Comp. & Maths. with Appls. Vol. 7, No. 5, pp. 425–430, 1981
Printed in Great Britain.

0097–4943/81/050425–06$02.00/0

A PROOF OF THE ISOMORPHISM OF *wxyz*-TRANSFORMALS AND 2 × 2 INTEGER MATRICES

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Communicated by E. Y. Rodin

(*Received January* 1980)

Abstract—This paper proves the conjecture that the family of transformals in Gosper's calculus of series rearrangements is isomorphic to 2 × 2 non-negative integer matrices under multiplication. A natural mapping is exhibited, and shown to be an isomorphism. Splitting pairs are defined as a weakening of the notion of commutativity. The proof does not rely on series rearrangement, and can be applied to splitting pairs over any monoid.

INTRODUCTION

Gosper[1] introduced R-notation as a more convenient way of representing series for certain manipulations. The notation is analogous to Horner's rule, characterizing the series by the ratio of successive terms as much as possible. Gosper defines the notation by:

$$b_n \mathop{\mathbf{R}}_{n=0}^{M} c_n = b_0 + c_0(b_1 + c_1(\cdots(b_{M-1} + c_{M-1}b_M)\cdots)) \qquad \text{(R-def).}$$

More formally, we can use the following inductive definition:

$$b_n \mathop{\mathbf{R}}_{n=m}^{m-1} c_n = 0 \qquad \text{(Inductive R-def).}$$

$$b_n \mathop{\mathbf{R}}_{n=m}^{M} c_n = b_m + c_m\Big(b_n \mathop{\mathbf{R}}_{n=m+1}^{M} c_n\Big).$$

One of the most fruitful ways to view R-notation is as a special case of matrix multiplication:

$$\prod_{n=m}^{M} \begin{pmatrix} c_n & b_n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \prod_{n=m}^{M} c_n & b_n \mathop{\mathbf{R}}_{n=m}^{M} c_n \\ 0 & 1 \end{pmatrix} \qquad \text{(Matrix R-def).}$$

The following abbreviations are useful:

$$b_n \mathop{\mathbf{R}}_{n\geq m} c_n = \lim_{M\to\infty} b_n \mathop{\mathbf{R}}_{n=m}^{M} c_n$$

$$\mathop{\mathbf{R}}_{n=m}^{M} c_n = 1 \mathop{\mathbf{R}}_{n=m}^{M} c_n.$$

If a series has no terms equal to zero, it is determined by the first term and the term ratios:

$$\sum_{n=m}^{M} a_n = a_n \mathop{\mathbf{R}}_{n=m}^{M} 1 = a_m \mathop{\mathbf{R}}_{n=m}^{M} \frac{a_n+1}{a_n} = a_m \mathop{\mathbf{R}}_{n=m}^{M} r_n.$$

where r_n denotes the term ratio

$$\frac{a_n+1}{a_n}.$$

Using (Inductive R-def), we can easily confirm the following manipulations with R-notation. Note that the differentiations are done formally, without regard to convergence for power series.

$$\mathop{\mathbf{R}}_{n=m}^{M} 0 = 1 \qquad (\text{if } M \geq m)$$

$$(c_n - 1) \mathop{\mathbf{R}}_{n=m}^{M} c_n = \left(\prod_{n=m}^{M} c_n\right) - 1$$

$$a_n b_n \mathop{\mathbf{R}}_{n=m}^{M} c_n = a_m b_n \mathop{\mathbf{R}}_{n=m}^{M} \frac{a_n + 1}{a_n} c_n$$

$$a(b_n \mathop{\mathbf{R}}_{n=m}^{M} d_n) + c_n \mathop{\mathbf{R}}_{n=m}^{M} d_n = (ab_n + c_n) \mathop{\mathbf{R}}_{n=m}^{M} d_n \qquad \text{(left-linearity)}$$

$$a_n \mathop{\mathbf{R}}_{n=2m}^{2M+1} c_n = (a_{2n} + c_{2n} a_{2n+1}) \mathop{\mathbf{R}}_{n=m}^{M} c_{2n} c_{2n+1}$$

$$\frac{d}{dx} \mathop{\mathbf{R}}_{n \geq m} f_n(x) = \left(\sum_{k=m+1}^{n} \frac{f_k'(x)}{f_k(x)}\right) \mathop{\mathbf{R}}_{n \geq m} f_n(x)$$

$$\frac{d}{dx} \mathop{\mathbf{R}}_{n \geq m} c_n x = c_m \mathop{\mathbf{R}}_{n \geq m+1} \left(\frac{n - m + 1}{n - m}\right) c_n x$$

$$\mathop{\mathbf{R}}_{n \geq m} x = (1 + x) \mathop{\mathbf{R}}_{n \geq m} x^2 = (1 + x)(1 + x^2) \mathop{\mathbf{R}}_{n \geq m} x^4 = \cdots.$$

More examples of R-notation can be found in the original paper (Gosper[1]).

