



Local detectors for high-resolution spectral analysis: Algorithms and performance [☆]

Morteza Shahram ^{*}, Peyman Milanfar

Electrical Engineering Department, University of California, Santa Cruz, CA, USA

Available online 20 January 2005

Abstract

This paper develops local signal detection strategies for spectral resolution of frequencies of nearby tones. The problem of interest is to decide whether a received noise-corrupted and discrete signal is a single-frequency sinusoid or a double-frequency sinusoid. This paper presents an extension to M. Shahram and P. Milanfar (On the resolvability of sinusoids with nearby frequencies in the presence of noise, *IEEE Trans. Signal Process.*, to appear, available at <http://www.soe.ucsc.edu/~milanfar>) the case where the noise variance is unknown. A general signal model is considered where the frequencies, amplitudes, phases and *also the level of the noise variance* is unknown to the detector. We derive a fundamental trade-off between SNR and the minimum detectable difference between the frequencies of two tones, for any desired decision error rate. We also demonstrate that the algorithm, when implemented in a practical scenario, yields significantly better performance compared to the standard subspace-based methods like MUSIC. It is also observed that the performance for the case where the noise variance is unknown, is very close to that when the noise variance is known to the detector.

© 2005 Elsevier Inc. All rights reserved.

Keywords: Resolution; Rayleigh limit; Detection theory; Generalized likelihood ratio test; Subspace; MUSIC

[☆] This work was supported in part by NSF CAREER Award CCR-9984246, and AFOSR grant F49620-03-1-0387.

^{*} Corresponding author.

E-mail addresses: shahram@ee.ucsc.edu (M. Shahram), milanfar@ee.ucsc.edu (P. Milanfar).

1. Introduction and problem setup

Resolving sinusoidal signals with nearby frequencies has been of a significant interest in array processing and in particular direction finding, where two incoherent plane waves are incident upon a linear equi-spaced array of sensors [2]. The previous works done in this area fit in two categories; some have researched and developed novel methods and algorithms (see Refs. [3,4] for a list of the related literature and Refs. [5–7] for more recent works), whereas many others have focused on the performance analysis of the developed algorithms [2,8–16]. Some common approaches in the latter group have been to derive a sensitivity measure for the algorithms to the noise level [14], or to determine the “threshold” SNR at which a minimum resolvability (in statistical terms) can be obtained [2,15]. In any event, the trade-off between resolvability and SNR has been consistently used as a performance figure for spectral estimators.

We consider the signal of interest to be

$$s(x; \delta_1, \delta_2) = a_1 \sin(2\pi(f_c - \delta_1)x + \phi_1) + a_2 \sin(2\pi(f_c + \delta_2)x + \phi_2) \quad (1)$$

with $x \in [-B/2, B/2]$, where the frequencies of sinusoids ($f_c - \delta_1$ and $f_c + \delta_2$) are around a “center” frequency f_c . This center frequency can be assumed to be known or estimated beforehand by applying one of the existing spectral estimation methods. Assuming that the measured signal is corrupted by noise and sampled at rate of f_s , we can write it as

$$\begin{aligned} f(k; \delta_1, \delta_2) &= s(k; \delta_1, \delta_2) + w(k) \\ &= a_1 \sin\left(2\pi(f_c - \delta_1)\frac{k}{f_s} + \phi_1\right) \\ &\quad + a_2 \sin\left(2\pi(f_c + \delta_2)\frac{k}{f_s} + \phi_2\right) + w(k), \end{aligned} \quad (2)$$

where the integer index k is in the range $k \in \{-(N-1)/2, \dots, (N-1)/2\}$ and $N = Bf_s$ is the total number of samples. The term $w(k)$ is a zero-mean Gaussian white noise process with unknown variance σ^2 .

The spectral representation (i.e., the discrete-time Fourier transform (DTFT)) of the signal in (3) consists of two overlapping patterns centered at $f_c - \delta_1$ and $f_c + \delta_2$. According to the so-called Rayleigh criterion [10], these two peaks in the frequency domain are barely resolvable if

$$\delta_1 + \delta_2 = 1/B. \quad (4)$$

Hence for any signal with a frequency separation ($\delta_1 + \delta_2 < 1/B$), the main-lobe of the Fourier transform of the (sum of) two sinusoids is located in the same DTFT bin. In other words, the two frequency components are, in the Rayleigh sense, *unresolvable*. This is the scenario of interest in this paper. In the remainder of this paper, we use the phrase “signals with short observation interval” to identify signals in which the values of B , δ_1 , and δ_2 satisfy the inequality $\delta_1 + \delta_2 < 1/B$.

