Statistical Performance Analysis of Superresolution Image Reconstruction

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***Abstract*— Recently, there has been much work developing super-resolution algorithms for combining a set of low quality images to produce a set of higher quality images. In most cases, such algorithms must first register the collection of images to a common sampling grid and then reconstruct the high resolution image. While many such algorithms have been proposed to address each one of these subproblems, no work has addressed the overall performance limits for this joint estimation problem. In this paper, we analyze the performance limits from statistical first principles using the Cramer-Rao bound. We offer insight into the fundamental bottlenecks limiting the performance of multiframe image reconstruction algorithms and hence super-resolution.**

I. Introduction

In the last decade, several papers have proposed algorithms addressing the problem of super-resolution. We refer the interested reader to [1] for a broad review of the work in this area. In general, the problem of super-resolution can be expressed as that of combining a set of noisy, aliased, low-resolution, blurry images to produce a higher resolution image or image sequence. Commonly, it is assumed that we are given a set of low resolution images which consist of noisy, warped, blurred, and downsampled versions of an unknown high resolution image. The simplest, and perhaps most commonly utilized, warping model assumes that the motion between frames is captured by a global shift or a translation. We focus on this type of motion for the remainder of this paper.

We represent the forward process by the linear measurement model

$$\mathbf{y}_k = \mathbf{DHF}(\mathbf{v}_k)\mathbf{x} + \mathbf{e}_k. \tag{1}$$

The vectors $\mathbf{y}_k$ represent the samples of the measured images raster scanned to form N_L dimensional vectors. Likewise, $\mathbf{x}$ represents the unknown original high resolution image similarly scanned to form a N_H dimensional vector. The matrix $\mathbf{D}$ captures the downsampling operation, $\mathbf{H}$ the blurring operation due to the imaging system point spread function (PSF), and $\mathbf{F}(\mathbf{v}_k)$ the translational motion operation with $\mathbf{v}_k = [v_{k_1}, v_{k_2}]^T$ being the unknown translation parameters for a particular frame. Finally, $\mathbf{e}_k$ represents the vector of additive white Gaussian measurement noise with variance σ^2.

We make several assumptions concerning our forward model. First, we assume that the downsampling ratio $\frac{N_L}{N_H} = \frac{1}{M}$ where M is a known integer. Second, we assume that the blurring operation and hence the imaging system's PSF is spatially invariant and can be represented by a convolution operation with a known space-invariant kernel. From this assumption, the blurring operator $\mathbf{H}$ becomes a block circulant matrix. Finally, in our formulation, we suppose that $K+1$ aliased low resolution images are available. Without loss of generality, we assume that the initial image $\mathbf{y}_0$ dictates the coordinate system so that $\mathbf{F}_0 = \mathbf{I}$ and hence we only have to estimate K unknown translation vectors $\mathbf{v}_k$ during the super-resolution process for a given set of $K+1$ low resolution frames.

When the motion is global translation, it has been shown that re-parameterizing the problem into two sequential estimation problems can substantially improve computational efficiency [1], [2]. Using the property that the circulant matrices $\mathbf{H}$ and $\mathbf{F}_k$ commute, a natural transformation of the unknown image $\mathbf{x}$ is given by

$$\mathbf{z} = \mathbf{H}\mathbf{x} \tag{2}$$

which maps the unknown high resolution image into a blurry version of the high resolution images $\mathbf{z}$. In both works, this transformation was used to break the problem of super-resolution into the two subproblems of multi-frame image reconstruction (estimating $\mathbf{z}$ from the data $\{\mathbf{y}_k\}$) and image restoration (estimating $\mathbf{x}$ from $\hat{\mathbf{z}}$). Such an approach is justified by the invariance property of the Maximum Likelihood estimator [3]. For the remainder of this paper, we focus on the problem of multi-frame image reconstruction which represents the core component to super-resolution.

