Accuracy Analysis for Correlation-Based Image Registration Algorithms

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Abstract – The behaviour of two well-known image registration techniques, namely the cross-correlation and the phase-correlation algorithms, for input signals corrupted by additive white noise, is studied in the frequency domain. Useful expressions of the displacement measurement error variance are presented, which allow for the analysis of the accuracy of both methods.

I. Introduction

Estimating the displacement (or *misregistration*) vector associated to the shifted version of a given signal is a fundamental issue in one-dimensional signal processing (radar), as well as in image processing and computer vision. Guidance systems, satellite multispectral aquisition, assembly operations, stereopsis, image sequences stabilization, television motion measurements, are practical fields of application of image registration algorithms.

An important feature is the possibility of accurate subpixel measurement, attainable only if the input images are correctly band-limited prior to sampling. On the other side, many factors, such as the sensor noise and geometrical image distortions, may lower the accuracy of any registration algorithm.

The optimal least squares solution to the registration problem is attained by the "traditional" cross-correlation technique. The effectiveness of such an algorithm for the detection of large area displacement, and its robustness to noise, are well known [1],[2]. Interpolation procedures for measuring subpixel displacements have been studied extensively [3]–[5].

One disadvantage of the direct cross-correlation technique, is that sometimes the presence of secondary maxima makes it difficult to select the actual maximum; moreover, the cross-correlation peak may be rather broad, so that the accuracy in determining its position may be poor [6].

A way of sharpening the cross-correlation peak is to "whiten" the input signals. A statistical approach to the whitening problem was proposed by Pratt [6], while a deterministic technique is the basis of the so-called *phase-correlation* technique [7]–[10]. Such a method can be shown to be equivalent to adopting a whitening filter of FIR type, designed by means of the frequency sampling technique.

The aim of this work is testing the robustness of the cross-correlation and phase-correlation techniques to small amounts of additive white noise. Subpixel interpolation is obtained by means of the efficient interpolation algorithm of [7]. Useful expressions of the measurement error (as the difference between the peak position and the "true" displacement vector) are presented, which are instrumental for the worst-case analysis, as well as for the estimation of the error variance.

For the sake of notation simplicity, the analysis is developed only for one-dimensional signals. The results may be easily extended to the two-dimensional case.

II. Cross-correlation and Phase-correlation Algorithms

Let $x_1(t)$ and $x_2(t)$ be the input signals, sampled with period T, and assume the input sequences are made of L samples each. If $X_1(k) = |X_1(k)|e^{j\varphi_1(k)}$, $X_2(k) = |X_2(k)|e^{j\varphi_2(k)}$ are the corresponding DFT sequences, the direct cross-correlation [2] and the phase-correlation algorithms [8] may be stated as follows:

Cross-correlation: Find the position of the maximum of sequence

$$\mathrm{DFT}^{-1}\left(X_1^*(k)X_2(k)\right) \tag{1}$$

Phase-correlation: Find the position of the maximum of sequence

$$\mathrm{DFT}^{-1}\left(\frac{X_1^*(k)X_2(k)}{|X_1^*(k)X_2(k)|}\right) \tag{2}$$

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As a matter of fact, (1) represents the expression of the *circular* cross-correlation of $x_1(nT)$ and $x_2(nT)$. In order to get the "true" cross-correlation sequence, it is necessary to pad out the input sequences by a suitable set of zero-valued samples [2].

Note the (2) represents the circular cross-correlation of two sequences, obtained by filtering $x_1(nT)$ and $x_2(nT)$ by "whitening" zero-phase FIR filters $H_1\left(e^{j\omega}\right)$ and $H_2\left(e^{j\omega}\right)$ respectively. Their length is L, and they satisfy the "whitening" conditions

$$H_i\left(e^{j\omega_k}\right) = 1/\left|X_i(k)\right| \ , \ i = 1,2 \ , \ \omega_k = \frac{2\pi k}{L} \qquad (3)$$

An effective procedure for interpolating the cross-correlation sequence (1) and the phase-correlation sequence (2), in order to get subpixel registration accuracy (virtually, to any extent), is that of padding out the sequences which are the arguments of the DFT^{-1} operator in (1) and (2) by a suitable set of zero-valued samples [7]. Consider for example expression (1), which yields L samples, spaced by period T, of the circular cross-correlation of sequences $x_1(nT)$ and $x_2(nT)$. Assume signals $x_1(t)$ and $x_2(t)$ are correctly band-limited. For obtaining the values of the "continuous" cross-correlation function of $x_1(t)$ and $x_2(t)$ at $nT/2$, one has to pad out sequence $X_1^*(k)X_2(k)$ with L zero-valued samples before performing the inverse DFT operation.

Having this notion in mind, the accuracy analysis will be performed considering the "continuous" cross-correlation or phase-correlation function, respectively obtained by interpolating (1) or (2).