As we are to decide whether the received signal is a single-sinusoid (one peak in the spectral domain) or double-sinusoid (two peaks in the spectral domain), the problem of resolution can be posed as the following hypothesis test

$$\begin{cases} \mathcal{H}_0: \delta_1 = 0 \text{ and } \delta_2 = 0, \\ \mathcal{H}_1: \delta_1 > 0 \text{ or } \delta_2 > 0, \end{cases} \quad (5)$$

where \mathcal{H}_0 and \mathcal{H}_1 denote the null hypothesis (one peak is present) and alternative hypothesis (two peaks are present), respectively. Since we consider the case where δ_1 and δ_2 are unknown to the detector, (5) represents a composite (but one-sided) hypothesis testing problem [17]. We treated this problem in the case where the noise variance σ^2 was known in [1], and the contribution of this paper consists of extending the earlier analysis to the unknown noise variance case.

Almost all the previously developed techniques in this area employ the structure of the second order statistics of the signal. The key assumption is that of independent uniformly distributed phases of each sinusoid. Assuming ϕ_1 and ϕ_2 to be independent and uniformly distributed random variables in the range of $[0, 2\pi]$, the resulting hypothesis test from (5) using the signal covariances is given by

$$\begin{cases} \mathcal{H}_0: \mathbf{f} \sim \mathcal{N}(0, \mathbf{R}_0 + \sigma^2 \mathbf{I}), \\ \mathcal{H}_1: \mathbf{f} \sim \mathcal{N}(0, \mathbf{R}_1 + \sigma^2 \mathbf{I}), \end{cases} \quad (6)$$

where \mathbf{R}_0 and \mathbf{R}_1 are the autocorrelation matrices of the signal $s(k)$ in (2),

$$\mathbf{R}_0 = \frac{(a_1 + a_2)^2}{2} \text{Re}[\mathbf{r}(f_c) \mathbf{r}^T(f_c)], \quad (7)$$

$$\mathbf{R}_1 = \frac{a_1^2}{2} \text{Re}[\mathbf{r}(f_c - \delta_1) \mathbf{r}^T(f_c - \delta_1)] + \frac{a_2^2}{2} \text{Re}[\mathbf{r}(f_c + \delta_2) \mathbf{r}^T(f_c + \delta_2)], \quad (8)$$

where $\text{Re}[\cdot]$ denotes the real part and $\mathbf{r}(\cdot)$ is the vector form¹ of

$$r(k; f_c) = \exp\left(j2\pi f_c \frac{k}{f_s}\right). \quad (9)$$

For the most idealistic case where all the parameters in (7) and (8) are known to the detector, an NP detector for (6) decides \mathcal{H}_1 if

$$T_c(\mathbf{f}) = \mathbf{f}^T [(\mathbf{R}_1 + \sigma^2 \mathbf{I})^{-1} - (\mathbf{R}_0 + \sigma^2 \mathbf{I})^{-1}] \mathbf{f} > \gamma, \quad (10)$$

where subscript “c” denotes the “completely known” case. In practice, however, since only the time series are available, an estimate of the autocorrelation matrix derived from the signal samples is used. Furthermore, since finding the maximum likelihood (ML) estimation of the autocorrelation matrix is a highly complicated task, other suboptimal alternatives are used. The so-called subspace methods (e.g., MUSIC) for spectral estimation are based on the eigen-decomposition of the autocorrelation matrix into orthogonal signal and noise subspaces [2–4, 18–20].

Our proposed analysis is based on the model for the measured signal in (2) instead of relying on the second order statistic (i.e., covariance structure) of the signal. It is useful to mention that our methodology assumes the phases of sinusoid to be deterministic unknown variables, whereas the subspace methods treat the phases of sinusoids as uniformly distributed random variables.

¹ Superscript “T” in (7) and (8) denotes conjugate transpose.

Our approach is to quantify a measure of resolution in statistical terms by addressing the following question: “What is the minimum separation between frequencies of two nearby tones (maximum attainable resolution) that is detectable at a particular signal-to-noise ratio (SNR), and for pre-specified probabilities of detection and false alarm (P_d and P_f)?”

Addressing the above question will provide a fundamental performance bound which can be used to understand the effect of SNR and also other parameters on the achievable resolution in spectral analysis. Furthermore, the final computed performance limit is simply the result of employing a local detector which can be indeed used and implemented in practice. For this purpose, we put forward some comparisons between a practical setup of the proposed algorithm and the MUSIC algorithm. We demonstrate that the proposed detectors yield noticeably improved performance in resolving the spectra of nearby tones.

In our earlier work [1], we studied the problem in the case where the noise variance is *known* to the detector. In this paper, the case of *unknown* noise variance is considered, which is perhaps a more practical scenario. The main result of the forgoing analysis, as we shall see, is that there is little loss in performance when the noise variance is unknown.

In Section 2 we introduce our approach for the most general case, where the amplitudes and phases are unequal and unknown to the detector. In this section we also develop the corresponding detection strategies and characterize their performance. In Section 3 we present some results and comparisons of the proposed method with existing subspace methods. Finally, in Section 4, we summarize the results and present some concluding remarks.