In general, the problem of super-resolution is an example of a separable nonlinear least squares problem. Typically, the estimation problem is divided into the tasks of first registering the low resolution images (a highly nonlinear estimation problem) followed by reconstruction the low-resolution data and finally deblurring and interpolating to produce the final high resolution image (an ill-conditioned linear estimation problem). Historically, most research in super-resolution tended to focus on the latter stages assuming that generic image registration algorithms could be trusted to produce estimates with a high degree of accuracy. Relatively recently, researchers have noted the importance of solving the estimation problems of image registration and reconstruction/restoration in a joint fashion [4], [5], [6]. Conversely, the only paper (to our knowledge) concerning registration of aliased (sub-Nyquist) images [7], does not directly address the problem of image reconstruction jointly with registration. Instead, it focuses on mitigating generic (not image specific) effects of aliasing on the registration algorithm. In this paper, we study the relationship between the task of image registration and image reconstruction.

In the current paper, we analyze the joint problem of aliased image registration and high resolution image reconstruction in the context of fundamental statistical performance limits. Little work has addressed performance limits for the problem of super-resolution [8], [9]. Both works study the problem of super-resolution from an algebraic perspective reducing all super-resolution algorithms to that of solving large systems of linear equations. Furthermore, both works make the overly simplistic assumption that the image registration is an independently performed operation. One observation noted in [9] is that for most imaging applications, the enhancement factor of 1.6 is ”unbreakable”. In retrospect, given the recent success of several approaches in this field, this statement seems inappropriate [1].

We study the performance limits of multi-frame image reconstruction from a statistical perspective enabling us to bound estimator performance in terms of Mean Square Error (MSE) using the Cramer-Rao (CR) bound. In general, the CR bound provides the lower bound on the MSE of *any* unbiased estimator $\hat{\theta}$ of an unknown parameter vector θ from a given set of data represented by the data vector Φ [10]. Specifically, the CR bound on the error correlation matrix $E[(\hat{\theta}-\theta)(\hat{\theta}-\theta)^T]$ for any unbiased estimator is given by

$$MSE(\theta) = E[(\hat{\theta}-\theta)(\hat{\theta}-\theta)^T] \geq \mathbf{J}^{-1}(\theta) \tag{3}$$

where the matrix $\mathbf{J}(\theta)$ is referred to as the Fisher Information Matrix (FIM). The inequality indicates that the difference between the MSE matrix (left side) and the CR bound matrix (right side) will be a positive semi-definite matrix. The elements of the FIM are defined by

$$\{\mathbf{J}\}_{i,j} = -E\left[\frac{\partial^2 l(\Phi,\theta)}{\partial\theta_i\partial\theta_j}\right]$$

where $l(\Phi,\theta)$ is the joint log-likelihood of the observed data Φ and the unknown parameters θ.

Often, it is convenient that the set of unknown parameters θ be divided into distinct subsets denoted in vector form by θ_a and θ_b. In such form, the FIM is partition as

$$\mathbf{J}(\theta_a,\theta_b) = \begin{bmatrix} \mathbf{J}_{aa} & \mathbf{J}_{ab} \\ \mathbf{J}_{ab} & \mathbf{J}_{bb} \end{bmatrix}$$

We can focus on the bound for one particular set of parameter by way of

$$\mathbf{J}^{-1}(\theta_a) = \left(\mathbf{J}_{aa} - \mathbf{J}_{ab}\mathbf{J}_{bb}^{-1}\mathbf{J}_{ab}\right)^{-1} \tag{4}$$

For multi-frame image reconstruction, the set of unknown parameters are the unknown high resolution image $\mathbf{z}$ and the set of unknown motion parameters defined by $\underline{\mathrm{v}} = [\mathbf{v}_1,\ldots,\mathbf{v}_K]^T$. Because each set shares the same units, a natural scalar representation of the CR bound is given by

$$\overline{rmse}(\theta_a) \geq T(\theta_a) \tag{5}$$

where we define $\overline{rmse}(\theta_a) = \left[\frac{Tr(MSE(\theta_a))}{N_a}\right]^{\frac{1}{2}}$ and $T(\theta_a) = \left[\frac{Tr(\mathbf{J}^{-1}(\theta_a))}{N_a}\right]^{\frac{1}{2}}$ and N_a is the dimension of the parameter vector θ_a. Such a representation shows the average overall performance bound over the set of parameters while maintaining the units of the unknown parameters.