Assuming the input signals energies are normalized to the unit, by means of the Parseval theorem it can be seen [10] that the cross-correlation and phase-correlation algorithms are equivalent to the following procedures in the frequency domain:

<u>Cross-correlation</u>: Find $\tau \in R$ which minimizes form

$$\sum_{k=0}^{\frac{L}{2}-1} \left| X_1^*(k)X_2(k) - e^{-j2\pi\frac{k}{L}\frac{\tau}{T}} \right|^2 \qquad (4)$$

<u>Phase-correlation</u>: Find $\tau \in R$ which minimizes form

$$\sum_{k=0}^{\frac{L}{2}-1} \left| e^{j(\varphi_2(k)-\varphi_1(k))} - e^{-j2\pi\frac{k}{L}\frac{\tau}{T}} \right|^2 \qquad (5)$$

The minima of (4) and (5) correspond to the zero-crossing of their corresponding derivatives with respect to τ. Therefore, the estimated displacement τ_0 is a solution of equation

$$\sum_{k=0}^{\frac{L}{2}-1} k\left|X_1(k)\right|\left|X_2(k)\right| \sin\left(\varphi_2(k) - \varphi_1(k) + 2\pi\frac{k}{L}\frac{\tau}{T}\right) = 0 \qquad (6)$$

for the direct cross-correlation case, and of equation

$$\sum_{k=0}^{\frac{L}{2}-1} k \sin\left(\varphi_2(k) - \varphi_1(k) + 2\pi\frac{k}{L}\frac{\tau}{T}\right) = 0 \qquad (7)$$

for the phase-correlation case.

III. Accuracy Analysis

In the "ideal case", $x_2(t)$ is a shifted version of $x_1(t)$:

$$x_2(t) = x_1(t - t_0) \ \text{ for some } t_0 \qquad (8)$$
$$X_2(k) = X_1(k)e^{-j2\pi\frac{k}{L}\frac{t_0}{T}}$$

therefore the solution of both (6) and (7) is $\tau_0 = t_0$.

In practice, condition (8) is hardly met, due to many causes (windowing, geometric distorsion, noise, etc.). In this paper, we consider the effect of gaussian additive white noise $w(t)$, uncorrelated with the input signals:

$$x_2(t) = x_1(t - t_0) + w(t) \qquad (9)$$

Let $W(k)$ be the DFT sequence of $w(nT)$, and put

$$\varphi_n(k) = \varphi_2(k) + 2\pi\frac{k}{L}\frac{t_0}{T} \qquad (10)$$

Term $\varphi_n(k)$ will be referred to as the *phase noise*.

Under the hypothesis of small noise variance, it is possible to linearize equations (6) and (7) in the neighbourhood of t_0. Approximate expressions for the registration error $\Delta_0 = \tau_0 - t_0$ are therefore:

<u>Cross-correlation</u>

$$\Delta_0 \approx \frac{LT}{2\pi}\frac{\sum_{k=1}^{\frac{L}{2}-1} k\left|X_1(k)\right|^2(1+\rho(k))\varphi_n(k)}{\sum_{l=1}^{\frac{L}{2}-1} l^2\left|X_1(l)\right|^2(1+\rho(k))} \qquad (11)$$

<u>Phase-correlation</u>

$$\Delta_0 \approx \frac{LT}{2\pi}\frac{\sum_{k=1}^{\frac{L}{2}-1} k\varphi_n(k)}{\sum_{l=1}^{\frac{L}{2}-1} l^2} \qquad (12)$$

whith $\rho(k) = \left(\left|X_2(k)\right| - \left|X_1(k)\right|\right)/\left|X_1(k)\right|$. Since $\left|\rho(k)\right| \leq \left|W(k)\right|/\left|X_1(k)\right|$, if the DFT magnitude of the noise samples is small with respect to the corresponding values $X_1(k)$ (which may be assumed to hold true in the case of small noise variance), terms $\rho(k)$ may be neglected. Therefore a simpler expression for the registration error is:

Cross-correlation

$$\Delta_0 \approx \frac{LT}{2\pi} \frac{\sum_{k=1}^{\frac{L}{2}-1} k\,|X_1(k)|^2 \varphi_n(k)}{\sum_{l=1}^{\frac{L}{2}-1} l^2\,|X_1(l)|^2} \tag{13}$$

Equations (13) and (12) are instrumental for the accuracy analysis. We first give the guidelines for the worst-case analysis. It is easy to see that

$$|\varphi_n(k)| \leq \arcsin\left(\frac{|W(k)|}{|X_1(k)|}\right) \tag{14}$$

Under the previously stated hypothesis ($|W(k)| \ll |X_1(k)|$), the approximation

$$\arcsin\left(\frac{|W(k)|}{|X_1(k)|}\right) \approx \frac{|W(k)|}{|X_1(k)|} \tag{15}$$

may be used. Hence, from (13) and (12) it is possible to derive upper bounds for the measurement error magnitude:

Cross-correlation

$$|\Delta_0| \leq \frac{LT}{2\pi} \frac{\sum_{k=1}^{\frac{L}{2}-1} k\,|X_1(k)|\,|W(k)|}{\sum_{l=1}^{\frac{L}{2}-1} l^2\,|X_1(l)|^2} \tag{16}$$

