Reflections

on

Finite Model Theory

Phokion G. Kolaitis

IBM Almaden Research Center
(on leave from UC Santa Cruz)
What is finite model theory?

It is the study of logics on classes of finite structures.

- **Logics:**
  - First-order logic FO and various extensions of FO:
    - Fragments of second-order logic SO.
    - Logics with fixed-point operators.
    - Logics with generalized quantifiers.

- **Classes of finite structures:**
  - All finite structures $A = (A, R_1, \ldots R_m)$ over a fixed vocabulary.
  - All ordered finite structures $A = (A, <, R_1, \ldots, R_m)$.
  - Restricted classes of finite structures of combinatorial or of algorithmic interest (trees, planar graphs, partial orders, ...).
Contrast with traditional focus of logic

- Study of logics on the class of all structures
  - Gödel’s Completeness Theorem
    Truth in FO on the class of all (finite & infinite) structures

- Study of logics on a fixed infinite structure
  - Gödel’s Incompleteness Theorem
    Truth in FO on the structure $\mathbb{N} = (\mathbb{N}, +, x)$ of the integers
  - Tarski’s Theorem
    Truth in FO on the structure $\mathbb{R} = (\mathbb{R}, +, x)$ of the reals.
Brief History

- Late 1940s to 1970:
  - Early scattered results and problems about FO in the finite.

- Early 1970s to present:
  - Steady development of finite model theory in its own right.
  - Extensive interaction with computational complexity, database theory, asymptotic combinatorics, automated verification, constraint satisfaction.

- Finite model theory has had a constant presence in LICS.
  - At least five times the Kleene Award for Best Student Paper has been given for work in finite model theory.
Aims of this Talk

To reflect on finite model by
- Highlighting some of its successes;
- Examining obstacles that were encountered;
- Discussing some open problems that have resisted solution.

This talk is

- neither
- a comprehensive survey of finite model theory
- a “personal perspective” on the development of finite model theory.
Early Beginnings: a theorem and two problems.

**Theorem:** Trakhtenbrot – 1950
First-order finite validities **cannot** be axiomatized:
The set of finitely valid first-order sentences is **not** recursively enumerable.

- “Anti-completeness” theorem
- Sharp contrast with Gödel’s Completeness Theorem: first-order validities **can** be axiomatized.
The Spectrum Problem

Definition:
A set $S$ of positive integers is a spectrum if there is a
FO-sentence $\phi$ such that
$$S = \{m: \phi \text{ has a finite model with } m \text{ elements }\}$$

Example: The set of all powers of primes is a spectrum.

The Spectrum Problem
- Scholz – 1952: Characterize all spectra
- Asser – 1955: Are spectra closed under complement?
  Is the complement of a spectrum a spectrum?
Preservation under Substructures

- **Theorem:** Łoś- Tarski – 1948
  If a FO-sentence $\psi$ is preserved under substructures on all (finite and infinite) structures, then there is a universal FO-sentence $\psi^*$ that is equivalent to $\psi$ on all structures.

- **Conjecture:** Scott and Suppes – 1958
  The Łoś- Tarski Theorem holds in the finite:
  If a FO-sentence $\psi$ is preserved under substructures on all finite structures, then there is a universal FO-sentence $\psi^*$ that is equivalent to $\psi$ on all finite structures.
Main Themes in Finite Model Theory

- **Descriptive complexity:**
  computational complexity vs. uniform definability.

- **Expressive power of logics in the finite:**
  What can and what cannot be expressed in various logics on classes of finite structures.

- **Logic and asymptotic probabilities on finite structures**
  0-1 laws and convergence laws.

- **Classical Model theory in the finite:**
  Do the classical results of model theory hold in the finite?
Notation and Terminology

- $\sigma$: a fixed relational vocabulary $\{R_1, \ldots, R_m\}$
- $\mathcal{C}$: a class of finite $\sigma$-structures closed under isomorphisms.

