Cost-Per-Impression and Cost-Per-Action Pricing in Display Advertising with Risk Preferences

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Online advertising has grown over the past decade to over $26 billion in recorded revenue in 2010, IAB (2010). The revenues generated are based on different pricing models that can be fundamentally grouped into two types: cost per (thousand) impressions (CPM) and cost per action (CPA), where an action can be a click, signing up with the advertiser, a sale, or any other measurable outcome. A web publisher generating revenues by selling advertising space on its site can offer either a CPM or CPA contract.

Web publishers and advertisers typically use the action probability (the so-called click-through rate in the case of cost-per-click) to convert between CPM and CPA prices. This simple conversion rule assumes that both parties are risk neutral. However, it is well known that publishers usually prefer CPM pricing as they find it less risky while advertisers prefer CPA pricing, which is performance based. Our paper is motivated by this misalignment between the two parties. We introduce a new conversion rule that takes risk into account and explains the preferences observed in practice. We analyze the conditions under which the two parties agree on each contract type, under the assumption that both players are risk averse. We then explore several natural variations to the core conversion rules, including the cases where action probabilities are uncertain and the contracts take different forms.

1. Introduction

The Internet and advances in mobile technology have altered advertising strategy significantly. Advertisers and publishers have both benefited from effective tracking of customer responses, allowing for outcome-based pricing.

Online advertising can be divided between sponsored search advertising and display advertising. Sponsored search advertising involves ads being posted on search engine websites related to the search queries performed, whereas display advertising involves banners posted on web publishers’ websites. The pricing of sponsored search ads is quite well established with prices mostly charged per action through auctions, whereas pricing of display ads is less developed and simple approaches are often used. In this paper we focus on display advertising.

Many publishers offer both Cost-Per-Impression (CPM) prices and Cost-Per-Action (CPA) prices for display advertising. Advertisers, especially those that are keen on driving traffic to their website, usually prefer CPA pricing, which is performance based. However, publishers often prefer CPM pricing because it provides a guaranteed income given the traffic to the site (the ad inventory).
Currently, a simple conversion rule is used in industry to convert between CPM and CPA prices based on scaling by the action probability (the click-through probability in the case of cost-per-click), which leads to an equivalent expected payment between the two contracts. Given the existing misalignment in the preferences of publishers for CPM contracts and advertisers for CPA contracts, it is evident that this simple conversion rule does not capture the true equivalence between these two pricing schemes. Rather, a risk premium should be charged for the riskier contract.

This paper is motivated by this misalignment. We consider a web publisher displaying ads on its website and an advertiser wanting to post an ad campaign. Both CPM and CPA contracts are considered. We develop a new conversion rule between the CPM and CPA prices that takes into account the risk associated with the two contracts and confirms the preference existing in the industry. Furthermore, we determine the conditions under which the two parties agree on a contract and show how it depends on the action probability and the risk attitude. The analysis is done under some standard utility models, which reflect risk aversion by both parties. We then extend this model to allow for common dynamics where the contracts and prior knowledge take different forms.

To our knowledge, this is the first paper to provide a conversion rule between CPM and CPA contracts based on the risk preferences of both advertisers and publishers. It is also the first paper to explicitly recognize the link between the distribution of individual advertising outcomes, the overall distribution of a series of ads, and the corresponding utility for both parties. We show that sometimes a mixed contract can be preferred by both parties over a CPA or CPM contract. We also analyze the role of uncertainty, and find that lack of knowledge of the action probabilities can make it very difficult for parties to agree on either contract. These contributions and insights make the negotiating of advertising contracts a much more scientific process, giving advertisers and publishers a better understanding of their own preferences and negotiation position. This will help them to decide between offered contracts and set the prices they are willing to accept.

This paper is structured as follows. First, we review the existing literature in Section 2, and introduce the model, including the role of utility functions and risk preferences in online advertising in Section 3. We analyze different contract structures to find the preferred contracts under different utility combinations. Next, we consider alternate contract structures including mixed contracts and commit-to-action contracts in Section 4. In Section 5 we analyze the role of control and uncertainty of the action probabilities. We conclude with Section 6.

2. Literature Review

Extensive research has been done on multiple topics in online advertising. Ha (2008) provides an overview of the research published in advertising journals and Evans (2008) summarizes the economics of the online advertising industry. Our paper focuses on comparing the different pricing schemes and Novak and Hoffman (2000) provide an overview of the commonly used ones.

Our research involves display advertising with, e.g., banners posted on webpages. However, sponsored search advertising is a research area with a large body of literature (see, for example, Ghose and Yang (2009), Edelman et al. (2007) and references within).

Scheduling of online ads is a popular topic. Adler et al. (2002) are among the first to address this problem and provide an efficient algorithm for their problem, which is a variant of the bin packing problem. Kumar et al. (2006) provide a good overview of the literature on online ad scheduling and propose a hybrid genetic algorithm that performs better than existing ones.

Our paper focuses on the comparison between CPM and CPA pricing schemes. The papers closest to ours in terms of content are Mangáni (2003) and Fjell (2008). Mangáni (2003) compares the expected revenues from CPM and CPA pricing schemes using a simple deterministic model from the perspective of the publisher. The simple conversion rule used in practice is derived from the revenue maximization model presented. The risk associated with actions is not considered.
Mangàni (2003) concludes that the ratio between the amount of CPM and CPA contracts depends on the elasticity of visits with respect to the amount of advertising as well as on the elasticity of actions with respect to the amount of advertising. Fjell (2008) extends the work of Mangàni (2003) and points out that a publisher should always choose to either charge according to a CPM or a CPA scheme but not a mixture of both. The only case where a mixture is optimal is when the common conversion rule holds, which he considers a strict assumption. Our new conversion rule, which takes risk into account, can be considered to support Fjell’s reasoning for using either a CPM or a CPA pricing scheme. However, we show that a mixed contract, where the publisher charges for both impressions and actions can be optimal. Karmarkar and Dutta (2010) consider a multi-period version of the problem analyzed in Mangàni (2003) and Fjell (2008), again based on risk neutral settings.

Several papers have considered demand and supply uncertainty with risk neutral decision making. Araman and Fridgersdottir (2011) develop a pricing framework for web publishers facing uncertain demand and uncertain advertising inventory (number of impressions). Using a novel steady-state queuing model, they determine the optimal prices to quote to heterogeneous advertisers. Building on their work, Fridgersdottir and Najafi-Asadolahi (2010) characterize the optimal pricing strategy when advertisers are impatient, which results in new types of queuing models. Both these papers consider only CPM pricing. Fridgersdottir and Najafi-Asadolahi (2011) consider optimal CPA pricing strategies with comparison between the two pricing models. Roels and Fridgersdottir (2009) consider dynamic admission control and delivery of advertising contracts over a finite horizon. The problem is formulated as a dynamic program and a Certainty Equivalent Control heuristic is proposed and tested.

Dellarocas (2010) focuses on an interesting feature of CPA pricing. He shows that advertisers advertising under a CPA scheme distort the prices of their goods compared to the optimal prices in a CPM setting, leading to a reduction in the social welfare, joint profit (with the publisher), and consumer surplus. We do not take this feature into account but focus on the trade-off between the two pricing schemes taking risk into account.

