38)$$

Because the product $\nu^2\lambda$ is constant, the solution to this equation is the same as that for dW, but with $\nu^2\lambda$ replacing σ^2:

$$u(x) = \frac{e^{\frac{2mx}{\nu^2\lambda}} - 1}{e^{\frac{-2mC}{\nu^2\lambda}} - 1}. \quad (39)$$

This result means that for small ν the Poisson increment will yield good approximations to Brownian Motion. However, I cannot say analytically what happens when ν is not small, of which equation (34) is evidence. Hence, the **Numerical Results** section.

V. **Numerical Results**

In this section I produce the numerical analogue of Figure 4 for both the Poisson and Brownian Motion Processes. Here, I am most interested in the question that I couldn't answer in the analytical section: what happens when ν is large? To answer this question numerically, I ran simulations of both processes, holding $\sigma^2 = \nu^2\lambda = 1$. Both of my simulations use the four indexing variables: simulation number (i), repetition number (j), initial holdings value (k) and time (t). Their ranges are

$$\begin{aligned} i &= 1 \text{ to } 15 \\ j &= 1 \text{ to } 100 \\ k &= 0, 20, 40, 60, 80, 100 \\ t &= 1 \text{ to } 10{,}000 \qquad \text{(in increments of dt=0.1).} \end{aligned}$$

In every simulation, I generated 100 paths for each of the 6 initial holdings values. The time length is relevant only in that it needed to be long enough to ensure that that every path hit either 100 or 0. Using dW as my perturbation term, holdings for simulation i, repitition j, initial holdings k and time t, $X[i, j, k, t]$ are

$$X[i, j, k, t] = \begin{cases} k & \text{if } t = 1 \\ X[i, j, k, t-1] - mdt + N \sim (0, dt) & \text{if } t > 1 \end{cases}$$

To compute approximate probabilities I took the proportion of the repetitions that hit 100 before 0 for each initial holdings and simulation. Figure 5 depicts the stochasticity around the analytical result from my simulation for dW. Here, each data point represents the computed proportion for a given simulation and starting point.

My simulations using $d\Pi$ follow a similar form but I compute holdings, $X[i, j, k, t]$, slightly differently. As before

$$X[i, j, k, 1] = k.$$

For $t > 1$, I draw two numbers from the standard normal distribution:

$$\begin{aligned} u_1 &\in U(0, 1) \\ u_2 &\in U(0, 1). \end{aligned}$$

If

$$u_1 > 1 - e^{-\lambda dt},$$

then $d\Pi = 0$ and there is no jump:

$$X[i, j, k, t] = X[i, j, k, t-1] - mdt$$

The complementary case of

$$u_1 < 1 - e^{-\lambda dt},$$

results in a perturbation of either $+\nu$ or $-\nu$. Since, they have equal probability, I assign a positive jump if $u_2 > .5$ and a negative jump if $u_2 < .5$:

$$X[i,j,k,t] = \begin{cases} X[i,j,k,t-1] - mdt + \nu & \text{if } u_2 > .5 \\ X[i,j,k,t-1] - mdt - \nu & \text{if } u_2 < .5 \end{cases}$$

I approximate probabilities from the simulations using $d\Pi$ in the same way. In Figure 6 I show the results for $\lambda = 0.1, \nu = \sqrt{10}$. For comparison, I have plotted the simulation data points over the analytical results for dW. Although ν is relatively far from the limit value of 0, the probabilities from the Poisson simulation appear to be close to the analytical probabilities derived using Brownian motion. In Figure 7, I show the residualized version of Figure 6 and plot the average deviations from the Brownian motion analytical result as the red data points. While one would certainly be naive to claim from this figure that these samples are drawn from the same probability distribution, the Poisson increment is undeniably yielding a good approximation.