A \textbf{k-ary query on $\mathcal{C}$} is a mapping $Q$ defined on $\mathcal{C}$ such that
- If $A \in \mathcal{C}$, then $Q(A)$ is a $k$-ary relation on $A$;
- $Q$ is invariant under isomorphisms:
  - if $f: A \to B$ is an isomorphism, then $Q(B) = f(Q(A))$.

\textbf{Example:} TRANSITIVE CLOSURE of a graph $G = (V,E)$

A \textbf{Boolean query on $\mathcal{C}$} is a mapping $Q$: $\mathcal{C} \to \{0, 1\}$
that is invariant under isomorphisms

\textbf{Example:} CONNECTIVITY, 3-COLORABILITY, ...
Complexity vs. Definability

- **Computational complexity** is concerned with the computational resources (model of computation, time, space) needed to compute queries.

- **Logical definability** is concerned with the logical resources (type of quantification, number of variables, operators extending the syntax of first-order logic, ...) needed to express queries.

- **Descriptive complexity** studies the connections between computational complexity and logical definability.
Descriptive Complexity

Main Finding:

All major computational complexity classes, including P, NP, and PSPACE, can be characterized in terms of definability in various logics on classes of finite structures.

- Reinforces the unity of computation and logic.
- Yields machine-independent characterizations of computational complexity classes.
Descriptive Complexity: Characterizing NP

**Theorem:** Fagin – 1974

Let $F$ be the class of all finite $\sigma$-structures and let $Q$ be a query on $F$. Then the following are equivalent:

- $Q$ is is NP.
- $Q$ is definable by an existential second-order formula
  \[ \exists S_1 \ldots \exists S_k \phi(S_1, \ldots, S_k). \]

In symbols, $\text{NP} = \text{ESO on } F$.

**Example:** 3-COLORABILITY of a graph $(V,E)$ is definable by

\[ \exists B \exists R \exists G ((B,R,G) \text{ form a partition of } V \land \forall x \forall y (E(x,y) \rightarrow x, y \text{ are in different parts})). \]
Descriptive Complexity: Characterizing NP

**Corollary:** The following are equivalent:

- NP is closed under complement (i.e., NP = coNP).
- ESO is closed under complement on the class $G$ of all finite graphs.
- NON 3-COLORABILITY is ESO-definable on $G$.

**Proof:** Fagin’s Theorem and NP-completeness of 3-COLORABILITY.
The following are equivalent for a set $S$ of positive integers in binary notation:
- $S$ is a spectrum.
- $S$ is in NEXPTIME.

Corollary:  The following are equivalent:
- Spectra are closed under complement.
- NEXPTIME is closed under complement.

Conclusion:  Asser’s question is equivalent to a major open problem in computational complexity.
Descriptive Complexity: Characterizing P

**Theorem:** Immerman – 1982, Vardi – 1982
Let $O$ be the class of all ordered finite $\sigma$-structures $A = (A, <, R_1, \ldots, R_m)$ and let $Q$ be a query on $O$. Then the following are equivalent:

- $Q$ is in $P$.
- $Q$ is definable in least-fixed point logic LFP.

In symbols, $P = \text{LFP on } O$.

**Note:** LFP = (FO + Least fixed-points of positive FO-formulas)

**Example:** The TRANSITIVE CLOSURE query is definable by the least fixed point of the FO-formula $E(x,y) \lor \exists z(E(x,z) \land T(z,y))$

$$T(x,y) \equiv E(x,y) \lor \exists z(E(x,z) \land T(z,y))$$
Descriptive Complexity Results

Two groups of results:

**Group I:** A complexity class (typically, NP or higher) can be characterized in terms of uniform definability in a logic on the class $F$ of all finite $\sigma$-structures (and, hence, on all subclasses of $F$).

**Group II:** A complexity class (typically, P or lower) can be characterized in terms of definability in a logic on the class $O$ of all ordered finite $\sigma$-structures $A = (A, <, R_1, ..., R_m)$.