It will become apparent that the action probability (or the click-through rate in the case of cost-per-click) plays an important role in our results. Chickering and Heckerman (2003) address the problem of maximizing the click-through rate given inventory constraints.

3. Model

We consider the interaction between a web publisher and an advertiser when agreeing on showing an ad to \(N\) visitors to the publisher’s website. For the sake of simplicity we assume one webpage with a single banner slot, but the results can be generalized to multiple web pages with multiple slots.

We characterize the contracts by the following parameters. Firstly, \(r\) is the probability of an action when an ad is shown (the click-through rate in the case of cost-per-click), and \(X\) is the (random) number of actions resulting from the \(N\) impressions. We denote \(p_a\) as the price per action - the revenue (cost) to the web publisher (advertiser) per action if a CPA contract is used, and \(p_i\) as price per impression - the revenue (cost) to the web publisher (advertiser) per impression if a CPM contract is used. In addition, we denote \(v_a\) and \(v_i\) to be the value to the advertiser of a single action and impression, respectively. For example \(v_a\) may represent the profit associated with each action, whereas \(v_i\) may be the value an advertiser places on exposure of their brand (a similar approach to advertising through more traditional advertising channels). Figure 1 illustrates the value and payoff functions.

For a CPM contract the advertiser will pay \(p_i N\) to the publisher, whereas for a CPA contract they will pay \(p_a X\). If the contract is for a single impression, \(N = 1\), then \(X\) is a bernoulli random variable with rate \(r\), but if \(N > 1\) then \(X\) is a binomial random variable with parameters \((N,p)\).
This assumes that each visitor has identical and independent action probabilities, which remain constant over time.

3.1. Contract Options

Web publishers often offer both CPM and CPA contracts. Advertisers who can choose between the two contracts usually prefer CPA contracts as in those they only pay for what they receive, while publishers often prefer the security of a CPM contract. Given this difference in preferences, we seek to understand when a publisher and an advertiser agree on a contract.

Let us consider the risk factors associated with the two contracts and two players. First, we focus on the web publisher. Under a CPM contract they generate deterministic revenues (given their ad inventory of $N$ visitors) while under a CPA contract the revenues are random, depending on the number of actions.

The advertiser’s setting is slightly more complicated depending on his goal. When focusing on raising brand awareness he sometimes only cares about how many viewers the ad is displayed to. In this setting he does not incur any risk associated with the value derived from a CPM contract nor the cost. However, in the case of a CPA contract there is risk associated with the cost.

If the advertiser’s goal is to drive traffic to his website, e.g., to sell a product or register a new customer, then it is not enough to display the ad, rather an action is needed to generate value. In the case of a CPM contract the value generated is uncertain while the cost is deterministic. However, for a CPA contract both the value and the cost are uncertain, driven by how many viewers respond with the appropriate action (see Figure 1).

The utility function of a publisher or advertiser reflects their risk preference. For online advertising, there are several scenarios, which would lead to variations in the form of the utility function. These functions deal with the relative importance of variation or risk in either the profit or the cost of a contract. For example, an advertiser may be concerned with the potential variation in cost, and the level of concern may or may not be linked to how the value earned from the actions in the contract are correlated to the fluctuation in cost.
Here we focus on two well known forms of utility function to represent the core values of the advertiser and publisher. The exponential utility function represents constant absolute risk aversion, whereas the piecewise linear function can be used to represent different risk aversion rates for different outcome values ranges.

In general, an advertiser will seek to maximize expected utility for contract $C$, which is given by

$$E \left[ U_A(N, X, C) \right] = \sum_{k=0}^{N} P[X = k] U_A(N, X, C)$$

$$= \sum_{k=0}^{N} \frac{N!}{k!(N-k)!} r^k (1-r)^{N-k} U_A(N, X, C),$$

because the outcomes follow a binomial distribution. We analyze when each of the two players, the advertiser and publisher, would prefer one contract structure over the other, and for what prices they are indifferent between the two. An advertiser or publisher is indifferent when the expected utility of the CPA and CPM contracts is the same.

### 3.2. Exponential utility functions

Exponential utility functions are one of the most well understood representations of risk averse utility, corresponding to the situation where a player shows constant absolute risk aversion. They are popular for analyzing financial decisions in the presence of uncertainty because they are equivalent to the mean-variance utility function, which is represented as the difference between the expected value and a multiple of the variance for that random variable.

#### 3.2.1. The Publisher’s Utility

The publisher receives revenue from either impressions or actions, depending on the contract choice. We assume that the publisher has an exponential utility function where the parameter $\gamma_P \geq 0$ represents the strength of their risk aversion. For the two contract types, the publisher has the following utility functions, which depend on the number of impressions and actions:

$$U_P(N, X, CPM) = 1 - e^{-\gamma_P N p_i}$$

$$U_P(N, X, CPA) = 1 - e^{-\gamma_P p_a X}.$$  

To analyze preferences between these two contracts, we take advantage of two important relationships amongst binomial, normal, and exponential distributions. When $N$ is large, the binomial distribution with parameters $N$ and $r$ converges to a normal distribution, with mean $Nr$ and variance $Nr(1-r)$. Further, when $X$ has a normal distribution with parameters $(\mu, \sigma^2)$, then

$$E \left[ 1 - e^{-\gamma X} \right] = 1 - e^{(-\gamma \mu + \frac{\gamma^2 \sigma^2}{2})}. \quad (1)$$

It is important to note that the CPM contract has nonnegative utility when $p_i \geq 0$, and the CPA contract has nonnegative expected utility when $0 \leq p_a \leq \frac{2}{\gamma_P (1-r)}$, based on Equation (1).

When $N$ is large, the publisher will prefer the CPM contract over the CPA contract if and only if

$$E [U_P(N, X, CPM)] \geq E [U_P(N, X, CPA)]$$

$$1 - e^{-\gamma_P N p_i} \geq E \left[ 1 - e^{(-\gamma_P p_a X)} \right]$$

$$p_i \geq p_a r - \frac{\gamma_P}{2} r (1-r) p_a^2. \quad (2)$$

We see that the publisher is indifferent between the two contracts when Equation (2) holds with equality, and we refer to the set of the $(p_a, p_i)$ points, which satisfy the equality, as the publisher’s indifference curve.
Note that for any $\gamma_p > 0$, the indifference curve is strictly dominated by the line $p_i = p_ar$ (the commonly used translation in industry), which implies that the publisher would always prefer a CPM contract if the two contracts were offered with $(p_i, p_a)$ values based on the transformation $p_i = r p_a$. We further observe that the indifference curve diverges from the line $p_i = r p_a$, meaning that the commonly used transformation becomes less accurate as the contract price increases. Further, for the new transformation structured in Equation (2), as $\gamma_p$ increases, the threshold for $p_i$ at which the publisher prefers CPM for a fixed value of $p_a$ decreases.