2. Detection theoretic approach

We consider the most general case of the signal model of (3), with unknown amplitudes, phases, and unknown frequency parameters (δ_1 and δ_2) and also unknown level of noise variance (σ^2). The case of known and equal amplitudes and phases has been studied in [1] when σ^2 is known, the result of which has been shown to lead to a uniformly most powerful test.

When two spectral peaks corresponding to the frequencies of sinusoids ($f_c - \delta_1$ and $f_c + \delta_2$) are located in the same bin (below the Rayleigh limit), the range of interest for the values of δ_1 and δ_2 is small ($\delta_1, \delta_2 < 1/2B$). Hence, it is quite appropriate for the purposes of our analysis to consider approximating the model of the signal around $(\delta_1, \delta_2) = (0, 0)$. The second-order Taylor expansion of (2) about $(\delta_1, \delta_2) = (0, 0)$, with all other variables fixed, is

$$s(k; \delta_1, \delta_2) \approx \sum_{i=0}^2 \alpha_i p_i(k) + \beta_i q_i(k), \quad (11)$$

where

$$p_i(k) = \left(\frac{k}{f_s}\right)^i \sin\left(2\pi f_c \frac{k}{f_s}\right), \quad (12)$$

$$q_i(k) = \left(\frac{k}{f_s}\right)^i \cos\left(2\pi f_c \frac{k}{f_s}\right), \quad (13)$$

and

$$\alpha_i = \frac{(2\pi)^i}{i!} \left[a_1 \delta_1^i \cos\left(\phi_1 + i \frac{\pi}{2}\right) + a_2 \delta_2^i \cos\left(\phi_1 + i \frac{\pi}{2}\right) \right], \quad (14)$$

$$\beta_i = \frac{(2\pi)^i}{i!} \left[a_1 \delta_1^i \sin\left(\phi_1 + i \frac{\pi}{2}\right) + a_2 \delta_2^i \sin\left(\phi_1 + i \frac{\pi}{2}\right) \right]. \quad (15)$$

We elect to keep terms up to order 2 of the above Taylor expansion. This gives a more accurate representation of the signal since in some cases the first-order terms ($p_1(k)$ and $q_1(k)$) would be very small or even would simply vanish. Rewriting (11) in vector form will result in

$$\mathbf{s} \approx \sum_{i=0}^2 \alpha_i \mathbf{p}_i + \beta_i \mathbf{q}_i, \quad (16)$$

where

$$\mathbf{p}_i = [p_i(-(N-1)/2), \dots, p_i((N-1)/2)]^T, \quad (17)$$

$$\mathbf{q}_i = [q_i(-(N-1)/2), \dots, q_i((N-1)/2)]^T. \quad (18)$$

Now, the hypotheses in (5) appear in the following form:

$$\begin{cases} \mathcal{H}_0: \mathbf{z} = \alpha_0 \mathbf{p}_0 + \beta_0 \mathbf{q}_0 + \mathbf{w}, \\ \mathcal{H}_1: \mathbf{z} = \sum_{i=0}^2 \alpha_i \mathbf{p}_i + \beta_i \mathbf{q}_i + \mathbf{w}, \end{cases} \quad (19)$$

where \mathbf{z} denotes the approximate measured signal model. Equation (19) leads to a linear model for testing the parameter set $\boldsymbol{\theta}$ defined as follows:

$$\mathbf{z} = \mathbf{H}\boldsymbol{\theta} + \mathbf{w}, \quad (20)$$

$$\mathbf{H} = [\mathbf{p}_0 \mid \mathbf{q}_0 \mid \mathbf{p}_1 \mid \mathbf{q}_1 \mid \mathbf{p}_2 \mid \mathbf{q}_2], \quad (21)$$

$$\boldsymbol{\theta} = [\alpha_0 \ \beta_0 \ \alpha_1 \ \beta_1 \ \alpha_2 \ \beta_2]^T, \quad (22)$$

where \mathbf{H} and $\boldsymbol{\theta}$ are an $N \times 6$ matrix, and a 6×1 vector, respectively. The corresponding hypotheses are

$$\begin{cases} \mathcal{H}_0: \mathbf{A}\boldsymbol{\theta} = \mathbf{0}, \ \sigma^2 > 0, \\ \mathcal{H}_1: \mathbf{A}\boldsymbol{\theta} \neq \mathbf{0}, \ \sigma^2 > 0, \end{cases} \quad (23)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (24)$$