Finally, to address the utility of the CR bound in studying general estimation problems, we note that the overall usefulness of a performance limit depends on its ability not only to limit, but predict actual estimator performance. For example, we might trivially bound MSE performance as $MSE(\theta) \geq 0$. While such a bound is provably correct, it offers no useful information about the estimation problem. The CR bound, however, can be shown theoretically to be asymptotically attainable by the class of Maximum Likelihood (ML) estimators. While there is no guarantee that such estimators are realizable, it does offer hope for predicting performance for a wide class of estimators.

In this paper, we study the performance limits for multi-frame image reconstruction in its entirety which includes analysis of both the image registration and reconstruction problems. To simplify the presentation, we derive and analyze the CR bounds for the 1-D analogue of the multi-frame image reconstruction problem which has a similar form as (1), with the exception that the translation parameter is given by the *scalar* v_k. In both cases, we study both the Fisher information associated with each estimation problem as well as the overall CR bound on joint estimation performance. In each section, we analyze the performance bound as it relates to the image content $\mathbf{z}$, motion parameters $\underline{\mathrm{v}}$, downsampling factor M, noise power σ^2 and the number of frames $K+1$. Ultimately, our analysis offers new insights into the fundamental performance tradeoffs inherent to multi-frame image reconstruction and hence super-resolution imaging.

II. CR Bound on the Registration of Multiple Aliased Images

In this section, we analyze the performance bound for the problem of registering a collection of aliased images. Studying the overall performance bound $T(\underline{\mathrm{v}})$ we can observe the relationship between registering aliased images and image reconstruction. To date, the problem of registering aliased images has not been studied in relation to that of image reconstruction. In [11], the performance bound on registering a pair of non-aliased images was studied. In that scenario, the estimation problem was independent of the image reconstruction problem. We will show that when the images are sampled below the Nyquist rate (hence are aliased), image registration and reconstruction are tightly coupled problems. Furthermore, we show that the ideal algorithm will solve the registration problem using the entire set of observed low-resolution images.

First, we examine $\mathbf{J}_{\underline{\mathrm{vv}}}$, the sub-matrix of the FIM related to image registration. As we shall show, there is much intuition about the estimation problem to acquire through direct examination of the FIM. We observe that $\mathbf{J}_{\underline{\mathrm{vv}}}$ is a diagonal matrix whose elements are given by

$$\begin{aligned} \left[\mathbf{J}_{\underline{\mathrm{vv}}}\right]_{kk} &= \frac{1}{\sigma^2}\left(\mathbf{z}^T\mathbf{C}^T\mathbf{F}^T(v_k)\mathbf{D}^T\mathbf{D}\mathbf{F}(v_k)\mathbf{C}\mathbf{z}\right) \\ &= \frac{1}{\sigma^2}\left(\mathbf{d}^T\mathbf{Q}(v_k)\mathbf{d}\right) \qquad (6) \end{aligned}$$

where $\mathbf{C}$ represents the matrix form for an ideal spatial derivative operator. The derivative (or gradient for the 2-D case) can be represented by a linear operator because we assume that the image $\mathbf{z}$ is sampled above the Nyquist rate. We simplify the expression by defining $\mathbf{d} \equiv \mathbf{C}\mathbf{z}$ (first derivative signal) and $\mathbf{Q}(v_k) \equiv \mathbf{F}^T(v_k)\mathbf{D}^T\mathbf{D}\mathbf{F}(v_k)$. In other words, the information necessary for registration depends on the energy in the spatial derivatives of the unknown signal $\mathbf{d}$ projected into the lower dimensional measurement sub-space via the projection operator $\mathbf{Q}(v_k)$. As one would expect, we see that the information is inversely proportional to the noise power σ^2.