Phase-correlation

$$\Delta_0 \approx \frac{LT}{2\pi} \frac{\sum_{k=1}^{\frac{L}{2}-1} k\frac{|W(k)|}{|X_1(k)|}}{\sum_{l=1}^{\frac{L}{2}-1} l^2} \tag{17}$$

We now consider the statistical approach, in order to achieve an expression of the variance of Δ_0 (as in [11]). Under the hypothesis $w(t)$ is white and gaussian, one can show that the random variables $\varphi_n(k)$ are uncorrelated. Since $w(t)$ is assumed to be uncorrelated with the (deterministic) input signals, it is possible to obtain simple expressions for the registration error variance $\sigma^2_{\Delta 0}$:

Cross-correlation

$$\sigma^2_{\Delta 0} = \left(\frac{LT}{2\pi}\right)^2 \frac{\sum_{k=1}^{\frac{L}{2}-1} k^2\,|X_1(k)|^4\,\sigma^2_{\varphi n(k)}}{\left(\sum_{l=1}^{\frac{L}{2}-1} l^2\,|X_1(l)|^2\right)^2} \tag{18}$$

Phase-correlation

$$\sigma^2_{\Delta 0} = \left(\frac{LT}{2\pi}\right)^2 \frac{\sum_{k=1}^{\frac{L}{2}-1} k^2 \sigma^2_{\varphi n(k)}}{\left(\sum_{l=1}^{\frac{L}{2}-1} l^2\right)^2} \tag{19}$$

where $\sigma^2_{\varphi n(k)}$ is the variance of $\varphi_n(k)$. Note that the values $\sigma^2_{\varphi n(k)}$ are dependent on $X_1(k)$.

Expressions (18) and (19) allow for the easy estimation of the displacement measurement error variance in both methods, for a given sequence $x_1(nT)$, in light of the fact that one can easily relate terms $\sigma^2_{\varphi n(k)}$ with the noise variance σ^2_w and the DFT sample values $X_1(k)$.

IV. Results and Final Remarks

By means of expressions (16) and (17) (worst-case analysis) or expressions (18) and (19) (statistical analysis), it is possible to check the sensitivity of the cross-correlation and phase-correlation algorithms with repect to small amount of additive white gaussian noise. Such an analysis may be adopted to estimate the tolerance in determining the displacement between two images, as well as to compare the two methods with respect to the measuring accuracy.

The analyses have been carried out both for low-pass and high-pass inputs. The main results are:

- Both methods are more sensitive to noise for low-pass inputs (blurred images) than they are for high-pass inputs (textured, sharp images)
- Phase-correlation technique is more sensitive to noise than direct cross-correlation is, both for low-pass and high-pass inputs

The latter result is of significance, since it states that increasing the robustness in detecting the correlation maximum (by sharpening the correlation peaks), comes at the expense of increased sensibility to noise of the computed maximum position.

References

[1] J.K. Aggarwal, L.S. Davis and W.N. Martin, "Correspondence Processes in Dynamic Scene Analysis", *Proceedings of the IEEE*, vol. 69, no. 5, pp. 562-571, 1981

[2] P.E. Anuta, "Spatial Registration of Multispectral and Multitemporal Digital Imagery Using Fast Fourier Transform Techniques", *IEEE Trans. Geosci. Electron.*, vol. 8, no. 4, pp. 353–368, 1970

[3] V.N. Dvornichenko, "Bounds on (Deterministic) Correlation Function with Application to Registration", *IEEE Trans. Pattern Anal. Machine Intell.*, vol. 5, no. 2, pp. 206–213, 1983

[4] Q. Tian and M.N. Huhns, "Algorithms for Subpixel Registration", *Computer Vision, Graphics and Image Processing*, 35, pp.220–233, 1986

[5] J.A. Parker, R.V. Kenyon, D.E. Troxel, "Comparison of Interpolating Methods for Image Resampling", *IEEE Trans. Med. Imag.*, vol. 2, no. 1, pp. 31–39, 1983

[6] W.K. Pratt, "Correlation Techniques for Image Registration", *IEEE Trans. Aerosp. Electron. Syst.*, vol. 10, no. 3, pp. 353–358, 1974,

[7] J.J. Pearson, D.C. Hines, Jr., S. Golosman, C.D. Kuglin, "Video-rate Image Correlation Processor", *SPIE* Vol. 119 *Application of Digital Image Processing (IOCC 1977)*, pp. 197–205

[8] C.D. Kuglin, A.F. Blumenthal, J.J. Pearson, "Map-matching Techniques for Terminal Guidance Using Fourier Phase information", *SPIE* Vol. 186 *Digital Processing of Aerial Images (1979)*, pp. 21–29

[9] G.A. Thomas, B.A. Hons, "Television Motion Measurement for DATV and Other Applications", *BBC Research Department* Report No. BBC RD 1987/11

[10] S. Alliney and C. Morandi, "Digital Image Registration Using Projections", *IEEE Trans. Pattern Anal. Machine Intell.*, vol. 8, no. 2, pp. 222–233, 1986

[11] C.D. McGillem, M. Svedlow, "Image Registration Error Variance as a Measure of Overlay Quality", *IEEE Trans. Geosci. Electron.*, vol. 14, no. 1, pp. 44–49, 1976