Optimum Follow the Leader Algorithm

Dima Kuzmin and Manfred K. Warmuth

University of California - Santa Cruz

Consider the following setting for an on-line algorithm (introduced in [FS97]) that learns from a set of experts: In trial t the algorithm chooses an expert with probability p_i^t. At the end of the trial a loss vector[1] $L^t \in [0,R]^n$ for the n experts is received and an expected loss of $\sum_i p_i^t L_i^t$ is incurred. A simple algorithm for this setting is the *Hedge* algorithm which uses the probabilities $p_i^t \sim \exp^{-\eta L_i^{<t}}$. This algorithm and its analysis is a simple reformulation of the randomized version of the Weighted Majority algorithm (WMR) [LW94] which was designed for the absolute loss. The total expected loss of the algorithm is close to the total loss of the best expert $L_* = \min_i L_i^{\leq T}$. That is, when the learning rate is optimally tuned based on L_*, R and n, then the total expected loss of the Hedge/WMR algorithm is at most

$$L_* + \sqrt{\mathbf{2}}\sqrt{L_* R \log n} + O(\log n).$$

The factor of $\sqrt{\mathbf{2}}$ is in some sense optimal [Vov97].

A new randomized algorithm for choosing the expert was given in [KV05]: *perturb* the losses of the experts by adding noise ν_i to $L_i^{<t}$ and then choose the expert with minimum perturbed loss. This *Following the Perturbed Leader* (FPL) algorithm has the same total expected loss bound except that the $\sqrt{\mathbf{2}}$ factor is replaced by **2**.

So the first question is whether there is an alternate way to perturb the losses in FPL which realizes WMR (with the optimal factor on the second term). In FPL the noise parameters of the additive noise only depend on the overall learning rate. But if you replace the entire loss $L_i^{<t}$ by a randomized loss depending on the learning rate **and** $L_i^{<t}$, then WMR can be realized:

Lemma 1. *Let Z_i be independent exponential random variables with parameters $\lambda_i = e^{-\eta L_i^{<t}}$ and $I = \arg\min_i Z_i$. Then $P(I = i) \sim \exp^{-\eta L_i^{<t}}$.*

Proof. $\min_i Z_i$ is an exponential random variable with parameter $\sum_i \lambda_i$ and $P(I = i) = \frac{\lambda_i}{\sum_j \lambda_j}$. □

However we know of no efficient implementation of WMR for the following representative application introduced in [TW03]: We have a directed graph with a source and a sink. The experts are the acyclic source to sink paths and the loss of a path is *additive* in the sense that it is the sum of the losses of its edges. The goal is again to incur loss close the best expert/path.

It is easy to implement the original FPL algorithm [KV05] (which has the worse constant): the loss of each edge is perturbed by some additive noise and

[1] It suffices to require that $(\max_i L_i^t - \min_i L_i^t) \leq R$.

the algorithm predicts with the path of minimum loss. However, we don't know how to implement WMR without computing the following type of quantity: the sum of acyclic paths from a start vertex to the sink. In [TW03] this problem is avoided by enlarging the set of path experts to *all* source to sink paths[2]. With the enlarged pool of experts, the path weights can be summed via dynamic programming.

So the natural open problem is whether there is a way to perturb the losses of the edges so that choosing the shortest path realizes WMR. That is, does there exist a distribution D, parameterized by L, satisfying the following two conditions.

1. If $Z_1 \sim D(L_1), \ldots, Z_n \sim D(L_n)$ are independent random variables, then
$$\mathrm{P}(\arg\min(Z_1, \ldots, Z_n) = i) \sim \exp^{-L_i}$$
2. If $Z_1 \sim D(L_1)$ and $Z_2 \sim D(L_2)$ are independent random variables, then
$$Z_1 + Z_2 \sim D(L_1 + L_2)$$

There are many distributions that satisfy one of the two properties (e.g. the exponential satisfies 1 and the gamma satisfies 2). The first condition seems to correspond to closure of the distribution under the minimum operation. Specifically, all the distributions that we know to satisfy 1, also satisfy $\min(Z_1, \ldots, Z_n) \sim D(e^{-L_1} + \ldots + e^{-L_n})$. We did not however establish formally whether closure under minimum and condition 1 are equivalent.

The solution has to avoid the following caveat. Consider for instance a case where losses of all edges (and paths) are zero. Then choosing a shortest path yields a random variable whose parameter is related to the number of paths. So if the parameter can be *accurately* estimated by sampling, then this solves a #P-complete problem [Val79].

Essentially, the original FPL is efficient if an expert with minimum perturbed loss can be found efficiently. Is the same true for Hedge/WMR which has the optimum constant before the square root term? Or, what is the best constant achievable by any algorithm that efficiently computes the minimum w.r.t. a perturbed loss?

[FS97] Yoav Freund and Robert E. Schapire. A decision-theoretic generalization of on-line learning and an application to boosting. *Journal of Computer and System Sciences*, 55(1):119–139, August 1997.

[KV05] Adam Kalai and Santosh Vempala. Efficient algorithms for online decision problems. *J. Computer System Sci*, 2005. To appear.

[LW94] N. Littlestone and M. K. Warmuth. The weighted majority algorithm. *Information and Computation*, 108(2):212–261, 1994.

[TW03] E. Takimoto and M. K. Warmuth. Path kernels and multiplicative updates. *Journal of Machine Learning Research*, 4:773–818, October 2003.

[Val79] Leslie Valiant. The complexity of enumeration and reliability problems. *SIAM Journal on Computing*, 8:410–421, 1979.

[Vov97] V. Vovk. A game of prediction with expert advice. *J. Computer System Sci*, 1997.

[2] It suffices to consider paths of length at most ℓ, where ℓ is the longest acyclic path from the source to the sink.