The hypothesis test in (23) is a problem of detecting a deterministic signal with unknown parameters ($\boldsymbol{\theta}$ and σ^2). The generalized likelihood ratio test (GLRT) is a well-known approach to solving this type of “composite” hypothesis testing problem [21]. The GLRT

uses the maximum likelihood (ML) estimates of the unknown parameters to form the standard Neyman–Pearson (NP) likelihood ratio detector. The GLRT for (23) [21] gives the following test statistic:

$$T = \frac{\hat{\boldsymbol{\theta}}^T \mathbf{A}^T [\mathbf{A}(\mathbf{H}^T \mathbf{H})^{-1} \mathbf{A}^T]^{-1} \mathbf{A} \hat{\boldsymbol{\theta}}}{\mathbf{z}^T [\mathbf{I} - \mathbf{H}(\mathbf{H}^T \mathbf{H})^{-1} \mathbf{H}^T] \mathbf{z}} > \gamma, \quad (25)$$

where \mathbf{I} is the identity matrix and²

$$\hat{\boldsymbol{\theta}} = (\mathbf{H}^T \mathbf{H})^{-1} \mathbf{H}^T \mathbf{z} \quad (26)$$

is the unconstrained maximum likelihood estimation of $\boldsymbol{\theta}$. For any given data set \mathbf{z} , we decide \mathcal{H}_1 if the statistic exceeds a specified threshold,

$$T(\mathbf{z}) > \gamma. \quad (27)$$

The choice of γ is motivated by the level of tolerable false alarm (or false-positive) in a given problem, but is typically kept very low. From (25), the performance of this detector is characterized by [21]

$$P_f = Q_{F_{4,N-6}}(\gamma), \quad (28)$$

$$P_d = Q_{F'_{4,N-6}(\lambda)}(\gamma), \quad (29)$$

$$\lambda = \frac{1}{\sigma^2} \boldsymbol{\theta}^T \mathbf{A}^T [\mathbf{A}(\mathbf{H}^T \mathbf{H})^{-1} \mathbf{A}^T]^{-1} \mathbf{A} \boldsymbol{\theta}, \quad (30)$$

where $Q_{F_{4,N-6}}$ is the right tail probability for a central F distribution with 4 numerator degrees of freedom and $N - 6$ denominator degrees of freedom, and $Q_{F'_{4,N-6}(\lambda)}$ is the right tail probability for a non-central F distribution with 4 numerator degrees of freedom and $N - 6$ denominator degrees of freedom, and non-centrality parameter λ . For a desired P_d and P_f , we can compute the required value for the non-centrality parameter from (28) and (29). We call this value of the non-centrality parameter $\lambda(P_f, P_d)$ as a function of desired probability of detection and false alarm rate. This notation is key in illuminating a very useful relationship between the SNR and the smallest separation (i.e., (δ_1, δ_2)) which can be detected with high probability, and low false alarm rate. From (30) we can write

$$\sigma^2 = \frac{1}{\lambda(P_f, P_d)} \boldsymbol{\theta}^T \mathbf{A}^T [\mathbf{A}(\mathbf{H}^T \mathbf{H})^{-1} \mathbf{A}^T]^{-1} \mathbf{A} \boldsymbol{\theta}. \quad (31)$$

Also, by defining the output SNR as

$$\text{SNR} = \frac{1}{\sigma^2} \boldsymbol{\theta}^T \mathbf{H}^T \mathbf{H} \boldsymbol{\theta} \quad (32)$$

and replacing the value of σ^2 with the right hand side of (31) the relation between the parameter set $\boldsymbol{\theta}$ and the required SNR can be made explicit,

$$\text{SNR} = \lambda(P_f, P_d) \frac{\boldsymbol{\theta}^T \mathbf{H}^T \mathbf{H} \boldsymbol{\theta}}{\boldsymbol{\theta}^T \mathbf{A}^T [\mathbf{A}(\mathbf{H}^T \mathbf{H})^{-1} \mathbf{A}^T]^{-1} \mathbf{A} \boldsymbol{\theta}}. \quad (33)$$

² Note that $(\mathbf{H}^T \mathbf{H})^{-1} \mathbf{H}^T$ is the pseudo inverse of \mathbf{H} .

It is instructive to simplify (33) by approximating the elements of the matrix $\mathbf{H}^T\mathbf{H}$. These approximations yield (see Appendix A for details)

$$\mathbf{H}^T\mathbf{H} \approx \begin{bmatrix} \frac{N}{2} & 0 & 0 & -\frac{N}{4}\mu & \frac{N^3}{24f_s^2} & 0 \\ 0 & \frac{N}{2} & -\frac{N}{4}\mu & 0 & 0 & \frac{N^3}{24f_s^2} \\ 0 & -\frac{N}{4}\mu & \frac{N^3}{24f_s^2} & 0 & 0 & -\frac{N^3}{16}\mu \\ -\frac{N}{4}\mu & 0 & 0 & \frac{N^3}{24f_s^2} & -\frac{N^3}{16}\mu & 0 \\ \frac{N^3}{24f_s^2} & 0 & 0 & -\frac{N^3}{16}\mu & \frac{N^5}{160f_s^4} & 0 \\ 0 & \frac{N^3}{24f_s^2} & -\frac{N^3}{16}\mu & 0 & 0 & \frac{N^5}{160f_s^4} \end{bmatrix},$$

where $\mu = \cos(\frac{2\pi f_c}{f_s} N) / \sin(\frac{2\pi f_c}{f_s})$. To gain further insight, we consider a special case by assuming $a_1\delta_1 \approx a_2\delta_2$, which results from a proper choice of the center frequency f_c (see Ref. [1]), and simultaneously considering the case where the value of ϕ_1 is close to that of ϕ_2 . After some algebra, replacing N/f_s by B and neglecting non-dominant terms, for small δ_1 and δ_2 (i.e., $\delta_1, \delta_2 \ll 1/B$) (33) will reduce to:

$$\text{SNR} \approx \frac{45}{\pi^4} \frac{\lambda(P_f, P_d)}{B^4} \frac{(a_1 + a_2)^2}{(a_1\delta_1^2 + a_2\delta_2^2)^2}. \tag{34}$$

