Learning rotations with little regret

Satyen Kale (Yahoo! Research)

Joint work with
Elad Hazan (IBM Almaden) and Manfred Warmuth (UCSC)
Input: pairs of unit vectors in $\mathbb{R}^n$: $(x_1, y_1), (x_2, y_2), \ldots, (x_T, y_T)$

Assumption: $y_t = Rx_t + \text{noise}$, where $R$ is an unknown rotation matrix

Problem: find “best-fit” rotation matrix for the data, i.e.

$$\arg \min_R \sum_t \|Rx_t - y_t\|^2$$
How to Solve the Batch Problem

- \[ \|Rx_t - y_t\|^2 = \|Rx_t\|^2 + \|y_t\|^2 - 2(y_t x_t^\top) \cdot R \]
  
  \[ = 2 - 2(y_t x_t^\top) \cdot R. \]

- \[ \text{arg min}_R \sum_t \|Rx_t - y_t\|^2 = \text{arg max}_R \sum_t y_t x_t^\top \cdot R \]

- Computing \( \text{arg max}_R M \cdot R \): “Wahba’s problem”
- Can be solved using SVD of \( M \)
Choose rot matrix $R_1$
Predict $R_1x_1$
$L_1(R_1) = \|R_1x_1 - y_1\|^2$

Choose rot matrix $R_2$
Predict $R_2x_2$
$L_2(R_2) = \|R_2x_2 - y_2\|^2$

Choose rot matrix $R_T$
Predict $R_TX_T$
$L_T(R_T) = \|R_TX_T - y_T\|^2$

Goal: Minimize regret:
Regret = $\sum_t L_t(R_t) - \min_R \sum_t L_t(R)$

Open problem
from COLT 2008
[Smith, Warmuth]
Rotation Matrices

- Rot matrix ≡ orthogonal matrix of determinant 1
- Set of rot matrices, $SO(n)$:
  - Non-convex: so online convex optimization techniques like gradient descent, exponentiated gradient, etc. don’t apply directly
  - Lie group with Lie algebra = set of all skew-symmetric matrices
  - Lie group gives universal representation for all Lie groups via a conformal embedding
Previous Work

- [Arora, NIPS ’09] using Lie group/Lie algebra structure

- Based on matrix exponentiated gradient: matrix exp maps Lie algebra to Lie group

- Deterministic algorithm

- $\Omega(T)$ lower bound on any such deterministic algorithm, so randomization is crucial
Assume for convenience that $n$ is even.

Bad example: $x_t = e_1$, $y_t = -R_t x_t$.

$L_t(R_t) = \|R_t x_t - y_t\|^2 = \|2y_t\|^2 = 4$. So total loss $= 4T$.

Since $n$ is even, both $I$, $-I$ are rot matrices, and
\[
\sum_t L_t(I) + L_t(-I) = \sum_t 2 \|y_t\|^2 + 2 \|x_t\|^2 = 4T.
\]

Hence, $\min_R \sum_t L_t(R) \leq 2T$.

So, Regret $\geq 2T$. 

Adversary can compute $R_t$ since alg is deterministic.
Our Results

- Randomized algorithm with expected regret $O(\sqrt{nL})$, where $L = \min_R \sum_t L_t(R)$

- Lower bound on regret of any online learning algorithm for choosing rot matrices of $\Omega(\sqrt{nT})$

- Using Hannan/Kalai-Vempala’s Follow-The-Perturbed-Leader technique based on linearity of loss function
Sample noise matrix $N$ with i.i.d entries distributed uniformly in $[-1/\eta, 1/\eta]$.

In round $t$, use $R_t = \arg \min_R \sum_{1}^{t-1} L_i(R) - N \cdot R$.

Thm [KV’05]: Regret $\leq O(n^{5/4} \sqrt{T})$. Using SVD solution to Wahba’s problem.
Optimal Algorithm: Follow-The-Spectrally-Perturbed-Leader (FSPL)

Sample \( n \) numbers \( \sigma_1, \sigma_2, ..., \sigma_n \) i.i.d. from the exponential distribution of density \( \eta \exp(-\eta \sigma) \)

Sample 2 orthogonal matrices \( U, V \) from the uniform Haar measure

Set \( N = U \Sigma V^T \), where \( \Sigma = \text{diag}(\sigma_1, \sigma_2, ..., \sigma_n) \).

In round \( t \), use \( R_t = \arg \min_R \sum_{i=1}^{t-1} L_i(R) - N \cdot R \).
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Set $N = U \Sigma V^T$, where

In round $t$, use $R_t = \arg \min_R \sum_{1}^{t-1} L_i(R) - N \cdot R$. 

E.g. using QR-decomposition of matrix with i.i.d. standard Gaussian entries
Optimal Algorithm: Follow-The-Spectrally-Perturbed-Leader (FSPL)

Sample $n$ numbers $\sigma_1, \sigma_2, \ldots, \sigma_n$ i.i.d. from the exponential distribution of density $\eta \exp(-\eta \sigma)$.