The publisher's indifference curve of $p_i(p_a)$ is concave (see Figure 2). Hence, the higher the impression price is, the more the action price needs to increase by in order to compensate for an impression price increase. This reflects the more risky nature of the CPA contract from the publisher's perspective.

### 3.2.2. The Advertiser’s Utility

We assume that the advertiser is also risk averse, with an exponential utility function for both impressions and actions, and a risk parameter $\gamma_A \geq 0$. We consider the two important cases.

1. First, we assume the advertiser gains value from impressions and actions for both CPM and CPA contracts. The value generated for both contracts is the same while the payments differ. The utility of a contract, which delivers $N$ impressions and $X$ actions for each of the contract options is given by

$$U_A(N, X, CPM) = 1 - e^{-\gamma_A[(v_i - p_i)N + v_aX]}$$

$$U_A(N, X, CPA) = 1 - e^{-\gamma_A[v_iN + (v_a - p_a)X]}$$

This is a common case probably more common than the second case below, because usually some value is attained through impressions independent of whether or not there is an action, and some additional value is usually attained from an action. Even if value is primarily from one or the other, one can set $v_a = 0$ or $v_i = 0$ to reflect this. The second case below considers the case where the values themselves depend on the contract choice.

2. In the second case, the advertiser gains value only for impressions in the case of a CPM contract and gains value only for actions in the case of a CPA contract. Hence, $v_a = 0$ for a CPM contract, and $v_i = 0$ for a CPA contract.

$$U_A(N, X, CPM) = 1 - e^{-\gamma_A(v_i - p_i)N}$$

$$U_A(N, X, CPA) = 1 - e^{-\gamma_A(v_a - p_a)X}$$

In these utility functions, the advertiser gains utility from either impressions or actions, but not both. This could arise, for example, when an advertiser is considering two different advertising campaigns. The first campaign is simply designed to increase brand awareness (and may not even have an action associated with it), whereas the second campaign is designed to attract a particular action (such as visitors to a website or signing up for a service). It could also arise when an advertising agency is rewarded differently depending on the contract type. While we expect that these situations are less likely than the first case (where one can set $v_a = 0$ or $v_i = 0$ for both contracts), they are sufficiently interesting to warrant investigation.

Observe that for the first case, the utility values for both contracts depend on the probability distribution for $X$, whereas for the second case the utility for a CPM contract is fixed by $N$ and the values of $v_i$ and $p_i$.

Using these relationships, we examine the two cases above to find the preferences. In particular, we are interested in finding values for the parameters $(p, N, \gamma_A, v_i, v_a, p_i, p_a)$ for which the advertiser would have a preference between the contracts, and how these lead to agreement with the publisher.
1. Let us first examine the prices at which the advertiser’s expected utility is nonnegative. For CPM, we must have

\[ p_i \leq v_i + v_a p - \frac{\gamma A}{2} v_a^2 r (1 - r) \]  

(5)

and for CPA we must have

\[ v_i + (v_a - p_a) p - \frac{\gamma A}{2} (v_a - p_a)^2 r (1 - r) \geq 0. \]  

(6)

Taking the roots of Equation (6) in terms of \((v_a - p_a)\) we find an upper bound of

\[ p_a \leq v_a + \frac{-p + \sqrt{r^2 + 2v_i \gamma A r (1 - r)}}{\gamma A r (1 - r)}. \]  

(7)

The upper bound for \(p_a\) is increasing in \(v_a, v_i,\) and \(r\) and decreasing in \(\gamma A\). This is expected as it is consistent with the change in the overall utility.

We now consider the preference between contracts. For large \(N\) we have a preference for a CPM contract when

\[ E[U_A(N, X, CPM)] \geq E[U_A(N, X, CPA)] \]

\[ p_i \leq [1 - \gamma A (1 - r) v_a] r p_a + \frac{\gamma A}{2} p_a^2 r (1 - r). \]  

(8)

The advertiser’s indifference curve, based on Equation (8) with equality, depends on \(v_a, p,\) and \(\gamma A,\) but is independent of \(v_i\) and \(N\). Independence of \(v_i\) is expected because both contracts give the same risk-free value to the advertiser in terms of impressions. However, independence on \(N\) is a result of the constant risk aversion given by the exponential utility function, and would not necessarily hold for other utility functions.

The advertiser’s indifference curve of \(p_i(p_a)\) is convex (see Figure 2). Hence, the higher the action price is, the more the impression price needs to increase by in order to compensate for the action price increase. Similarly to the publisher, this is due to the risky nature of the CPA contract and the fact that a higher price means less utility for the advertiser.

If the contracts would be based on the commonly used indifference curve, \(p_i = r p_a,\) we can see after inserting that relationship into Equation (8) that we would need \(v_a \leq \frac{p}{r}\) for the CPM contract to be preferred. This means that the value per action is less than half of the price, which would never be acceptable. In fact, for the indifference curve to cross the line \(p_i = r p_a\) at a value with nonnegative utility for the advertiser, we would need \(\gamma A \leq \frac{r(1-r)}{2}\), which represents very close to risk neutrality for the advertiser.

Hence, we can conclude that a risk-averse advertiser always prefers the CPA contract if the publisher offers CPM and CPA contracts based on the commonly-used transformation \(p_i = r p_a,\) which is in line with what can be observed in practice.

**Contract Agreement**

In Figure 2 we illustrate these preferences of both the advertiser and the publisher. Recall that the advertiser prefers contracts with low prices (the lower left), while the publisher prefers contracts with high prices (the upper right). A few observations are important. First, both of the indifference curves go through the origin, which means that as \(p_i\) or \(p_a\) go toward zero, both the utility functions have equal value (which is the worst for the publisher, and best for the advertiser).

Secondly, the curves are nonlinear with two important regions. For \(p_i\) or \(p_a\) less than a threshold value, both parties could agree on a CPA contract. Above that, both parties could agree on a CPM contract. Since low prices correspond to a reduction in the impact of overall risk, and also to the price range which advertisers prefer, this CPA agreement preference is expected. For situations
where the publisher has stronger negotiating power, the agreement is more likely to have higher prices and lead to a CPM contract.

The price $p_a$ at which the preferred contract agreement changes, is found where the two indifference curves cross:

$$p_a r - \frac{\gamma_p}{2} r (1 - r) p_a^2 = (1 - \gamma_A (1 - r) v_a) r p_a + \frac{\gamma_A}{2} p_a^2 r (1 - r)$$

$$p_a = \frac{2 \gamma_A v_a}{\gamma_p + \gamma_A}.$$

This point is particularly interesting because it is the point at which both the publisher and the advertiser are indifferent to both contracts. This is independent of $r$, which means that the probability of an action does not impact the point of indifference for $p_a$. However, through the indifference transformation (based on either Equation 2 or 8) it impacts the corresponding value of $p_i$. The indifference point for $p_a$ is linear in $v_a$, i.e., a higher value per action leads to a greater threshold of values where the agreed contract would be the CPA one resulting in a larger agreement region. Keeping $v_a$ fixed, the indifferent point is increasing in $\gamma_A$, with a limiting value of $2v_a$. This means that as the advertiser becomes more risk averse, the CPA contract becomes more likely as it carries less risk to the advertiser. Similarly, the indifference point is decreasing in $\gamma_p$, with a limiting value of 0. This means that the CPM contract is more likely, which is less risky for the publisher.