**Note:** LFP cannot express counting queries on $F$ (e.g., EVEN CARDINALITY).
The Quest for a Logic for P

**Problem:** Chandra and Harel – 1982
Is there an effective enumeration of all polynomial-time computable queries on the class $F$ of all finite $\sigma$-structures?

**Conjecture:** Gurevich – 1988
There is no logic that captures P on the class $F$ of all finite $\sigma$-structures.

**Note:**
If $P = NP$, then there is logic for P (namely, ESO).
The Quest for a Logic for P

- Has motivated numerous investigations in finite model theory:
  - Systematic study of various extensions of first-order logic, including generalized quantifiers and fixed-point operators.
  - Systematic development of tools to delineate the expressive power of extensions of first-order logic in the finite, such as Ehrenfeucht – Fraïssé games and their variants: Ehrenfeucht – Fraïssé games for ESO, pebble games, and games for logics with generalized quantifiers.

- However, Chandra and Harel’s Problem and Gurevich’s Conjecture remain outstanding open problems in finite model theory.
Restricted Classes of Finite Structures

- Progressive shift of emphasis from the class of all finite structures to restricted classes of finite structures.

- **Theorem:** Let \((\text{IFP} + C)\) be the extension of FO with inflationary fixed-points and counting quantifiers.
  - Grohe – 1998
    - \(P = (\text{IFP} + C)\) on the class \(\mathcal{P}\) of all planar graphs.
  - Grohe and Mariño – 1999
    - \(P = (\text{IFP} + C)\) on the class \(\mathcal{T}(k)\) of graphs of treewidth \(\leq k\).

- **Note:** Deeper properties of the restricted classes are used to find an \((\text{IFP} + C)\)-definable linear order on structures in the restricted class.
Reflecting on Descriptive Complexity

**Early Optimism:**
- Descriptive complexity results reduce the separation of complexity classes to the separation of logics in the finite.
- Combinatorial games (Ehrenfeucht – Fraïssé games and their variants) provide a sound and complete method for delineating the expressive power of logics in the finite.
- Use logic to resolve open problems in computational complexity.

**Example:** Recall that the following are equivalent:
- NP is not closed under complement (i.e., $\text{NP} \neq \text{coNP}$).
- NON 3-COLORABILITY is not ESO-definable on $G$. 
Reflecting on Descriptive Complexity

**Reality:** The implementation of this approach is confronted with seemingly insurmountable combinatorial obstacles.
- Combinatorial games have been successfully used to analyze the expressive power of monadic ESO
  \[ \exists S_1 \ldots \exists S_k \phi(S_1, \ldots, S_k), \]  where the \( S_i \)’s are unary symbols.
- The expressive power of binary ESO is poorly understood.

**Problem:** Fagin – 1990
Prove or disprove that there is a query \( Q \) on graphs such that
- \( Q \) is ESO-definable.
- \( Q \) is **not** definable in binary ESO with a single existentially quantified binary symbol
  \[ \exists S \phi(S), \]  where \( S \) is a binary relation symbol.
Reflecting on Descriptive Complexity

Reality:
- The expressive power of FO on the class $F$ of all finite structures is **well** understood.
- The expressive power of FO on classes of ordered finite structures $A = (A, <, R_1, ..., R_m)$ is **poorly** understood.

**The Ordered Conjecture:** K ... and Vardi – 1992
If $C$ is a class of ordered finite structures of arbitrarily large cardinalities, then $\text{FO} \neq \text{LFP}$ on $C$ (i.e., $\text{FO} \neq \text{P}$ on $C$).

**Note:** Either way of resolving the Ordered Conjecture has complexity-theoretic implications.
Main Themes in Finite Model Theory

✓ **Descriptive complexity:**
  computational complexity vs. uniform definability.

✓ **Expressive power of logics in the finite:**
  What *can* and what *cannot* be expressed in various logics on classes of finite structures.

- **Logic and asymptotic probabilities on finite structures**
  0-1 laws and convergence laws.