Sample $2$ orthogonal matrices $U, V$ from the uniform Haar measure.

Set $N = U \Sigma V^\top$, where $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$.

Effectively, we choose $N$ w.p. $\propto \exp(-\eta \|N\|_*)$, where $\|N\|_* = \text{trace norm, i.e. sum of singular values of } N$.

In round $t$, use $R_t = \arg \min_R \sum_{1}^{t-1} L_i(R) - N \cdot R$. 
Stability Lemma [KV’05]:
\[ E[\text{Regret}] \leq \sum_t E[L_t(R_t)] - E[L_t(R_{t+1})] + 2E[\|N\|_*] \]

Choose \( \eta = \frac{\sqrt{n}}{L} \), and we get
\[ E[\text{Regret}] \leq O\left(\sqrt{nL}\right). \]
\[ \sum_t E[L_t(R_t)] - E[L_t(R_{t+1})] \leq 2\eta T \]

- \( R_t = \text{arg max}_R \left( \sum_{1}^{t-1} y_i x_i^T + N \right) \cdot R \)
- \( R_{t+1} = \text{arg max}_R \left( \sum_{1}^{t} y_i x_i^T + N' \right) \cdot R \)

Re-randomization doesn’t change expected regret
\[ \sum_t E[L_t(R_t)] - E[L_t(R_{t+1})] \leq 2\eta T \]

- \( R_t = \arg \max_R (\sum_{i=1}^{t-1} y_i x_i^T + N) \cdot R \)
- \( R_{t+1} = \arg \max_R (\sum_1^t y_i x_i^T + N') \cdot R \)

- First sample \( N \), then set \( N' = N - y_t x_t^T \).
- Then \( R_t = R_{t+1} \), and so \( E_D[L_t(R_t)] - E_{D'}[L_t(R_{t+1})] = 0 \).

\( D = \text{dist of } N \),
\( D' = \text{dist of } N' \)
\[ \sum_t E[L_t(R_t)] - E[L_t(R_{t+1})] \leq 2\eta T \]

- \( R_t = \arg \max_R (\sum_{1}^{t-1} y_i x_i^T + N) \cdot R \)
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- However, \( \|D' - D\|_1 \leq \eta \).
- So \( E_D[L_t(R_{t+1})] - E_D[L_t(R_{t+1})] \leq 2\eta \).
\[
\sum_t E[L_t(R_t)] - E[L_t(R_{t+1})] \leq 2\eta T
\]

- \( R_t = \arg \max_R (\sum_1^{t-1} y_i x_i^T + N) \cdot R \)
- \( R_{t+1} = \arg \max_R (\sum_1^t y_i x_i^T + N') \cdot R \)

- First sample \( N \), then set \( N' = N - y_t x_t^T \).
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- However, \( \|D' - D\|_1 \leq \eta \).
- So \( E_{D'}[L_t(R_{t+1})] - E_{D}[L_t(R_{t+1})] \leq 2\eta \).

\[
\Pr_{D'}[N]/\Pr_D[N] \approx \exp(\pm \eta \|y_t x_t^T\|_*) \approx 1 \pm \eta.
\]
\[ E[\|N\|_*] = \frac{n}{\eta} \]

Because \( \sigma_i \) is drawn from the exponential distribution of density \( \eta \exp(-\eta \sigma) \)
Lower Bound on Any Algorithm

- Bad example: $x_t = e_{t \mod n}$, $y_t = \pm x_t$ w.p. $\frac{1}{2}$ each

- Opt rot matrix $R^* = \text{diag}(\text{sgn}(X_1), \ldots, \text{sgn}(X_n))$

  $X_i = \text{sum of } \pm \text{ signs over all } t \text{ s.t. } (t \mod n) = i.$

* ignoring $\det(R^*) = 1$ issue
**Lower Bound on Any Algorithm**

- **Bad example:** $x_t = e_{t \mod n}$, $y_t = \pm x_t$ w.p. $\frac{1}{2}$ each

- **Opt* rot matrix** $R^* = \text{diag}(\text{sgn}(X_1), \ldots, \text{sgn}(X_n))$

- **Expected total loss**
  
  $$2T - 2\sum_i E[|X_i|] \geq 2T - n \cdot \Omega(\sqrt{T/n}) = 2T - \Omega(\sqrt{nT})$$

- **But for any** $R_t$, $E[L_t(R_t)] = 2 - 2E[(y_t x_t^T) \bullet R_t] = 2$, and hence total expected loss of alg = $2T$.

- **So,** $E[\text{Regret}] \geq \Omega(\sqrt{nT})$.

* ignoring $\det(R^*) = 1$ issue
Conclusions and Future Work

- Optimal algorithm for online learning of rotations with regret $O(\sqrt{nL})$
- Based on FSPL

Open questions:
- Any other example of natural problems where FPL is the only known technique that works?

Thank you!