Note that from Equation (7) that the crossover point is within the feasible set of values for $p_a$ only if

$$(\gamma_A - \gamma_p) v_a \leq \frac{-p + \sqrt{r^2 + 2 v_A \gamma_A r (1 - r)}}{\gamma_A r (1 - r)}$$  \hspace{1cm} (9)

Similarly, for the publisher’s utility to be nonnegative we must have

$$\frac{\gamma_A \gamma_p}{\gamma_A + \gamma_p} v_a \leq \frac{1}{1 - r}.$$  \hspace{1cm} (10)

If these conditions are not satisfied, then the crossover (which is always nonnegative) will occur above the maximum acceptable $p_a$ value, and both parties can only agree on a CPA contract.

It is interesting to see that the commonly used indifference curve $p_i = r p_a$, is in the region where the advertiser prefers a CPA contract while the publisher prefers the CPM contract, which is the misalignment observed in practice.

In a two-party situation such as the one described, a final contract agreement is usually subject to negotiation. The outcome of this negotiation will depend on each party’s preferences, the maximum (minimum) price that the advertiser (publisher) is willing to pay (receive) for each advertisement, knowledge of the other party, and the relative negotiating power. If the advertiser knows the minimum values for $p_i$ and $p_a$, that the publisher would accept for a CPM or CPA contract respectively, then they can use Equation (8) to select their preferred contract from the available ones. Similarly, if the publisher knows the maximum prices that the advertiser is willing to pay, then they will select the contract which maximizes their own utility. Clearly there is an advantage to keeping preference information private, and the negotiation of contracts with imperfect information is an interesting subject for future research.

2. In the second case, value is derived from either actions or impressions, but not both. For large $N$, the CPM contract is preferred if

$$E[U_A(N, X, CPM)] \geq E[U_A(N, X, CPA)]$$

$$1 - e^{-\gamma_A (v_i - p_i) N} \geq \frac{1}{2} \left[ e^{-\gamma_A (v_i - p_a) N} \right]$$

$$v_i - p_i \geq (v_a - p_a) p - \frac{\gamma_A}{2} (v_a - p_a)^2 r (1 - r)$$

$$p_i \leq \frac{\gamma_A}{2} r (1 - r) p_a^2 + \left[ p - v_a \gamma_A r (1 - r) \right] p_a + v_i - v_a p + v_a^2 \frac{\gamma_A}{2} r (1 - r)$$
The CPM contract has nonnegative utility when $v_a \geq p_i$. The CPA contract has nonnegative utility when

$$0 \leq (v_a - p_a) - \frac{\gamma_A}{2} (v_a - p_a)^2 (1 - r) \leq p_a \leq v_a$$

Considering the roots of this equation, we must have

$$0 \leq v_a - p_a \leq \frac{2}{\gamma_A (1 - r)}$$

$$v_a - \frac{2}{\gamma_A (1 - r)} \leq p_a \leq v_a$$

Now we see that there is a quadratic relationship between $p_i$ and $p_a$ in the indifference curve, with the relationship not crossing through the origin. It can be viewed in terms of $d_i = v_i - p_i$ and $d_a = v_a - p_a$, which represent the added value of either an impression or an action. If $\gamma_A > 0$ then $\frac{\gamma_A}{2} (v_a - p_a)^2 r (1 - r) > 0$ and the price-equivalent point occurs at $d_i < p d_a$.

This time the curves will cross when

$$p_i = p_a r - \frac{\gamma_P}{2} r (1 - r) p_a^2 = \frac{\gamma_A}{2} r (1 - r) p_a^2 + (p - v_a \gamma_A r (1 - r)) p_a + v_i - v_a p + v_a^2 \gamma_A r (1 - r)$$

$$0 = (\gamma_P + \gamma_A) p_a^2 - 2 v_A \gamma_A p_a + \frac{v_a^2 \gamma_A r (1 - r) - 2 v_a p + 2 v_i}{r (1 - r)}.$$

Taking the roots of this equation will give the threshold for $p_A$ where the agreed preference switches from CPM to CPA and back to CPM.

$$p_a = \frac{v_A \gamma_A \pm \sqrt{-v_a^2 \gamma_P \gamma_A r (1 - r) - 2 v_a r (\gamma_P + \gamma_A) + 2 v_i (\gamma_P + \gamma_A)}}{\gamma_P + \gamma_A}$$
Note that for there to be a range of low price values with CPM agreement, the lower root must be nonnegative.

The relationship for this case is illustrated in Figure 3 for two scenarios. In the main case, the advertiser and the publisher can now agree on the CPM contract for low values of prices, then the CPA contract, and again on a CPM contract for high prices. Compared to case one the difference is the CPM agreement region for low prices. When the two parties only get a value for the impressions in the case of a CPM contract and actions in the case of a CPA contract (but not both), the CPM contract can be attractive for both parties if the value of an impression is high compared to the value of the action (see the intercept of the advertiser’s equivalence line, \( v_i - v_a p + \frac{v_a^2 \gamma_A}{2} r (1 - r) \)). In the second case, the value \( v_i \) is smaller relative to \( v_a \), and the advertiser indifference curve intercepts with \( p_i < 0 \). In this case there can only be CPA agreement.

3.3. Piecewise Linear Utility Functions

One feature, not captured explicitly by the exponential utility function, is that both the advertiser and publisher usually enter a contract with a certain expectation of the number of actions and level of payments. Sometimes reference values are announced, based on averages, which will have an effect on their relative utility. In line with this setting, we consider a piecewise linear utility that is based on the idea that the advertiser and publisher have a reference point representing their expectation for the number of actions received. Their risk aversion is then captured with a higher unit utility loss for under delivery than the utility gain for over delivery. A more comprehensive utility function could be considered with any number of linear segments, but we focus here on the fundamental characteristics and consider one inflection point.

3.3.1. Publisher

For the CPM contract, the publisher receives \( p_i N \) without any uncertainty. We assume that the publisher expects to receive \( N r \) actions, with the utility increasing by \( p_a \alpha^+ \)
for each action above the expected number \( N_r \) and decreasing by \( p_a \alpha^+ \) for each action below. We assume the publisher is risk averse, which means that \( \alpha^+ \leq \alpha^- \). For the CPA contract the utility of the payment received is:

\[
U_P(N, X, a) = p_a(N_r + \alpha^+[X - N_r]^+ - \alpha^- [N_r - X]^+) = p_a f_a(X).
\]

(12)

Let us calculate \( E[f_\beta(X)] \) when \( N \) is large, i.e., when \( X \) follows a Normal distribution with mean \( N_r \) and variance \( N_r(1-r) \).