VI. **Conclusion**

In limiting the scope to the gambler's ruin problem, I have shown that the Brownian Motion process and the Poisson increment do not differ significantly. That is, they give a gambler similar probabilities of breaking the bank before going broke. Although I was unable to demonstrate it analytically, my simulations show probabilities using $d\Pi$ that appear very close to those using

dW. I find this result quite interesting, if not somewhat counterintuitive. I can easily imagine a scenario in which as we get further away from the limit, the larger jumps make it more likely for a gambler to break the bank. However, from the results, I gather that the bigger jumps occur sufficiently less frequently to leave the probabilities largely unchanged. Essentially, though a gambler's holdings modeled with a large ν may exhibit more erratic behavior, the structure of the Poisson increment serves to balance out the chances of breaking the bank before going broke. To be prudent, I should note that I have shown dW and $d\Pi$ to be similar only in their results for $u(x)$ given $C = 100$ and $m = 0.01$. Certainly, in some aspects dW and $d\Pi$ are significantly different and I wish to maintain their similarity only in the context in which I have considered them. Additionally, it remains to be shown whether my results are consistent across different values of C and m. Despite these caveats, I find my results to be interesting on a theoretical level and believe they could potentially have implications for applications that I have yet to explore.

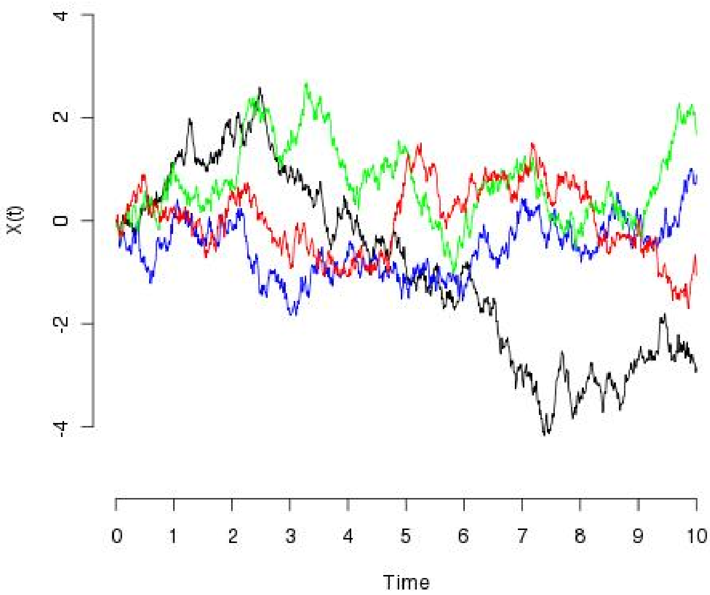

Figure 1: Brownian Motion Realizations

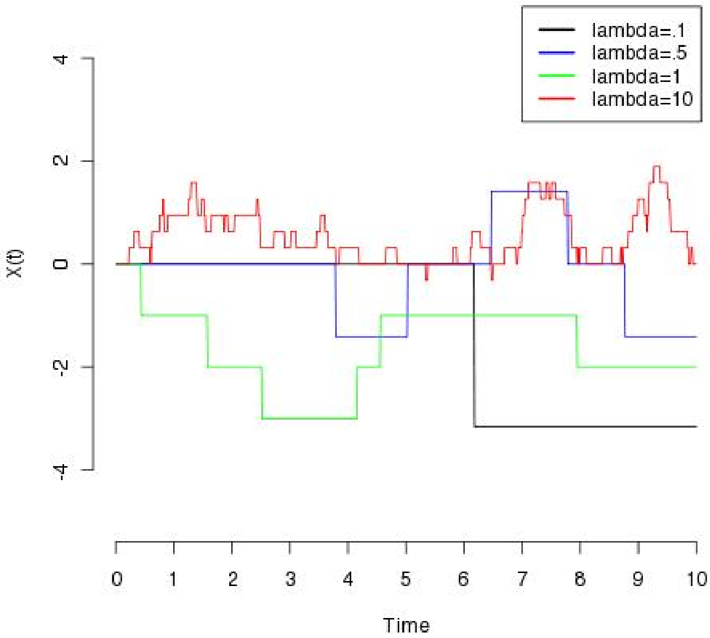

Figure 2: Poisson Increment Realizations

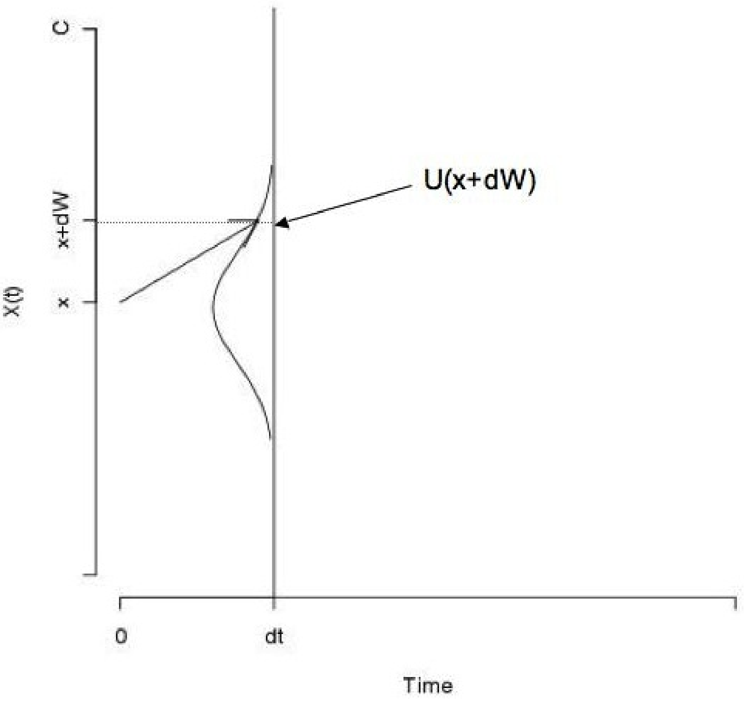

Figure 3: Change in Holdings

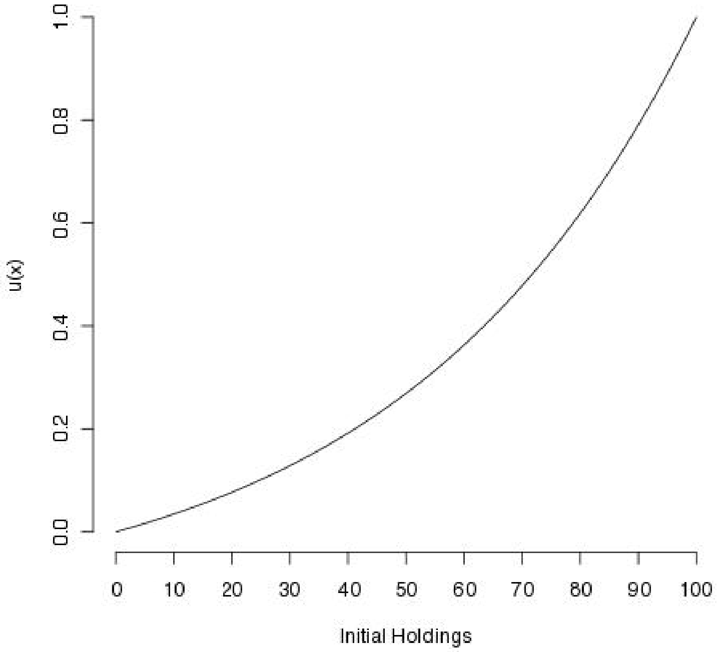

Figure 4: Analytical Gambler's Ruin Probabilities

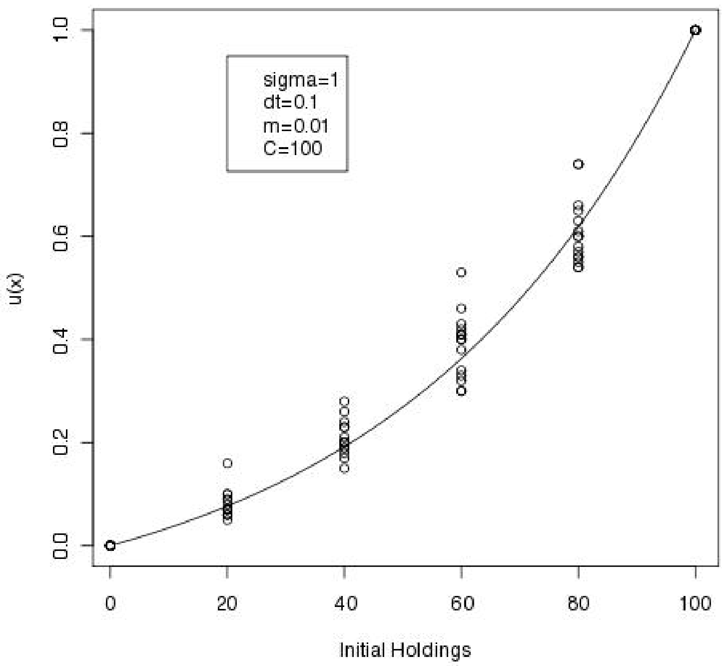

Figure 5: Numerical Gambler's Ruin Probabilities (dW)

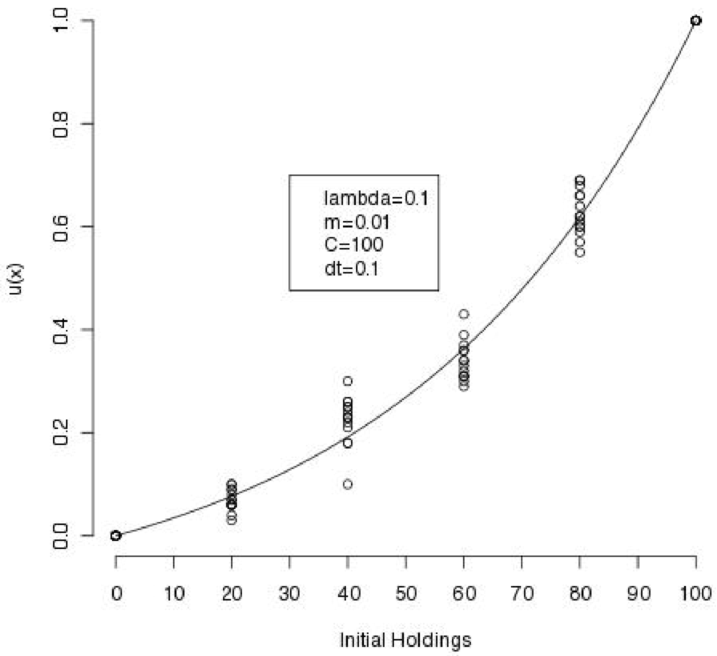

Figure 6: Numerical Gambler's Ruin Probabilities ($d\Pi$)

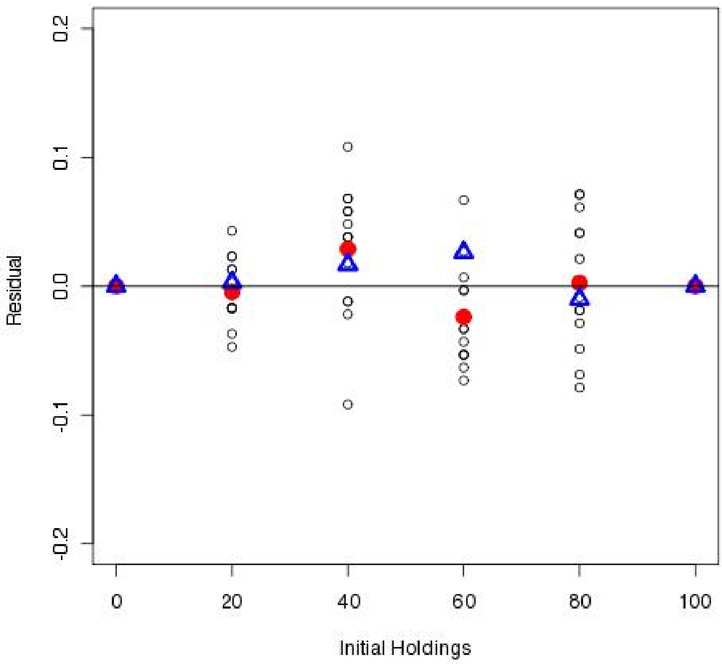

Figure 7: Residualized Comparison of $d\Pi$ and dW. The red dots and blue triangles are the average residuals for the $d\Pi$ and dW simulations, respectively.