The required SNR is minimized when $\delta_1 = \delta_2 = \delta$ (i.e., $a_1 = a_2 = 1$); and in this case we have

$$\text{SNR} \approx \frac{45}{\pi^4} \frac{\lambda(P_f, P_d)}{(B\delta)^4}. \tag{35}$$

An important question is to consider how different this obtained performance is from that of the ideal detector, to which all the parameters (amplitudes, phases, frequencies, and noise variance) are known. We first note that in this case the hypothesis test in (19) will be a standard Gauss–Gauss detection problem. Also, we can further simplify the problem by seeing that the term $\alpha_0\mathbf{p}_0 + \beta_0\mathbf{q}_0$ is a common known term under both hypotheses and can be removed. As a result, the following relationship can be verified for the completely known case:

$$\text{SNR}_{\text{id}} = \eta(P_f, P_d) \frac{\boldsymbol{\theta}^T \mathbf{H}^T \mathbf{H} \boldsymbol{\theta}}{\boldsymbol{\theta}^T \mathbf{A}^T \mathbf{A} \mathbf{H}^T \mathbf{H} \mathbf{A} \boldsymbol{\theta}}, \tag{36}$$

where the subscript “id” denotes the ideal case and $\eta(P_f, P_d)$ is the required deflection coefficient [21] computed as

$$\eta = (Q^{-1}(P_f) - Q^{-1}(P_d))^2, \tag{37}$$

where $Q^{-1}(\cdot)$ is the inverse of the right-tail probability function for a standard Gaussian random variable (zero mean and unit variance). Comparing the expression in (33) to that of (37),

$$\frac{\text{SNR}}{\text{SNR}_{\text{id}}} = \frac{\lambda(P_f, P_d)}{\eta(P_f, P_d)} \frac{\boldsymbol{\theta}^T \mathbf{A}^T \mathbf{A} \mathbf{H}^T \mathbf{H} \mathbf{A}^T \mathbf{A} \boldsymbol{\theta}}{\boldsymbol{\theta}^T \mathbf{A}^T [\mathbf{A} (\mathbf{H}^T \mathbf{H})^{-1} \mathbf{A}^T]^{-1} \mathbf{A} \boldsymbol{\theta}} \quad (38)$$

we notice that $\eta(P_f, P_d) < \lambda(P_f, P_d)$ provided $P_d > P_f$. Also, It can be proved that $\mathbf{A} \mathbf{H}^T \mathbf{H} \mathbf{A}^T - [\mathbf{A} (\mathbf{H}^T \mathbf{H})^{-1} \mathbf{A}^T]^{-1}$ is a positive definite matrix. As a result, as expected, $\text{SNR} > \text{SNR}_{\text{id}}$ always.

3. Numerical results and comparisons

We first compare the performance of the proposed detector for the unknown σ^2 case with those of the detector for the known σ^2 case [1] and the ideal detector (36). Figure 1 presents the performance figures of these detectors for the case where $a_1 = a_2$ and $\delta_1 = \delta_2 = \delta$ (curves of $2\delta B$ versus required SNR). We observe that knowledge of the noise variance makes little difference to the performance (around 1 dB in required SNR). It is worth mentioning that the estimate of the noise variance used in (25) is known to be unbi-