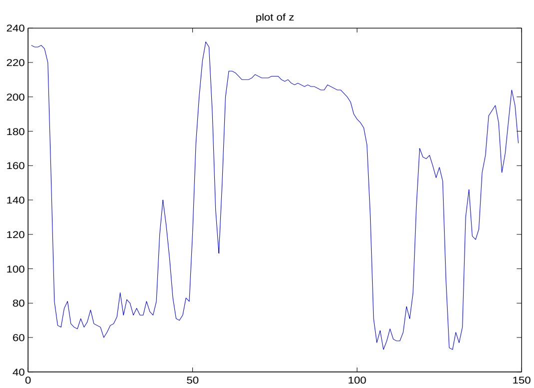

Fig. 1. Plot of the signal $\mathbf{z}$.

There are several observations that we can make about the Fisher information as it relates to the motion parameters v_k and the signal $\mathbf{z}$. First, we observe that any low pass filtering due to the blurring effect of the imaging system reduces the ability to register the images by damping the energy in the higher spatial frequencies (texture). This generalizes the observation introduced in [11] that higher frequency information or texture improves the ability to register images for the non-aliased scenario. Second, we note that the Fisher information is a periodic function of v_k with a period of M pixels. Intuitively, this can be interpreted as meaning that the information depends only on the sampling offset phase $\phi_k = \frac{v_k 2\pi}{M}$ for the kth measured low resolution image. For example, Figure 2 shows the value of $\mathbf{d}^T\mathbf{Q}(\phi)\mathbf{d}$ throughout the range of sample phase offsets ϕ for the signal $\mathbf{z}$ shown in Figure 1. The function is shown in polar coordinates about the sampling phase offset ϕ. In plotting the information as a polar function of ϕ, we can compare the information function for various downsampling factors. Immediately, we see that the information can vary quite

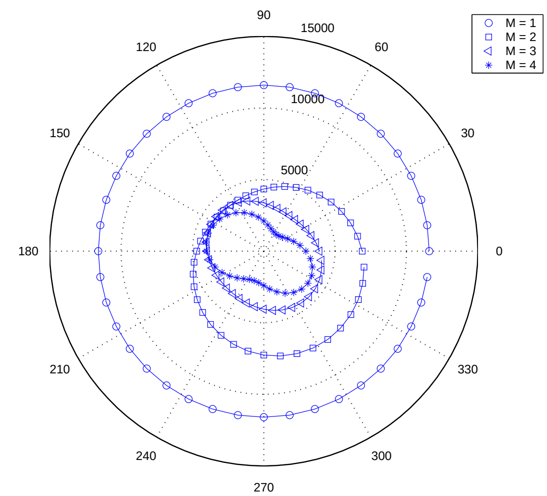

Fig. 2. Polar plot of $\mathbf{d}^T\mathbf{Q}(\phi)\mathbf{d}$ (in units of $\frac{gray\ levels^2}{pixel^2}$) verses ϕ (in degrees) for different downsampling factors.

dramatically for different sampling offsets ϕ when aliasing occurs ($M > 1$). Because the performance bound can vary so widely for different values of ϕ, it is important to explore the entire space of translations v when evaluating a particular registration algorithm. We note that this has generally not been the practice in the past, where typically algorithms are evaluated only for whole pixel motions (or in units of $\frac{1}{M}$ in the low-resolution image coordinates). Third, we note that the downsampling operation significantly reduces the overall amount of available information. As a rule of thumb, the downsampling operation reduces the overall information on the order of $\frac{1}{M}$.