Hence, using the properties of the truncated Normal distribution we have that

\[
E[f_\beta(X)] = E[N_r + \sqrt{N}r(1-r)(\beta^+ - \beta^-)\phi(0)]
\]

The publisher then prefers the CPM contract if

\[
p_i \geq p_a(p + (\alpha^+ - \alpha^-)\frac{\sqrt{r(1-r)}}{\sqrt{N}}\phi(0)).
\]

(13)

Given that \( \alpha^+ \leq \alpha^- \) this indifference curve has a slope less than \( r \), which means that the publisher would always prefer a CPM contract if the two contracts were offered with \( (p_i, p_a) \) values based on the commonly used \( p_i = rp_a \) transformation. We observe this preference in practice.

Note that as \( N \) increases, our proposed equivalence curve converges to, \( p_i = rp_a \). Hence, the risk is reduced with larger contracts. Furthermore, note that the difference between the two curves is not monotone in \( r \); it first increases and then decreases, with the maximum at \( r = 0.5 \). Overall, the ratio between \( p_i \) and \( p_a \) is increasing in \( r \) for large \( N \). However, this is not necessarily true for low values of \( N \). Furthermore, for large \( N \), the more likely it is that the ad will be clicked, the closer the prices of the two contracts are.

### 3.3.2. Advertiser

We assume that the advertiser expects to receive \( N_r \) actions as the publisher does. His utility increases by \( v_i \beta^+ \) for each click above the expected number \( N_r \) and decreases by \( v_i \beta^- \) for each click below. We consider two cases, which correspond to the same cases as for the exponential utility function:

1. First, we assume the advertiser gains value for both impressions and actions, for the CPM and CPA contracts. Hence, the value generated for both contracts is the same while the payments differ. The utility of the contracts, which deliver \( N \) impressions and \( X \) actions are

\[
U_A(N, X, CPM) = v_i N - p_i N + v_a(N_r + \beta^+ [X - N_r]^+ - \beta^- [N_r - X]^+)= v_i N - p_i N + v_a f_\beta(X)
\]

\[
U_A(N, X, CPA) = v_i N + (v_a - p_a)(N_r + \beta^+ [X - N_r]^+ - \beta^- [N_r - X]^+)= v_i N + (v_a - p_a)f_\beta(X)
\]

The advertiser will prefer the CPM contract if:

\[
E[U_A(N, X, CPM)] \geq E[U_A(N, X, CPA)]
\]

\[
v_i N - p_i N + v_a E[f_\beta(X)] \geq v_i N + (v_a - p_a)E[f_\beta(X)]
\]

\[
p_i \leq \frac{p_a}{N} E[f_\beta(X)]
\]

\[
= \frac{p_a}{N} (p + (\beta^+ - \beta^-)\frac{\sqrt{r(1-r)}}{\sqrt{N}}\phi(0)).
\]

Note that the indifference curve for the values \( (p_i, p_a) \) corresponding to the equality in the above expression goes through the origin. We observe that the advertiser will always prefer a CPA contract
if two contracts with \( p_i = r p_a \) are offered, since all such combinations lie below the indifference curve (see Figure 4). The case \( p_i = r p_a \) is valid when \( \beta^+ = \beta^- \). We observe this preference in practice. We also note that the publisher and advertiser would agree on a CPA contract if the advertiser is more risk averse than the publisher.

In Figure 4, we observe the same preference pattern as for the exponential case in Figure ?? . The two parties tend to agree on a CPA agreement for realistic parameter values. Since there is only one inflection point in the piecewise linear equations illustrated, the two indifference curves do not cross again as they do for the exponential case, but they would indeed do so if a convex piecewise linear utility with more line segments was considered.

2. In the second case, the advertiser gains value for only impressions in the case of a CPM contract and for only actions in the case of a CPA contract. The utility for the CPM contract is:

\[
U_A(N, X, CPM) = v_i N - p_i N. 
\]  

The utility for a CPA contract is:

\[
U_A(N, X, CPA) = (v_a - p_a)(N r + \beta^+[X - N r]^+ - \beta^-[N r - X]^+) \\
= (v_a - p_a) f_\beta(X). 
\]  

The advertiser will prefer the CPM contract if:

\[
p_i \leq v_i - \frac{1}{N} (v_a - p_a) E[f_\beta(X)] = v_i - (v_a - p_a) (p + (\beta^+ - \beta^-) \sqrt{\frac{r(1-r)}{N}} \phi(0)). 
\]  

Note that in this case the value of impressions and actions do play a role and the price indifference curve for the values \((p_i, p_a)\) is shifted by \( v_i - v_a (p + (\beta^+ - \beta^-) \sqrt{\frac{r(1-r)}{N}} \phi(0)) \) from the origin with the slope of the curve the same as for Case 1.
Another way to view this is that Case 2 is equivalent to Case 1 with $p_i$ replaced by $p_i - v_i$ and $p_a$ replaced by $p_a - v_a$, i.e., Case 1’s indifference is price driven while Case 2 is driven by the marginal utility.

We can write the inequality in (15) as

$$p_i \leq p_a(p + \Delta_1) + \Delta_2$$

where

$$\Delta_1 = (\beta^+ - \beta^-) \frac{r(1-r)}{\sqrt{N}} \phi(0)$$

$$\Delta_2 = v_i - v_a(p + (\beta^+ - \beta^-) \frac{r(1-r)}{\sqrt{N}} \phi(0))$$

Since $\beta^+ \leq \beta^-$ we have $\Delta_1 < 0$. Hence, we can see that if $\Delta_2 \leq 0$ then offering prices according to the $p_i = r p_a$ equivalence means that the advertisers always choose the CPA contract (see Case 2a in Figure 2). However, if $\Delta_2 \geq 0$ then if prices are offered according to the $p_i = r p_a$ equivalence the advertisers choose the CPA contract if $p_a > p_a^0$ and the CPM contract if $p_a < p_a^0$ where $p_a^0 = -\frac{\Delta_2}{\Delta_1}$ (see Case 2a in Figure 2). Furthermore, we see that the publisher and advertiser could agree only on a CPA contract as in Case 2a or agree on a CPM contract for low prices and a CPA contract for higher prices depending on the parameter values.

4. Alternative contracts

The base case that we have described in this paper captures some of the most important elements of contract preference and risk. However, in reality some of the assumptions behind this analysis
may not hold in certain situations. Here we explore alternative forms of the contract. We restrict our analysis to the exponential utility function as well as the first case where value is derived from both actions and impressions. Note that similar observations hold for the other utilities considered in the previous section.