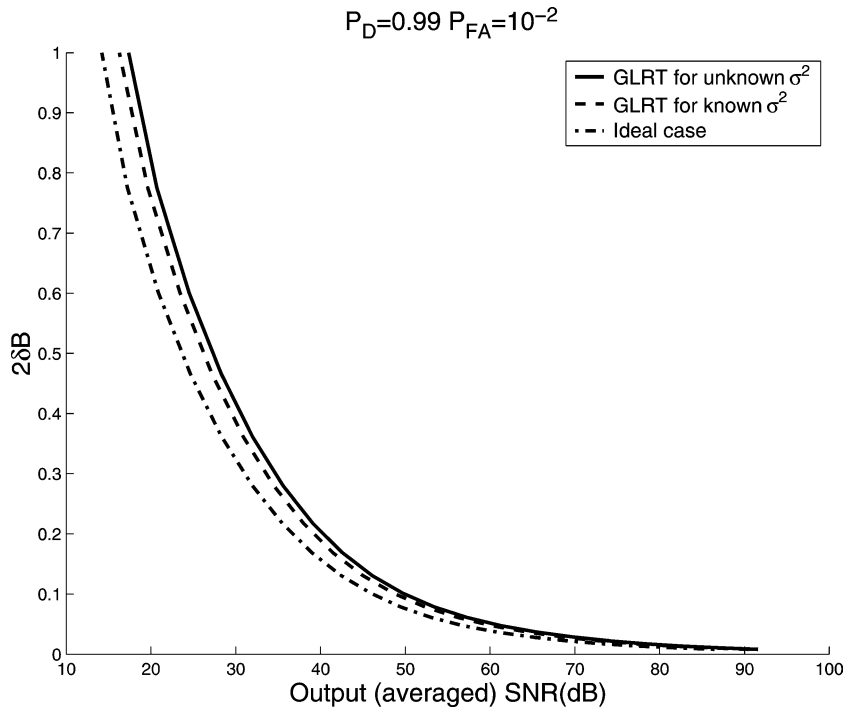


Fig. 1. $2\delta B$ vs required SNR for known and unknown noise variance, $a_1 = a_2$ and $\delta_1 = \delta_2 = \delta$.

ased [21]. Comparing the ideal (unrealizable) detector, the GLR detector in (25) requires 3–5 dB more SNR to achieve the same resolvability.

As for the comparison to existing methods, we first note that for subspace detectors, the phase is typically assumed to be a uniformly distributed random variable in $[0, 2\pi]$. However the “required SNR” computed in Section 2 is in general a function of the phases of the sinusoids. Hence in order to compare the obtained results to that of subspace methods, we carry out the following averaging for the required SNR over the possible range of ϕ_1 and ϕ_2 :

$$\text{SNR}_{\text{avg}} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \text{SNR} \, d\phi_1 \, d\phi_2, \tag{39}$$

where subscript “avg” denotes the averaged value and the integrand (SNR) is the right-hand side of (33).

Next, we simulated the behavior of the MUSIC algorithm for resolving sinusoids with nearby frequencies. In simulation of MUSIC, the signal is declared to be resolvable if the output of MUSIC produces two distinct peaks within an interval around the true frequencies ($f_c \pm \delta$). The simulations for MUSIC are carried out for cases in which either a single

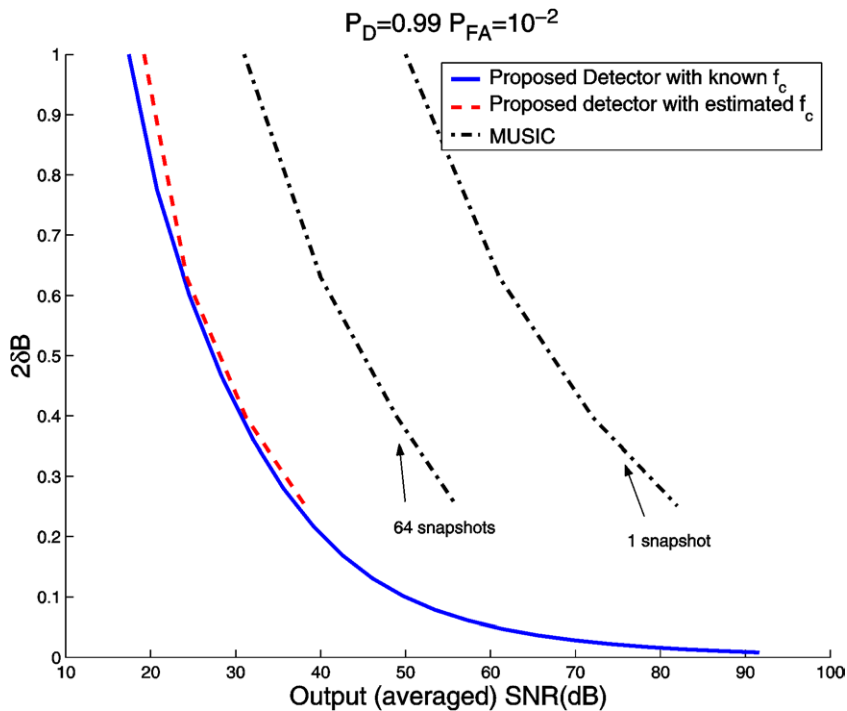


Fig. 2. $2\delta B$ vs required output SNR for the MUSIC algorithm and the proposed detector, unknown noise variance, $a_1 = a_2$ and $\delta_1 = \delta_2 = \delta$.

snapshot, or multiple snapshots, are available. Naturally, we consider the output SNR in the latter case as the sum of SNR's of each snapshot.

We develop two different comparison procedures. First, we compare the performance of MUSIC against the detector in (25), where we assume that the center frequency f_c , at which we perform the hypothesis test, is known a priori. Since this might be seen as an unfair comparison, we have put forward an alternative (perhaps more practical) scenario, too. In this scenario, we first seek assistance from MUSIC to estimate the center frequency (f_c) and then apply the proposed detector in (25) centered at the peak estimated by MUSIC.