- **Classical Model theory in the finite:**
  Do the classical results of model theory hold in the finite?
Logic and Asymptotic Probabilities

**Notation:**
- Q: Boolean query on the class $F$ of all finite structures
- $F_n$: Class of finite structures of cardinality $n$
- $\mu_n$: Probability measure on $F_n$, $n \geq 1$
- $\mu_n(Q)$ = Probability of $Q$ on $F_n$ with respect to $\mu_n$, $n \geq 1$.

**Definition:** Asymptotic probability of query $Q$
\[
\mu(Q) = \lim_{n \to \infty} \mu_n(Q) \quad \text{(provided the limit exists)}
\]

**Examples:** For the uniform measure $\mu$ on finite graphs $G$:
- $\mu(G \text{ contains a triangle}) = 1$.
- $\mu(G \text{ is connected}) = 1$.
- $\mu(G \text{ is 3-colorable}) = 0$.
- $\mu(G \text{ has even cardinality})$ does not exist.
0-1 Laws and Convergence Laws

**Question:** Is there a connection between the definability of a query $Q$ in some logic $L$ and its asymptotic probability?

**Definition:** Let $L$ be a logic

- The **0-1 law** holds for $L$ w.r.t. to a measure $\mu_n$, $n \geq 1$, if
  \[ \mu(Q) = 0 \text{ or } \mu(Q) = 1, \]
  for every $L$-definable Boolean query $Q$.

- The **convergence law** holds for $L$ w.r.t. to a measure $\mu_n$, $n \geq 1$, if $\mu(Q)$ exists, for every $L$-definable Boolean query $Q$. 
0-1 Law for First-Order Logic

**Theorem:** Glebskii et al. – 1969, Fagin – 1972
The 0-1 law holds for FO w.r.t. to the uniform measure.

**Transfer Theorem:** Fagin – 1972
There is a unique countable graph $\mathbf{R}$ such that for every FO-sentence $\psi$, we have that
$$\mu(\psi) = 1 \text{ if and only if } \mathbf{R} \models \psi.$$  

**Note:**
- $\mathbf{R}$ is Rado’s graph: the unique countable, homogeneous, and universal graph.
- $\mathbf{R}$ is characterized by a set of first-order extension axioms.
Problem: Given a FO-sentence $\psi$, tell whether $\mu(\psi) = 0$ or $\mu(\psi) = 1$.

Note:
- By the Transfer Theorem, this is equivalent to deciding first-order truth on $\mathbb{R}$.
- Fagin’s proof shows it is a decidable problem.

Theorem: Grandjean – 1983
The decision problem for the 0-1 law for FO is PSPACE-complete.
## FO Truth vs. FO Almost Sure Truth

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Description</th>
<th>Complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Everywhere true (valid)</td>
<td>First-Order Truth Testing if a FO-sentence is true on all finite graphs is an <strong>undecidable</strong> problem.</td>
<td></td>
</tr>
<tr>
<td>Somewhere true &amp; Somewhere false</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Everywhere false (contradiction)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Almost surely true</td>
<td>Almost Sure First-Order Truth Testing if a FO-sentence is almost surely true on all finite graphs is a <strong>decidable</strong> problem; in fact, it is PSPACE-complete.</td>
<td></td>
</tr>
<tr>
<td>Almost surely false</td>
<td></td>
<td></td>
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</tbody>
</table>

- **First-Order Truth** Testing if a FO-sentence is true on all finite graphs is an **undecidable** problem.
- **Almost Sure First-Order Truth** Testing if a FO-sentence is almost surely true on all finite graphs is a **decidable** problem; in fact, it is PSPACE-complete.
Three Directions of Research on 0-1 Laws

- 0-1 laws for extensions of FO w.r.t. the uniform measure.

- 0-1 laws for FO on restricted classes of finite structures

- 0-1 laws on graphs under variable probability measures.
0-1 Laws for Fragments of ESO

Fact:
- The convergence law fails for ESO
  - EVEN CARDINALITY is ESO-definable.
- Many natural NP-complete problems have probability 0 or 1:
  - 3-COLORABILITY
  - HAMILTONIAN PATH
  - SATISFIABILITY
  - KERNEL
  - ...