We now examine the actual CR bound on overall estimation performance in estimating the set of motion parameters $\underline{\mathrm{v}}$. Because of the complicated structure of the CR bound, henceforth we compute the bounds numerically for a given signal, set of translations, and noise power. For example, Figure 3 shows the overall performance bound $T(\underline{\mathrm{v}})$, over the set of unknown motions for the signal shown in Figure 1. Each point in the plot indicates the performance bound for a set of $K+1$ frames with equally-spaced translations defined by $v_k = \frac{kM}{K+1}$ with a noise power of $\sigma^2 = 1$. We note that increasing the number of frames does not affect the performance bound for the non-aliased scenario when ($M = 1$). This suggests that an algorithm that performs pairwise registration could conceivably work as well as a more complicated algorithm which estimates a set of registration parameters using the entire collection of low resolution images. This is not the case, however, when the low resolution images contain aliasing. For downsampling factors greater than $M = 1$, we see that increasing the number of measured frames improves the overall performance bound.

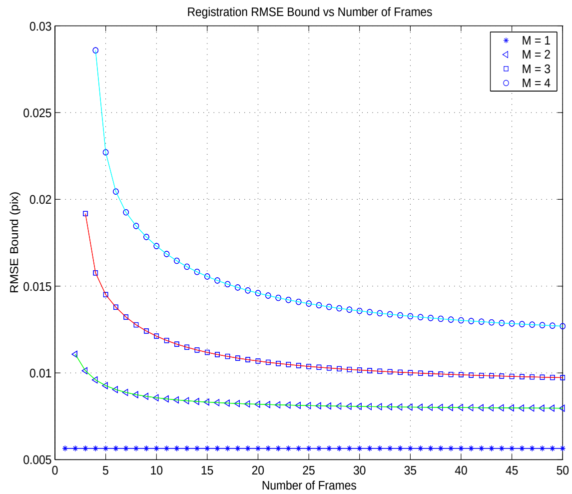

Fig. 3. Plot of the CR bound $T(\underline{v})$ vs number of frames $K + 1$ for equally spaced translations.

In some cases, the presence of additional frames cuts the overall performance bound in half. We can interpret this to mean that optimal aliased image registration algorithms must estimate the set of translations $\{v_k\}$ from a set of low resolution measurements in a joint fashion. Estimating translations in using subsets of the collection of measured images $\{\mathbf{y}_k\}$ will necessarily result in a poorer performance bound. We shall see an example of such performance degradation shortly. After a certain number of frames, however, the benefit of observing more low-resolution images does little to improve the average performance bound.

We now compare the estimator performance of the aliased image registration algorithm [7] with the corresponding CR bounds on multi-frame aliased image registration. The Stone et.al. algorithm [7] was specifically proposed to address the problem of registering a pair of aliased images. In deriving the algorithm, the authors make several heuristic observations which they use to motivate the algorithm. In particular, the algorithm applies a nonlinear weighting of zeros and ones (a mask) to prune away portions of the image spectrum where the negative effects of aliasing are assumed to worsen estimation performance. For our experiments, we used parameter settings recommended in [7].

We perform our experiments using the Tree image shown in Figure 4. We conduct experiments using an equally-spaced translation on a 2-D grid. The translations is defined by the set of ordered pairs $\{\ldots \frac{kM}{K+1} \ldots\} \times \{\ldots \frac{kM}{K+1} \ldots\}$ In order that the estimation problem be well conditioned, we use $K+1 = 8$ frames for $M = 2$ and $K + 1 = 16$ frames for $M = 3$. Such offset locations guarantee that the FIM is well conditioned for both downsampling factors.