The results of these experiments are shown in Fig. 2. First, we observe that the proposed detector significantly outperforms MUSIC in both cases (using known or estimated center frequency). More interestingly, we see that the result of the proposed detector with estimated center frequency (provided by MUSIC) is very close to the performance of the same detector with known center frequency. This implies that the MUSIC algorithm does a very promising job in locating the center frequency (i.e., the candidate location where we can perform a refinement step using our proposed approach). Intuitively, the reason for this behavior is that for the case where a high probability of resolution (0.99) is considered, a fairly high value of SNR should be provided. This value of SNR will effectively guarantee a condition under which the MUSIC algorithm will produce the peak in its spectrum within the range of $[f_c - \delta, f_c + \delta]$. This observation is essentially in agreement with what has been observed in the past about the stability of MUSIC for single-sinusoid signals.

4. Conclusion

The problem of interest in this paper has been to establish a statistical analysis of attainable spectral resolution and to propose its associated detection algorithm. We formulate the problem as a hypothesis test, the aim of which is to distinguish whether the received signal contains a single-tone or double-tone. We have considered the most general case where the amplitudes, frequencies and phases of sinusoids and also the value of noise variance is unknown to the detector. This paper is different and more general than [1] in that it assumes σ^2 is unknown.

By utilizing a quadratic approximation, we in fact carried out the analysis in the context of locally optimal detectors, and developed corresponding detection strategies. The performance figure of resolution has been quantified by the following practical question: "What is the minimum detectable frequency difference between two sinusoids at a given signal-to-noise ratio?"

We clearly observed that noise variance being unknown has little effect on the detection performance. This is a useful observation, since in practice the variance is often unknown to the receiver.

Also, the proposed locally optimal detectors yield significantly improved detection of very nearby frequencies, as compared to the existing subspace methods. In terms of implementing the suggested detection algorithm, we merely need to estimate the center frequency. Fortunately, as we confirmed by some experiments, this task can be very effectively performed by using one of the myriad of existing methods for spectral estimation

and then running the proposed detector at the estimated peak. The performance of such a detector is nearly identical to that of the detector with a known center frequency.

Appendix A. Computing the energy terms

In this appendix, we explain the general process for the approximate computation of the energy terms. We will utilize the following identities for the calculation:

$$\sum_{k=0}^L x^k = \frac{1 - x^{L+1}}{1 - x}, \tag{A.1}$$

$$\sum_{k=0}^L k^p x^k = \sum_{m=1}^p x^m \frac{\partial^m}{\partial x^m} \left(\frac{1 - x^{L+1}}{1 - x} \right), \tag{A.2}$$

$$\sum_k k^{p+1} \sin(xk) \cos(xk) = \frac{1}{2} \frac{\partial}{\partial x} \sum_k k^p \sin^2(xk). \tag{A.3}$$

Instead of showing all the calculations, for the sake of brevity we discuss, as an example, the calculation of the term $\mathbf{h}_0^T \mathbf{h}_0$:

$$\begin{aligned} \mathbf{h}_0^T \mathbf{h}_0 &= 4 \sum_{k=-(N-1)/2}^{(N-1)/2} \sin^2 \left(\frac{2\pi f_c}{f_s} k \right) \\ &= \sum_{k=-(N-1)/2}^{(N-1)/2} \left[\exp \left(j \frac{2\pi f_c}{f_s} k \right) - \exp \left(-j \frac{2\pi f_c}{f_s} k \right) \right]^2 \\ &= \sum_{k=-(N-1)/2}^{(N-1)/2} 2 - \exp \left(j \frac{4\pi f_c}{f_s} k \right) - \exp \left(-j \frac{4\pi f_c}{f_s} k \right) \\ &= 2N - 2 \frac{1 - \exp(j \frac{2\pi f_c}{f_s} (N+1))}{1 - \exp(j \frac{4\pi f_c}{f_s})} - 2 \frac{1 - \exp(-j \frac{2\pi f_c}{f_s} (N+1))}{1 - \exp(-j \frac{4\pi f_c}{f_s})} + 2 \\ &= 2N + 2 - 2 \frac{1 - \cos(\frac{4\pi f_c}{f_s}) + \cos(\frac{2\pi f_c}{f_s} (N-1)) - \cos(\frac{2\pi f_c}{f_s} (N+1))}{1 - \cos(\frac{4\pi f_c}{f_s})} \\ &= 2N - 2 \underbrace{\frac{\sin(\frac{2\pi f_c}{f_s} N)}{\sin(\frac{2\pi f_c}{f_s})}}_C. \end{aligned} \tag{A.4}$$