4.1. Mixed contracts
Consider a mixed contract where the advertiser pays a fee per impression plus an additional fee per action. These contracts are offered in practice but their value has not been explored to our knowledge. (Mangāni (2003) and Fjell (2008) consider the problem of how many slots to offer with CPM pricing and how many with CPA pricing.) Using the previous notation and the exponential utility function we have the following utility functions for a mixed contract:

\[ U_A(N, X, MIX) = 1 - e^{-\gamma_A[(v_i - p_i)N + (v_a - p_a)X]} \]
\[ U_P(N, X, MIX) = 1 - e^{-\gamma_P(Np_i + Xp_a)} \]

for the advertiser and publisher, respectively. Note that both \( p_i \) and \( p_a \) can be expected to be lower than in the non-mix setting. For large \( N \), the expected utility will be

\[ E[U_A(N, X, MIX)] = 1 - e^{-\gamma_A[(v_i - p_i)N + (v_a - p_a)N] + \frac{2}{r^2} (v_a - p_a)^2 Nr(1 - r)} \]
\[ E[U_P(N, X, MIX)] = 1 - e^{-\gamma_P[Np_i + Np_aN] + \frac{2}{r^2} p_a^2 Nr(1 - r)} \]

The publisher has a nonnegative utility whenever

\[ p_a \geq \frac{P - \sqrt{r^2 + 2\gamma_P r(1 - r)p_i}}{\gamma_A r(1 - r)} \tag{16} \]

and the advertiser has a nonnegative utility when

\[ p_a \leq v_a + \frac{-P + \sqrt{r^2 + 2\gamma_A r(1 - r)(v_i - p_i)}}{\gamma_A r(1 - r)} \tag{17} \]

The set of \( (p_i, p_a) \) pairs, which all lead to the same utility for the publisher, will satisfy for some value \( \hat{U}_P > 0 \):

\[ p_i = \hat{U}_P - p_a r + \frac{\gamma_P}{2} p_a^2 r(1 - r). \]

Substituting this value into the advertiser’s utility, and transforming using the log function, the advertiser will seek to maximize

\[ v_i - \hat{U}_P - \frac{\gamma_P}{2} p_a^2 r(1 - r) + v_a p + \frac{\gamma_A}{2} (v_a - p_a)^2 r(1 - r). \tag{18} \]

Maximizing Equation (18) with respect to \( p_a \) leads to

\[ 0 = \gamma_P r(1 - r)p_a - \gamma_A (v_a - p_a) r(1 - r) \]
\[ p_a = \frac{\gamma_A v_a}{\gamma_P + \gamma_A}. \]

If this gives values for \( p_i \) and \( p_a \), which satisfy Equations (16) and (17) then the advertiser will select the corresponding mixed contract. However, if the maximum occurs for a value \( p_a \) outside of that range, then the advertiser will select one of the extremes, \( (p_i = 0, p_a = \frac{1}{\gamma_A(1 - r)}) \) or \( (p_i = v_i + v_a p - \frac{\gamma_A}{2} v_a^2 r(1 - r), p_a = 0) \). This of course is not a mixed contract, and shows a common case where the agreed mixed contract will be effectively a CPM or CPA contract.
Figure 6  The advertiser’s transformed utility 
\[ \hat{U}_A = -\gamma_A \log(1 - U_A) \]

is shown for a set of mixed contracts where the publisher is indifferent. Parameter values: \( r = 0.005, v_a = 4, v_i = 0.01, \gamma_A = 0.15, \gamma_P = 0.1 \).

Figure 6 illustrates the decision from the perspective of the advertiser, in a situation where the optimal mixed contract corresponds to a true mixed contract. If a publisher offers a set of mixed contracts, then the advertiser will select the mixed contract that optimizes their utility in Equation (18). The utility is plotted in Figure 6 with different values of \( p_a \).

Figure 7 illustrates a point of agreement between the advertiser and publisher. The figure shows, for particular utility values of each party, all of the mixed contracts for which each of them is indifferent to the agreed contract. The agreed contract is characterized by the common point of the two curves. Note that the advertiser possesses one equivalence curve for each utility value. The curve shown in Figure 7 represents the maximum utility obtained from the set of contracts along the corresponding equivalence curve for the publisher. The same applies to the publisher. Depending on the relative negotiating power and available alternatives, the agreed mixed contract will lie somewhere on the vertical line shown as \( p_a = \gamma_A v_a / (\gamma_P + \gamma_A) \).

4.2. Commitment to actions rather than impressions

In the contracts we have presented, the publisher commits to displaying a fixed number of impressions, \( N \). Alternatively, the publisher could commit to display as many impressions as required until a fixed number of actions has happened, which we denote \( M \). For example, this could arise if the advertiser has committed to a specified budget but prefers a CPA contract.

If the commitment is to actions rather than impressions, the publisher is taking on further risk, because of the uncertainty in the number of impressions that will be required to fulfill the action commitment. In particular, we will see that there is a much greater variance in the number of impressions required to fulfill a fixed number of actions than in the number of actions required for a fixed number of impressions.

To analyze this case we introduce a new cost \( c_P \), which represents the cost to the publisher of displaying one ad to a single customer, and a new variable \( Y \) representing the number of impressions required to fulfill the contract. If there are \( M \) actions in the commitment, then the advertiser would pay \( p_i Y \) under CPM and \( p_a M \) under a CPA contract. The revenue to the publisher would be the advertiser’s cost minus a variable cost of \( c_P Y \) (these costs can be ignored for fixed-impression contracts since they do not vary between CPA and CPM). This can be thought of as an opportunity cost that the publisher pays in denying the sale of advertising slots to other advertisers.
With a commitment to $M$ actions, the number of impressions $Y$ follows a negative binomial distribution, with success rate $(1-r)$. The probability distribution for a negative binomial distribution is given by $P[Y = k] = \binom{k + M - 1}{k} r^M (1 - r)^k$.

The expected number of impressions required is given by $E[Y] = \frac{M(1-r)}{p}$. Note in particular that if the contract specifies the publisher to show the ad until $Y = Nr$ actions happen, then the expected number of impressions required will be $N(1-r) < N$.

The variance of the negative binomial distribution is $\text{Var}(Y) = \frac{M(1-r)}{p^2}$. Again, if the commitment is to $M = Nr$ actions, then the variance is $\frac{N(1-r)}{p^2}$. Comparing this with the original contract, $\frac{\text{Var}(Y; \text{Actions})}{\text{Var}(X; \text{Impressions})} = \frac{N(1-r)}{p} \cdot \frac{1}{N r (1-r)} = \frac{1}{r^2 p}$. In other words, the variance has increased by a factor of $r^{-2}$, which can be very large for small values of $r$.

Unfortunately, the limiting distribution for the negative binomial distribution is not the normal distribution so we cannot follow the exact same analysis for this utility function. However if we instead assume explicitly that the utility function is the mean-variance utility then we can analyze this in a similar way.

Consider for example the publisher deciding between a CPM contract or a CPA contract, both committing to $M$ actions, where $r$ is the probability of action from each impression. Let their utility for a contract with revenue $R$ and cost $C$ be given by

$$U_P = E[R - C] - \gamma_P \text{Var}(R - C).$$

(19)

Note that this explicitly models the utility in terms of the expectation and variance of payments, rather than deriving the expected utility from the original distribution. For the two types of contracts we can now calculate the expected values for a commitment to $M$ actions:

$$U_P(\text{CPM}) = (p_i - c_P) \frac{M(1-r)}{p} - \gamma_P \frac{M(1-r)}{r^2} (p_i - c_P)^2$$

$$U_P(\text{CPA}) = p_a M - \frac{M(1-r)}{p} c_P - \gamma_P \frac{M(1-r)}{r^2} c_P^2.$$

Note that CPM has a nonnegative utility whenever $c_P \leq p_i \leq c_P + \frac{\gamma_P}{p} c_P$. The CPA contract has nonnegative utility when $p_a \geq \frac{1-r}{p} c_P + \frac{\gamma_P}{p} c_P^2$. 