We evaluated the estimator performances for SNR values ranging from 20 to 60 db. Both registration algorithms were applied in a pair-wise fashion assuming the same reference

Fig. 4. Tree test image.

frame. Figure 5 compares the performance of the two algorithms with the CR bound for the given set of images. Each point on the curve represents the value of $\overline{rmse}(\mathbf{v}_k)$ computed numerically for 500 Monte Carlo (MC) simluation runs. The

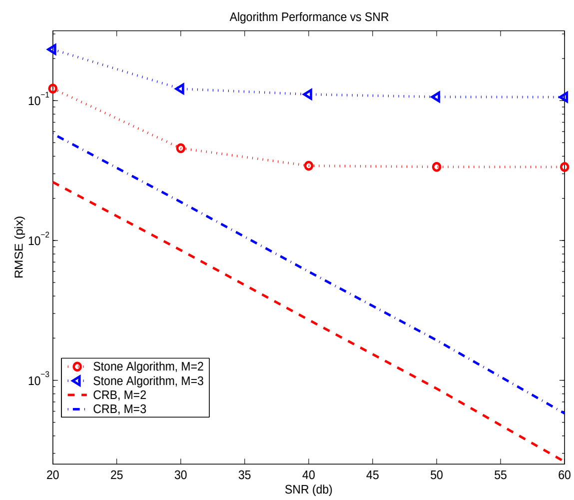

Fig. 5. Experimental $\overline{rmse}(\underline{v})$ versus CR bound for $M = 2$ and $M = 3$.

algorithm shows a flattening out of RMSE performance as SNR increases beyond 30 dB indicative of significant estimator bias. Presumably, this occurs due to the heuristic assumptions made while designing the algorithm. For a downsampling factor $M = 3$, the bias for the algorithm of Stone et. al. is greater than $\frac{1}{10}$ of a pixel. While such bias is highly dependent on the original image content, such estimator performance suggests that there is much work yet to do in the area of aliased image registration. Overall, we conclude from these experiments that the current approach to registering aliased images, utilizing a heuristically designed algorithm, leads to sub-optimal performance.

III. CR Bound on Multi-frame Image Reconstruction

In this section we analyze the CR bound on multi-frame image reconstruction performance. Much work has been done in the area of signal and image reconstruction from multiple aliased images including very comprehensive performance analysis [12]. Such work, however, has assumed that the sampling phase offsets were known prior to reconstruction. In this section, we examine the degradations when the motions or sampling offsets must be estimated from the observed data.

The FIM term associated with the image reconstruction problem is given by

$$\mathbf{J}_{\mathbf{zz}} = \frac{1}{\sigma^2}\left(\sum_{k=0}^{K}\mathbf{Q}(v_k)\right) \tag{7}$$

From (7) we see that the information pertaining solely to the unknown image $\mathbf{z}$ depends only on the motion parameters. It is primarily the condition number of $\mathbf{J}_{\mathbf{zz}}$ which is studied in papers such as [12]. In such works, it was shown that the condition number of a matrix of the form (7) is maximized with equally spaced sampling offsets on the high resolution *sampling grid*. Thus, were the sampling offsets known perfectly, the problem of image reconstruction would be a that of solving a linear system of equations. When the motions were equally spaced on the high resolution grid, one could expect a performance bound of the form $\frac{M\sigma^2}{K+1}$ independent of the signal $\mathbf{z}$.

When the motion parameters must also be estimated from the data, however, the performance bound depends on the unknown signal $\mathbf{z}$. We begin by analyzing the simple scenario where $M = 1$, or no downsampling (and hence no aliasing). The performance bound for this case characterizes the general behavior of the performance bound for $M > 1$. By way of the matrix-inversion lemma [13], we see that the general inverse $\mathbf{J}^{-1}(\mathbf{z})$ can be written as

$$\mathbf{J}^{-1}(\mathbf{z}) \quad = \quad \mathbf{J}_{\mathbf{zz}}^{-1} + \mathbf{J}_{\mathbf{zz}}^{-1}\mathbf{J}_{\mathbf{z}\underline{\mathrm{v}}}\mathbf{J}^{-1}(\underline{\mathrm{v}})\mathbf{J}_{\mathbf{z}\underline{\mathrm{v}}}^{T}\mathbf{J}_{\mathbf{zz}}^{-1} \tag{8}$$