Since $\sin(x) \geq 1 - |\frac{2}{\pi}x - 1|$ for $0 \leq x \leq \pi$, and $\frac{2\pi f_c}{f_s} < \pi$, by upper and lower bounding the numerator and the denominator of $|C|$, respectively, we have

$$|C| = 2 \left| \frac{\sin(\frac{2\pi f_c}{f_s} N)}{\sin(\frac{2\pi f_c}{f_s})} \right| \leq \frac{2}{\sin(\frac{2\pi f_c}{f_s})} \leq \frac{2}{1 - |\frac{4f_c}{f_s} - 1|}. \tag{A.5}$$

Thus for the range of $\epsilon \leq 2f_c/f_s \leq 1 - \epsilon$ (representing the range of f_s from just above the Nyquist rate ($2f_c$) to $1/\epsilon$ times the Nyquist rate), we will have $|C| < 1/\epsilon$ and therefore for $\epsilon < 5/N$

$$\mathbf{h}_0^T \mathbf{h}_0 \approx 2N. \quad (\text{A.6})$$

A similar approach can be followed to compute other energy terms.

References

- [1] M. Shahram, P. Milanfar, On the resolvability of sinusoids with nearby frequencies in the presence of noise, *IEEE Trans. Signal Process.*, to appear, available at: <http://www.soe.ucsc.edu/~milanfar>.
- [2] M. Kaveh, A.J. Barabell, The statistical performance of the MUSIC and the minimum-norm algorithms in resolving plane waves in noise, *IEEE Trans. Acoust. Speech Signal Process.* 34 (1986) 331–341.
- [3] S.M. Kay, *Modern Spectral Estimation: Theory and Application*, Prentice–Hall, 1988.
- [4] S.M. Kay, Spectrum analysis: A modern perspective, *Proc. IEEE* 69 (11) (1981) 1380–1418.
- [5] M.L. McCloud, L.L. Scharf, A new subspace identification algorithm for high-resolution DOA estimation, *IEEE Trans. Antenn. Propagat.* 50 (10) (2002) 1382–1390.
- [6] P. Charge, Y. Wang, J. Saillard, An extended cyclic MUSIC algorithm, *IEEE Trans. Signal Process.* 51 (7) (2003) 1605–1613.
- [7] E.G. Larsson, P. Stoica, High-resolution direction finding: the missing data case, *IEEE Trans. Signal Process.* 49 (5) (2001) 950–958.
- [8] V. Reddy, L. Biradar, SVD-based information theoretic criteria for detection of the number of damped/undamped sinusoids and their performance analysis, *IEEE Trans. Signal Process.* 41 (9) (1993) 2872–2881.
- [9] L. McWhorter, L. Scharf, Cramér-Rao bounds for deterministic modal analysis, *IEEE Trans. Signal Process.* 41 (1993) 1847–1862.
- [10] L.L. Scharf, P.H. Moose, Information measures and performance bounds for array processors, *IEEE Trans. Inform. Theory* 22 (1) (1976) 11–21.
- [11] H.V. Hamme, A stochastic limit to the resolution of least squares estimation of the frequencies of a double complex sinusoid, *IEEE Trans. Signal Process.* 39 (12) (1991) 2652–2658.
- [12] P. Stoica, A. Nehorai, MUSIC, maximum likelihood, and Cramér-Rao bound, *IEEE Trans. Acoust. Speech Signal Process.* 37 (1989) 720–741.
- [13] P. Stoica, A. Nehorai, MUSIC, maximum likelihood, and Cramér-Rao bound: Further results and comparisons, *IEEE Trans. Acoust. Speech Signal Process.* 38 (1990) 2140–2150.
- [14] A. Weiss, B. Friedlander, Effects of modeling error on the resolution threshold of the MUSIC algorithm, *IEEE Trans. Signal Process.* 42 (1) (1994) 147–155.
- [15] H.B. Lee, M.S. Wegrovitz, Resolution threshold of beamspace MUSIC for two closely spaced emitters, *IEEE Trans. Acoust. Speech Signal Process.* 38 (1990) 1545–1559.
- [16] A.S. Kayhan, M.G. Amin, Spatial evolutionary spectrum for DOA estimation and blind signal separation, *IEEE Trans. Signal Process.* 48 (3) (2000) 791–798.
- [17] L.L. Scharf, *Statistical Signal Processing, Detection, Estimation, and Time Series Analysis*, Addison-Wesley, Reading, MA, 1991.
- [18] L.L. Scharf, B. Friedlander, Matched subspace detectors, *IEEE Trans. Signal Process.* 42 (1994) 2146–2157.
- [19] J.A. Cadzow, Y.S. Kim, D.C. Shiue, General direction-of-arrival estimation: a signal subspace approach, *IEEE Trans. Aerospace Electron. Syst.* 25 (1) (1989) 31–47.
- [20] J.A. Cadzow, Direction finding: a signal subspace approach, *IEEE Trans. Syst. Man Cybernet.* 21 (5) (1991) 1115–1124.
- [21] S.M. Kay, *Fundamentals of Statistical Signal Processing, Detection Theory*, Prentice–Hall, 1998.