Question: Do 0-1 laws hold for fragments of ESO?
0-1 Laws for Fragments of ESO

**Idea:**
Pursue 0-1 laws for fragments of ESO obtained by restricting the quantifier pattern in the FO-part $\phi(S)$ of ESO-sentences $\exists S \phi(S)$.

**Guiding Principle:** Skolem Normal Form for ESO:

$$\exists S \exists x \forall y \exists z \theta(S, x, y, z),$$

where $S$ is a tuple of SO-variables, $x$, $y$, and $z$ are tuples of FO-variables, and $\theta(S, x, y, z)$ is a quantifier-free formula.

Thus, it suffices to consider first-order prefix classes that are subclasses of $\exists^* \forall^* \exists^*$. 
Theorem: K ... and Vardi – 1987
- For every ESO(∃*∀*)-sentence $\psi$, we have that $\mu(\psi) = 1$ if and only if $R \vDash \psi$.
- The 0-1 law holds for ESO(∃*∀*).

Theorem: K ... and Vardi – 1988
- For every ESO(∃*∀∃*)-sentence $\psi$, we have that $\mu(\psi) = 1$ if and only if $R \vDash \psi$.
- The 0-1 law holds for ESO(∃*∀∃*).

Theorem: Pacholski and Szwast – 1991
The convergence law fails for ESO(∀∀∃).
# 0-1 Laws for Fragments of ESO

<table>
<thead>
<tr>
<th>ESO Fragment</th>
<th>0-1 Law</th>
<th>Decision Problem</th>
</tr>
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<tbody>
<tr>
<td>ESO(∃<em>∀</em>)</td>
<td>Yes</td>
<td>NEXPTIME-complete</td>
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<td>Yes</td>
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</tr>
<tr>
<td>ESO(∀∀∃)</td>
<td>No</td>
<td>Undecidable</td>
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**Classification Theorem:**
The Bernays-Schönfinkel Class $∃^*∀^*$ and the Ackermann Class $∃^*∀∃^*$ are the **only** prefix classes $Ψ$ of FO such that the 0-1 law holds for the corresponding fragment ESO($Ψ$) of ESO.
0-1 Laws for Fragments of ESO

**Note:**
The Bernays-Schönfinkel Class $\exists^* \forall^* \exists^*$ and the Ackermann Class $\exists^* \forall \exists^*$ are the **only** prefix classes of FO (with equality) for which the satisfiability problem is decidable.

**Theorem:**  Gödel – 1932
The satisfiability problem for the prefix class $\forall \exists$ without equality is decidable.

**Theorem:**  Le Bars – 1998
The convergence law fails for $\text{ESO}(\forall \exists)$ without equality.
Reflecting on 0-1 Laws

On the positive side:

- 0-1 laws are new phenomena that are meaningful only in the context of finite structures.

- Finiteness is a feature, not a limitation.

- The study of 0-1 laws gave rise to an extensive interaction between finite model theory and asymptotic combinatorics (genuine two-way interaction; e.g., 0-1 laws for restricted classes of finite structures: partial orders, clique-free graphs).
Reflecting on 0-1 Laws

On the negative side:

- The study of 0-1 laws had less interaction with and impact on computer science than other areas of FMT.

  N. Immerman – 1999: 0-1 laws are “inimical to computation”.

- There was early speculation that the analysis of the asymptotic properties of logically definable queries may be useful in the average-case analysis of algorithms.

  This early optimism and expectation remains largely unrealized.
Main Themes in Finite Model Theory

✓ **Descriptive complexity:**
  computational complexity vs. uniform definability.

✓ **Expressive power of logics in the finite:**
  What **can** and what **cannot** be expressed in various logics on classes of finite structures.

✓ **Logic and asymptotic probabilities on finite structures**
  0-1 laws and convergence laws.