When $M = 1$, we have

$$\mathbf{J}^{-1}(\mathbf{z}) \quad = \quad \frac{1}{K+1}\mathbf{I} + \frac{K}{(K+1)}\frac{\mathbf{d}\mathbf{d}^T}{\mathbf{d}^T\mathbf{d}} \tag{9}$$

Thus, we see that when $M = 1$, the form for $\mathbf{J}_{\mathbf{zz}}$ is independent of the translations. The second term of (9) is a rank 1, unit eigenvalue matrix composed of outer product of the spatial derivative signal $\mathbf{d}$. Such a term reflects the idea that image reconstruction (and later restoration) is more difficult in the textured regions. Essentially, this reflects the intuitive observation that errors in motion estimation will be most detrimental to image restoration in highly textured or high spatial frequency areas. It has been noted in the past that poor registration during multi-frame image reconstruction causes an edge-like feature to be distorted, creating jagged artifacts [1].

For example, the graph of Figure 6 shows the variance bound (diagonal of $\mathbf{J}^{-1}(\mathbf{z})$) for estimating the gray level for each pixel for the signal shown in the graph of Figure 1. The bound was calculated assuming four measured low resolution images with the translations $\{0.5,\ 1,\ 2\}$, a downsampling factor of $M = 3$ and noise power $\sigma^2 = 1$. Here, we show the bound in the spatial domain to simplify its interpretation. The per-pixel variance bound has two distinct characteristics. First, the sawtooth-like periodic function reflects the amount of measured data associated with each pixel location in the high resolution image. This term is independent of the signal $\mathbf{z}$ and depends only on the number of low-resolution images and their respective sampling offsets. Second, the spikes in performance bound correspond to the locations of the 'edges' or high-frequency detail in the original spatial domain signal $\mathbf{z}$. In these regions, image reconstruction performance is degraded by the need to estimate the motion parameters in a joint fashion.

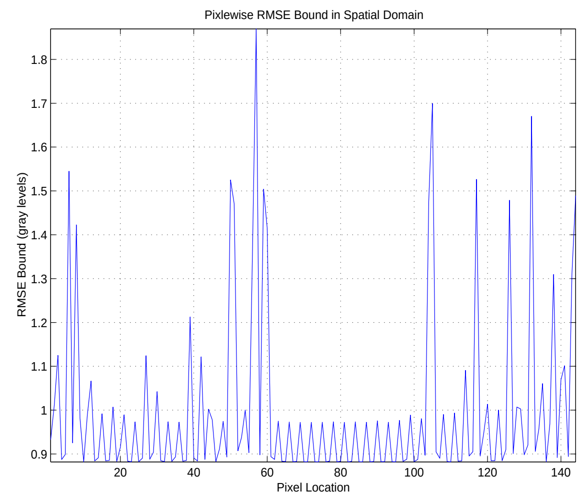

Fig. 6. Variance bounds on image reconstruction shown for every pixel.

To give an idea of the overall degradation in performance due to uncertainty about the motion parameters, we compare the complete CR bound $T(\mathbf{z})$ with a the performance bound under the conditions that the sampling offsets are known perfectly prior to image reconstruction. As mentioned earlier, for equally-spaced sampling offsets, this weak (optimistic) bound reduces to $\left(\frac{tr(\mathbf{J}_{\mathbf{zz}}^{-1})}{N_H}\right)^{\frac{1}{2}} = \frac{M\sigma^2}{K+1}$. In comparing these two performance bounds, we can quantify the expected degradation in performance when the motions must be estimated from the data. In Figure 7 we shows the degradation in the performance bound when the motions must be estimated from the low-resolution images for $M = 4$, $\sigma^2 = 1$ for our test signal. Here, we see that the loss of information can be quite significant. Relatively speaking, the performance ranges from 10 to 25 percent loss in gray level estimation accuracy as the number of available frames increase. We note, that such performance degradation is even more severe for the 2-D scenario when much more information is lost due to downsampling.