---

**Figure 7** Indifference curves for the publisher and advertiser are shown, meeting at the point where both parties agree on a mixed contract. The parameter values are the same as for Figure 6.
The CPM contract is preferred if
\[
U_p(\text{CPM}) \geq U_p(\text{CPA})
\]
\[
p_i(1 - r) - \gamma_p \frac{(1 - r)}{p} (p_i - c_p)^2 \geq p_a r - \gamma_p \frac{(1 - r)}{p} c_p^2
\]
\[
p_i p_r - \gamma p (p_i - c_p)^2 \geq p_a r - \gamma p c_p^2
\]
\[-\gamma_p p_i^2 + (p + 2 \gamma_p c_p) p_i - p_a \frac{r^2}{1 - r} \geq 0.
\]
Solving this quadratic equation gives the values for which this preference holds.

\[
\frac{p + 2 \gamma_p c_p - \sqrt{(p + 2 \gamma_p c_p)^2 - 4 \gamma p p_a \frac{r^2}{1 - r}}}{2 \gamma_p} \leq p_i \leq \frac{p + 2 \gamma_p c_p + \sqrt{(p + 2 \gamma_p c_p)^2 - 4 \gamma p p_a \frac{r^2}{1 - r}}}{2 \gamma_p}.
\] (20)

Note that for this region to be nonempty, the terms inside the square root must be nonnegative, which gives a bound on the value for \(\gamma_p\). In particular, for reasonable parameter values, the values for \(\gamma_p\) assumed in the previous section would give too much risk to the publisher, and never lead to a point of indifference. In this case the publisher will always prefer the CPM contract, because they gain income from each impression to offset the variable risk.

For an advertiser with a utility function of the same mean-variance form, the utility functions will be given by:

\[
U_A(\text{CPM}) = (v_i - p_i) \frac{M(1 - r)}{p} - \gamma_A \frac{M(1 - r)}{r^2} (v_i - p_i)^2
\]
\[
U_A(\text{CPA}) = (v_a - p_a) M.
\]

The advertiser has nonnegative utility for CPM when \(v_i - \frac{p}{\gamma_A} \leq p_i \leq v_i\). For CPA they require \(p_a \leq v_a\).

Again, the CPM contract is preferred if examining the preferences we find
\[
U_A(\text{CPM}) \geq U_A(\text{CPA})
\]
\[
(v_i - p_i)(1 - r)p - \gamma_A(1 - r)(v_i - p_i)^2 \geq (v_a - p_a)r^2
\]
\[-\gamma_A(1 - r)(v_i - p_i)^2 + (1 - r)r(v_i - p_i) - (v_a - p_a)r^2 \geq 0.
\]
Solving this for \((v_i - p_i)\) we must have
\[
v_i - p_i \geq \frac{r(1 - r) - \sqrt{r^2(1 - r)^2 - 4 \gamma_A(1 - r)(v_a - p_a)r^2}}{2 \gamma_A(1 - r)}
\]
\[
v_i - p_i \leq \frac{r(1 - r) + \sqrt{r^2(1 - r)^2 - 4 \gamma_A(1 - r)(v_a - p_a)r^2}}{2 \gamma_A(1 - r)}.
\]

In particular, this gives a range of \(p_i\) values for which CPM is preferred over CPA, in terms of \(p_a\). Since the second inequality always holds for allowable values, we have
\[
p_i \leq p_i + \frac{-r(1 - r) + \sqrt{r^2(1 - r)^2 - 4 \gamma_A(1 - r)(v_a - p_a)r^2}}{2 \gamma_A(1 - r)}
\] (21)

Figure 8 illustrates the contract preferences when the commitment is made to actions. The relative positions of the two indifferent curves depends on the risk parameters, with the illustrated case showing CPA agreement.

As for the previous cases, we highlight the most interesting parameter values where contract agreement may be negotiated.
5. Uncertainty and control of the action probabilities

In our core contract analysis, we assumed that the action probability $r$ is both fixed and known to both parties. In some scenarios this will not be the case. For example, the publisher may have control to place an advertisement according to the profile of site visitors, thereby influencing the likelihood that an action will result. Further, there may be disagreement or a lack of knowledge of the correct value of $r$, which may lead to different preferences. This is especially true for banner campaign advertising where banners including changing promotional content. Three of these cases are analyzed here, for exponential utility functions.

5.1. Different probability estimates

If the publisher and advertiser have different prior estimates of the action probability, their contract agreement may be impacted. This could arise in several scenarios, including a new relationship between advertiser and publisher, a new advertising campaign or a new product being advertised. These scenarios also highlight the nature of these contracts, where a commitment in advance to display $N$ ads could be costly for one of the parties. This leads to the indifference curves given by (2) and (8) being shifted by the appropriate amount. Naturally, each party will be self-consistent with the same estimate across contracts (and therefore equations (2) and (8) hold), but the value for $r$ differs between the publisher and advertiser. If the publisher over (under) estimates $r$ then the threshold for $p_i$ increases (decreases) and they are more (less) inclined to select the CPA contract. Similarly, if the advertiser over (under) estimates $r$ then the threshold increases (decreases) relative to $p_a$ and they become more (less) inclined to the CPM contract.

Figure 9 illustrates the impact of contract preferences when one of the parties (in this case the advertiser) has a different estimate for $r$. With the parameters illustrated, the parties could agree on either contract if they had the same estimate. However, if the advertiser overestimates then...
agreement is only possible with a CPM contract, and if they overestimate then agreement can only be with CPA.

5.2. Dependent probability values

Our analysis so far has assumed that the probability of action $r$ is independent of the agreed contract. Frequently in practice, however, a publishing site will be able to influence the value of $r$ by placing advertising more strategically, and therefore may behave differently given the additional reward offered by a CPA contract if the number of actions increases. A publisher with a portfolio of advertisers and visitors could strategically place each banner advertisement based on the contracts in place and the visitor behavior.

If the publisher will ensure that the probabilities are $r_a$ and $r_i$ for a CPA contract and CPM contract respectively, then the publisher’s utility will be governed by $r_a$, with the CPM contract preferred whenever

$$r_i \geq r_a p_a - \frac{\gamma_p}{2} r_a (1 - r_a) p_a^2.$$

This is independent of $p_i$. However, the advertiser prefers the CPM contract if

$$E[U_A(N, X, CPM)] \geq E[U_A(N, X, CPA)]$$

$$1 - e^{-\gamma_A[(v_i - p_i)N + v_a N r_i]} + \frac{\gamma_a^2}{2} v_a^2 r_i (1 - r_i) \geq 1 - e^{-\gamma_A(v_i N + (v_a - p_a) N r_a)} + \frac{\gamma_a^2}{2} (v_a - p_a)^2 N r_a (1 - r_a)$$

which leads to

$$p_i \leq (r_a - r_i) v_a \left[ \frac{\gamma_A v_a}{2} (1 - r_i - r_a) - 1 \right] + [1 - \gamma_A v_a (1 - r_a)] r_a p_a + \frac{\gamma_A}{2} r_a (1 - r_a) p_a^2.$$
Figure 10 Preferences when $r$ depends on the contract. In the above plot, value of $r$ is $r_a = 0.005$ if the contract is CPA, or $r_i = 0.0025$ if the CPM contract is selected. The advertiser is more inclined to select CPA because of the gain in $r$ as well as the reduction in risk relative to CPM. Parameters: $v_i = 0.01, v_a = 4$.