SPLITTING IN MATRIX NOTATION

Gosper[1] introduced term splitting, and showed how R-notation made it more convenient. The definition of splitting given here can be shown to specialize to Gosper's concept. If we let $A_{k+1,n} = T_{k,n} A_{k,n} S_{k,n+1}$ for some sequences of matrices A, T and S, then we get

$$\prod_{n=m}^{M} A_{k+1,n} = T_{k,m} A_{k,m} S_{k,m+1} T_{k,m+1} A_{k,m+1} S_{k,m+2} \cdots T_{k,M} A_{k,M} S_{k,M+1}.$$

This is not very interesting unless $S_{k,n} T_{k,n} = 1$, so we want $T_{k,n} = S_{k,n}^{-1}$. Alternatively, we could avoid the problem of non-invertible matrices with the following.

Definition. $[S, A]$ is a splitting pair if for some monoid $\mathcal{M}$, S and $A \in \mathcal{M}''$, where $\mathcal{M}''$ is the doubly-indexed sequences over $\mathcal{M}$, and $(\forall k, n) S_{k,n} A_{k+1,n} = A_{k,n} S_{k,n+1}$.

To get Gosper's concept of term splitting, we notice that $\mathop{\mathbf{R}}_{n=m}^{M} r_{k,n}$ corresponds to $\prod_{n=m}^{M} \begin{pmatrix} r_{k,n} & 1 \\ 0 & 1 \end{pmatrix}$ as in (Matrix R-def). If we let $A_{k,n} = \begin{pmatrix} r_{k,n} & 1 \\ 0 & 1 \end{pmatrix}$ and try to solve for $S_{k,n} = \begin{pmatrix} u_{k,n} & s_{k,n} \\ a_{k,n} & b_{k,n} \end{pmatrix}$, we get

$$\text{left hand side} = \begin{pmatrix} u_{k,n} r_{k+1,n} & u_{k,n} + s_{k,n} \\ a_{k,n} r_{k+1,n} & a_{k,n} + b_{k,n} \end{pmatrix}$$

$$\text{right hand side} = \begin{pmatrix} r_{k,n} u_{k,n+1} + a_{k,n+1} & r_{k,n} s_{k,n+1} + b_{k+1,n} \\ a_{k,n+1} & b_{k,n+1} \end{pmatrix}.$$

We can simplify by assuming that the solution we're interested in corresponds to something in R-notation, so $a_{k,n} = 0$ and $b_{k,n} = 1$. With this assumption and some manipulation we get:

$$u_{k,n} = 1 - s_{k,n} + r_{k,n} s_{k,n+1} \qquad \text{(udef)}$$

$$r_{k+1,n} = \frac{u_{k,n+1}}{u_{k,n}} r_{k,n} \qquad \text{(recur)}.$$

These formulas are exactly the same as the ones Gosper used to define term splitting. His generalization (Jsplit) can be restated as follows.

LEMMA 1. (Jsplit)
If $[S, A]$ is splitting pair, and $M-n$ and $J-j$ are integers, then

$$\prod_{k=j}^{J} S_{k,m} \prod_{n=m}^{M} A_{J+1,n} = \prod_{n=m}^{M} A_{j,n} \prod_{k=j}^{J} S_{k,M+1}.$$

The proof is trivial. As a special case we have

$$\prod_{k=j}^{J}\begin{pmatrix} u_{k,m} & s_{k,m} \\ 0 & 1 \end{pmatrix}\prod_{n=m}^{M}\begin{pmatrix} r_{J+1,n} & 1 \\ 0 & 1 \end{pmatrix} = \prod_{n=m}^{M}\begin{pmatrix} r_{j,n} & 1 \\ 0 & 1 \end{pmatrix}\prod_{k=j}^{J}\begin{pmatrix} u_{k,M+1} & s_{k,M+1} \\ 0 & 1 \end{pmatrix}.$$

If we abbreviate

$$r_{k,m} r_{k,m+1} \cdots r_{k,M} s_{k,M+1} = \frac{a_{k,M+1} s_{k,M+1}}{a_{k,m}} = t_{k,m}(M)$$

and simplify, we have the special case of J-split for series:

$$\mathop{\mathbf{R}}_{n=m}^{M} r_{j,n} = (s_{k,m} - t_{k,m}(M)) \mathop{\mathbf{R}}_{k=j}^{J} u_{k,m} + u_{j,m} u_{j+1,m} \cdots u_{J,m} \mathop{\mathbf{R}}_{n=m}^{M} r_{J+1,n} \qquad \text{(Jsplit)}.$$

Specializing still further, we can get a single splitting:

$$\mathop{\mathbf{R}}_{n=m}^{M} r_{k,n} = s_{k,m} - t_{k,m}(M) + u_{k,m} \mathop{\mathbf{R}}_{n=m}^{M} r_{k+1,n} \qquad \text{(1split)}.$$

TRANSFORMALS

Define: A *transformal* is a mapping from $\mathcal{M}'' \times \mathcal{M}'' \to \mathcal{M}'' \times \mathcal{M}''$ which takes splitting pairs into splitting pairs.