Finally, we would like to compare the performance of an actual multiframe image reconstruction algorithm with the derived bound. To do so, we implemented a two-step algorithm composed of first registering the collection of images using the method of [7] followed by a direct estimate of the image as

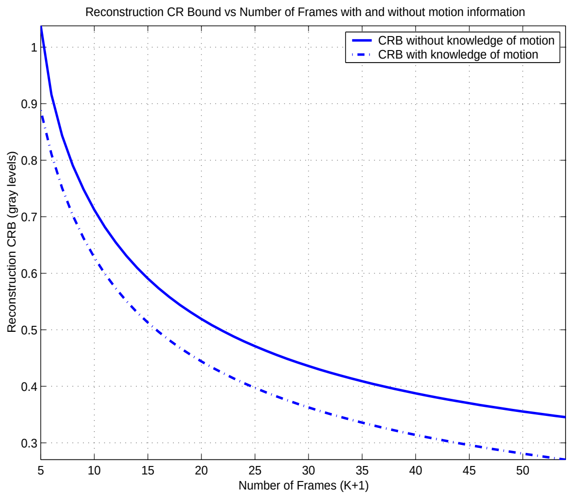

Fig. 7. Reconstruction CR bound $T(\mathbf{z})$ and the weak bound $T_{weak}(\mathbf{z})$ versus the number of frames $K+1$.

given by

$$\hat{\mathbf{z}} = \left(\sum_k \mathbf{Q}(\hat{v_k}) \right)^{-1} \left(\sum_k \mathbf{Q}(\hat{v_k})^T \mathbf{y}_k \right). \quad (10)$$

Again, the Tree image of Figure 4 was used as the signal and equally-spaced sampling offsets as in the previous section. Figure 8 compares the actual estimator performance with the CR bound as a function of the number of frames. Again, each point represents 500 MC runs at SNR values of 30, 40, and 50 dB. The downsampling factor is $M = 2$. Here, we see

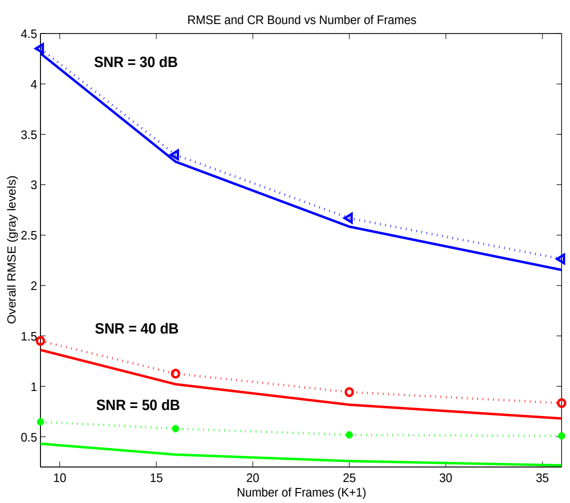

Fig. 8. Experimental $\overline{rmse}(\mathbf{z})$ (points) and CR bound (solid lines) versus the number of frames $K+1$.

that the CR bound predicts the actual estimator performance for SNR of 30 and 40 dB reasonably well. At this SNR, the reconstruction error due to mis-registration is masked by the error due to the additive noise. At 50dB, however, the bias of the registration algorithm prevents the actual estimator performance from tracking the CR bound as well as before. Overall, we observe that the CR bound offers an efficient mechanism for bounding and predicting actual estimator performance over a wide range of SNRs.

IV. Conclusion

In this paper, we have derived and explored the use of the Cramér-Rao inequality in bounding the MSE performance for the problem of multi-frame image reconstruction which is the backbone of the superresolution problem. We have shown for the case of translational motion how this problem naturally relates to the more general problem of super-resolution. We analyzed the relationships between the sub-problems of aliased image registration and image reconstruction and characterized the performance limits of each. We have compared modern estimators to these performance bounds. In doing so, we observe that the CR bound offers a useful means of bounding as well as predicting actual estimator performance. Moreover, we have shown the need for further algorithm development into the area of aliased image registration for multi-frame image reconstruction.

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