Note that this indifference curve is equivalent to the advertiser’s original indifference curve (8) plus a scalar term, which reduces to zero if $r_i = r_a$.

Figure 10 illustrates the impact on contract preferences when $r$ depends on the contract choice. In this case the advertiser has more incentive to select the CPA contract, since it increases the expected number of actions and therefore the expected value of the advertiser’s utility.

5.3. Uncertain probability values

For display advertising, it is often difficult to estimate $r$ at the beginning of an advertising campaign because of potential changes to the ad, the publishing website content, market timing, the customer base and customer behavior.

For example, consider a new advertising campaign to be run on a news site for a promotion on a popular brand of shoes. The publisher may know the typical viewers and responses to previous campaigns but the actual click-through rate will depend on how compelling the advertising is and what the news cycle is when the ads are shown. Different subsets of customers may view the website, influencing the actual effective click-through rates.

If the advertising slots were sold in single-view contracts through a mechanism such as an online auction, then the risk of committing to CPM or CPA contract is minimal, since it is a single decision. However, if there is a batch agreement in the style of those described in this paper, then the uncertainty in $r$ could be important.

Note that this case differs from the earlier scenario where the two parties disagree on the value of $r$ because both acknowledge the same uncertainty. Let the prior distribution for $r$ be a beta-distribution with parameters $\alpha$ and $\beta$. The beta distribution is a very general one, with special cases including uniform, unimodal and U-shaped distributions. In general, we would expect a unimodal distribution, which occurs where $\alpha > 1$ and $\beta > 1$. 

Given the parameters of the prior beta distribution, the key performance metrics of mean and variance are known.

\[ \text{Mean} = \frac{N\alpha}{\alpha + \beta}, \quad \text{Variance} = \frac{N\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}. \]

Comparing this to the case of known \( r \) value with \( p = \frac{\alpha}{\alpha + \beta} \), we see that the expected value is unchanged. The variance on the other hand has changed from \( Nr(1-r) \), which is the equivalent of \( \frac{N\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)} \). That is, the variance has increased by a factor of \( \frac{\alpha + \beta + N}{\alpha + \beta + 1} \).

Using these parameters, if we let the utility be defined as the mean-variance utility, then a publisher will prefer the CPM contract whenever

\[
U_P(N, X, CPM) \geq U_P(N, X, CPA) \\
p_iN \geq p_ar - \gamma rp_a \frac{N\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)} \\
p_i \geq rp_a - \gamma rp(1-r) \frac{\alpha + \beta + N}{\alpha + \beta + 1}
\]

where \( r \) is the expected value, \( \frac{N\alpha}{\alpha + \beta} \).

Similarly, the advertiser will prefer the CPM contract whenever

\[
U_A(N, X, CPM) \geq U_A(N, X, CPA) \\
p_i \leq p_ar + \gamma A(p_a^2 - 2v_ap_a)r(1-r) \frac{\alpha + \beta + N}{\alpha + \beta + 1}
\]

Notice that this implies that larger \( N \) means larger variance - the longer the commitment, the more risk is being taken on by each side. Therefore, if each party is confident of the value of \( r \), then risk goes down with commitment level, whereas with less confidence the opposite effect is observed.

Often the prior distribution for \( r \) will not be well understood, but using the method of moments, one can calculate the appropriate parameters of the beta distribution. Given a sample of \( N \) values, \( \bar{X} = \frac{1}{N} \sum X_i \) and \( V = \frac{1}{N} \sum (X_i - \bar{X})^2 \). Given these values, we have \( \alpha = \bar{X} \frac{(1-\bar{X})}{V} - 1 \) and \( \beta = (1 - \bar{X}) \frac{\bar{X}(1-\bar{X})}{V} - 1 \).

Figure 11 illustrates the impact on contract preferences when \( r \) is uncertain, taking a beta distribution. The uncertainty heavily penalizes both contracts for the advertiser, and the CPA contract for the publisher. No agreement can be reached for the parameters shown, which may be common if uncertainty is prevalent.

6. Conclusions and further research

We have modeled the choice of contracts available to advertisers and publishers in online display advertising, and characterized their optimal preferences under natural utility assumptions. We find that the conversion rule used in practice does not adequately account for risk tolerance and have illustrated a more reasonable method for finding an agreeable contract.

We provide a new conversion rule which allows advertisers and publishers to base contract decisions on their own risk preferences. In particular, we show how much a publisher should discount a CPM contract relative to a CPA contract with the same expected revenue, to account for the certainty of revenue provided.

We find that sometimes a mixed contract can be preferred by both parties over a CPA or CPM contract, providing the appropriate mix of expected revenue/cost and certainty. We also discuss other potential contract structures, but find that in general it would be difficult to base a contract on a fixed number of actions rather than a fixed number of impressions.

We have analyzed the role of both prior information and uncertainty, and find that a lack of knowledge of the action probabilities can make it very difficult for parties to agree on either
Figure 11  Preferences when $r$ takes a prior beta distribution. In the above plot, value of $r$ is uncertain, taking values from a beta distribution. The same parameters are assumed as for the previous studies, and we observe that no agreement would be reached. Specifically, for positive values of $p_a$ and $p_i$, the advertiser prefers a CPA contract and the publisher prefers a CPM contract. Parameters: $v_i = 0.01, v_a = 4$; Distribution parameters $\alpha = 12.4, \beta = 2474, N = 10,000$, with $E[p] = 0.005$ and $\text{Var}(p) = 2 \times 10^{-6}$.

contract. These contributions and insights make the negotiating of advertising contracts a much more scientific process for both advertisers and publishers.

There are several interesting and relevant extensions to the model presented here that warrant further investigation. Firstly, the situation for both the advertiser and publisher may be better described as a portfolio of contracts rather than a single contract selected in isolation. Under such conditions, the role of risk changes, and therefore the optimal combination of advertisers (from a publisher perspective) or publishers (from an advertiser perspective) would be an interesting area for study. Building upon that, a publisher may find themselves with several contract commitments and a range of ads that can be shown in a particular slot. Dynamically assigning advertisers based on the customer profiles and contract reward scheme would be particularly beneficial.

The second area for extension involves forecasting the action probabilities and accounting for changes over time. Since market dynamics can change rapidly, the fluctuations of the value $r$ may impact the contracts that each party is willing to commit to. Modeling this in more detail could lead to more sophisticated approaches for managing the contract and associated risks.

Finally, we suggest that the decision-making framework, and particularly the role of negotiating power and information sharing will become increasingly important in online display advertising. If one party has the ability to withhold information or more accurately forecast or control the responses to an advertising campaign then there are many opportunities to take advantage of this capability.

References