- **Classical Model theory in the finite:**
  Do the classical results of model theory hold in the finite?
Classical Model Theory in the Finite

- The Skolem-Löwenheim Theorem is meaningful in the finite.
- The Compactness Theorem fails in the finite.
- The Craig Interpolation Theorem fails in the finite: the EVEN CARDINALITY query is not FO-definable.

**Conjecture**: Scott and Suppes – 1958
The Łoś- Tarski Theorem holds in the finite:
If a FO-sentence $\psi$ is preserved under substructures on all finite structures, then there is a universal FO-sentence $\psi^*$ that is equivalent to $\psi$ on all finite structures
Classical Model Theory in the Finite

**Theorem:** Tait – 1959

The Łoś- Tarski Theorem fails in the finite.
(rediscovered by Gurevich and Shelah in the 1980s)

**Theorem:** Ajtai and Gurevich – 1987

Lyndon’s Positivity Theorem fails in the finite:
There is a FO-sentence $\psi(S)$ that is monotone in $S$ on all finite structures, but is not equivalent to any positive-in-$S$ FO-sentence on all finite structures.

**Question:** Do any of the classical results of model theory survive the passage to the finite?
Classical Model Theory in the Finite

**Preservation-under-Homomorphisms Theorem:**
If a FO-sentence $\psi$ is preserved under homomorphisms on all structures, then there is an existential positive FO-sentence $\psi^*$ that is equivalent to $\psi$ on all structures.

**Problem:** Does the preservation-under-homomorphisms theorem hold in the finite? Suppose that a FO-sentence $\psi$ is preserved under homomorphisms on all finite structures. Is there a FO-sentence $\psi^*$ that is equivalent to $\psi$ on all finite structures?

This problem had remained open for a long time ...
Classical Model Theory in the Finite

**Theorem:** Rossman – 2005
If a FO-sentence $\psi$ is preserved under homomorphisms on all finite structures, then there is an existential positive FO-sentence $\psi^*$ that is equivalent to $\psi$ on all finite structures.

- So, finally, we have a positive result about classical model theory in the finite.

- And there is more ...
Theorem: Atserias, Dawar, K ... – 2004

- Let $T(k)$ be the class of graphs of treewidth at most $k$. If a FO-sentence is preserved under homomorphisms on $T(k)$, then it is equivalent to some existential-positive FO-sentence on $T(k)$.

- If a FO-sentence is preserved under homomorphisms on all planar graphs, then it is equivalent to some existential-positive FO-sentence on all planar graphs.

Note: Preservation theorems do not relativize to subclasses.
Model Theory of Restricted Classes

**Theorem:** Atserias, Dawar, Grohe – 2005

- Let $T(k)$ be the class of graphs of treewidth at most $k$. If a FO-sentence is preserved under substructures on $T(k)$, then it is equivalent to some universal FO-sentence on $T(k)$.

- There is a FO-sentence that is preserved under substructures on all planar graphs, but it is **not** equivalent to any universal FO-sentence on all planar graphs.
Abstract Model Theory in the Finite

**Theorem:** Lindström – 1969  
First-order logic is a maximal logic possessing both the Compactness Theorem and the Skolem-Löwenheim Theorem.

**Problem:** K ... and Väänänen - 1992  
- Is there a Lindström-type characterization of first-order logic on finite structures?
- Is there a Lindström-type characterization of least fixed-point logic on finite structures?
Concluding Remarks

Many topics were not covered in this talk:

- Finite-variable logics and analysis of k-types.
- Logics with generalized quantifiers.
- Interaction with modal logics, connections with the $\mu$-calculus and automated verification.
- Applications to database theory and to constraint databases.
- Interaction with constraint satisfaction.
Concluding Remarks

- Finite model theory has come a long way from a collection of early sporadic results to a mature research area.

- There have been numerous successes, but also frustrations:
  - Lack of progress on resolving open problems in complexity.
  - Limited impact of 0-1 laws on other areas of CS.

- On the positive side,
  - Shift of focus on restricted classes of structures is bearing fruit.
  - Growing connections with constraint satisfaction.

- One can only hope that the next 30 years of finite model theory will be at least as fruitful as the past 30.