One simple transformal is the *splitformal*: If $B_{k,n}$ is invertible, then $[S, A] \mapsto [B_{k,n}^{-1} S_{k,n} B_{k+1,n}, B_{k,n}^{-1} A_{k,n} B_{k,n+1}]$ is a transformal. In Gosper's section "Simplifying Terms" (which he applies to his series for the second lemniscate constant), the splitting pair he works with is actually

$$\left[\begin{pmatrix} 1 & b_{k,n} q_{n-1} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} p_n/q_n & c_{k,n} \\ 0 & 1 \end{pmatrix}\right]$$

although he uses

$$\left[\begin{pmatrix} \frac{c_{k+1,n}}{c_{k,n}} & \frac{b_{k,n} q_{n-1}}{c_{k,n}} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{c_{k,n+1}}{c_{k,n}} \frac{p_n}{q_n} & 1 \\ 0 & 1 \end{pmatrix}\right],$$

since it is easier to represent in his notation. The two are obviously related by a splitformal with

$$B_{k,n} = \begin{pmatrix} c_{k,n} & 0 \\ 0 & 1 \end{pmatrix}.$$

We can now define an important family of transformals, and prove a basic theorem of the rearrangment calculus, conjectured by Gosper.

Define: $[S, A] \mapsto [S', A']$ is a *wxyz*-transformal if

$$S'_{k,n} = \prod_{j=0}^{w-1} S_{wk+yn+j,xk+zn} \prod_{j=0}^{x-1} A_{w(k+1)+yn,xk+zn+j}$$

and

$$A'_{k,n} = \prod_{j=0}^{y-1} S_{wk+yn+j,xk+zn} \prod_{j=0}^{z-1} A_{wk+y(n+1),xk+zn+j}.$$

LEMMA 2.

A $wxyz$-transformal is a transformal.

Proof:

$$\begin{aligned}
S'_{k,n}A'_{k+1,n} &= \left(\prod_{j=0}^{w-1} S_{wk+yn+j,xk+zn} \prod_{j=0}^{z-1} A_{w(k+1)+yn,xk+zn+j}\right)\left(\prod_{j=0}^{y-1} S_{w(k+1)+yn+j,x(k+1)+zn}\right. \\
&\qquad \left.\prod_{j=0}^{z-1} A_{w(k+1)+y(n+1),x(k+1)+zn+j}\right) \\
&= \prod_{j=0}^{w-1} S_{wk+yn+j,xk+zn} \prod_{j=0}^{y-1} S_{w(k+1)+yn+j,xk+zn} \prod_{j=0}^{x-1} A_{w(k+1)+y(n+1),xk+zn+j} \\
&\qquad \prod_{j=0}^{z-1} A_{w(k+1)+y(n+1),x(k+1)+zn+j} \\
&= \prod_{j=0}^{w+y-1} S_{wk+yn+j,xk+zn} \prod_{j=0}^{x+z-1} A_{w(k+1)+y(n+1),xk+zn+j} \\
&= \prod_{j=0}^{y-1} S_{wk+yn+j,xk+zn} \prod_{j=0}^{w-1} S_{wk+y(n+1)+j,xk+zn} \prod_{j=0}^{z-1} A_{w(k+1)+y(n+1),xk+zn+j} \\
&\qquad \prod_{j=0}^{x-1} A_{w(k+1)+y(n+1),xk+z(n+1)+j} \\
&= \left(\prod_{j=0}^{y-1} S_{wk+yn+j,xk+zn} \prod_{j=0}^{z-1} A_{wk+y(n+1),xk+zn+j}\right) \\
&\qquad \left(\prod_{j=0}^{w-1} S_{wk+y(n+1)+j,xk+z(n+1)} \prod_{j=0}^{x-1} A_{w(k+1)+y(n+1),xk+z(n+1)+j}\right) \\
&= A'_{k,n}S'_{k,n+1}.
\end{aligned}$$

THEOREM 1.

The $wxyz$-transformals form a monoid under composition, and the natural mapping from the non-negative 2×2 integer matrices under multiplication $\begin{pmatrix} w & x \\ y & z \end{pmatrix} \to wxyz$-transformal is a homomorphism.

Proof: **(1) Mapping onto—trivial by definition.**

(2) Mapping takes multiplication into composition

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mapsto ([S, A] \mapsto [S, A]) = \text{identity transformal.}$$

Now consider the transformal defined by

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} wa + xc & wb + xd \\ ya + zc & yb + zd \end{pmatrix}.$$

Let $i1 = wk + yn$, $i2 = xk + zn$, then

$$\begin{aligned}
S'_{k,n} &= \prod_{j=0}^{w-1}\left(\prod_{m=0}^{a-1} S_{a(i1+j)+c(i2)+m,b(i1+j)+d(i2)} \prod_{m=0}^{b-1} A_{a(i1+j+1)+c(i2),b(i1+j)+d(i2)+m}\right) \\
&\qquad \prod_{j=0}^{x-1}\left(\prod_{m=0}^{c-1} S_{a(i1+w)+c(i2+j)+m,b(i1+w)+d(i2+j)} \prod_{m=0}^{d-1} A_{a(i1+w)+c(i2+j+1),b(i1+w)+d(i2+j)+m}\right)
\end{aligned}$$

$$= \prod_{j=0}^{w-1}\left(\prod_{m=0}^{a-1} S_{a(i1+j)+c(i2)+m,b(i1)+d(i2)}\right)\prod_{j=0}^{w-1}\left(\prod_{m=0}^{b-1} A_{a(i1+w)+c(i2),b(i1+j)+d(i2)+m}\right)$$

$$\prod_{j=0}^{x-1}\left(\prod_{m=0}^{c-1} S_{a(i1+w)+c(i2+j)+m,b(i1+w)+d(i2)}\right)\prod_{j=0}^{x-1}\left(\prod_{m=0}^{d-1} A_{a(i1+w)+c(i2+x),b(i1+w)+d(i2+j)+m}\right.$$

$$= \prod_{j=0}^{wa-1} S_{a(i1)+c(i2)+j,b(i1)+d(i2)} \prod_{j=0}^{wb-1} A_{a(i1+w)+c(i2),b(i1)+d(i2)+j} \prod_{j=0}^{xc-1} S_{a(i1+w)+c(i2)+j,b(i1+w)+d(i2)}$$

$$\prod_{j=0}^{xd-1} A_{a(i1+w)+c(i2+x),b(i1+w)+d(i2)+j}$$

$$= \prod_{j=0}^{wa+xc-1} S_{a(i1)+c(i2)+j,b(i1)+d(i2)} \prod_{j=o}^{wb+xd-1} A_{a(i1+w)+c(i2+x),b(i1)+d(i2)+j}$$

$$= \prod_{j=0}^{wa+xc-1} S_{(wa+xc)k+(ya+zc)n+j,(wb+xd)k+(yb+zd)n} \prod_{j=0}^{wb+xd-1} A_{(wa+xc)(k+1)+(ya+zc)n,(wb+xd)k+(yb+zd)n+j}\,.$$

Note. $A'_{k,n}$ is handled analogously.

COROLLARY 1.

If the base monoid $\mathcal{M}$ has an element of infinite order then the mapping of theorem 1 is an isomorphism.

Proof. If $wxyz$-transformal = $abcd$-transformal, we need to show that $\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$. Let α be an element of infinite order in $\mathcal{M}$. Let $[S, A] = [\alpha, 1]$ or $[1, \alpha]$. Note that a pair of constants $s, a \in \mathcal{M}$ form a splitting pair $[s, a]$ iff s and a commute in $\mathcal{M}$.

$$\begin{array}{ll} [\alpha, 1] \mapsto [\alpha^w, \alpha^y] & [\alpha, 1] \mapsto [\alpha^a, \alpha^c] \\ [1, \alpha] \mapsto [\alpha^x, \alpha^z] & [1, \alpha] \mapsto [\alpha^b, \alpha^d] \end{array}$$

So $\alpha^w = \alpha^a$, $\alpha^x = \alpha^b$, $\alpha^y = \alpha^c$, $\alpha^z = \alpha^d$. Thus $w = a$, $x = b$, $y = c$, $z = d$, and the mapping is one-to-one.

Note. If the base monoid $\mathcal{M}$ is actually a group then the homomorphism can be extended to all 2×2 integer matrices. Since $\prod_{j=0}^{n} B_j B_{n+1} = \prod_{j=0}^{n+1} B_j$, we can use $\prod_{j=0}^{n} B_j = (\prod_{j=0}^{n+1} B_j)B_{n+1}^{-1}$ to inductively define the products for negative w, x, y, or z.

Here are some examples of $wxyz$-transformals. The names are taken from the corresponding transformals in Gosper's paper.

Name	Matrix	$[S, A] \mapsto [S', A']$ $[S'_{k,n}, A'_{k,n}]$
identity	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$[S_{k,n}, A_{k,n}]$
leftshift	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$[S_{k,n+k}A_{k+1,n+k}, A_{k,n+k}]$
rightshift	$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$	$[S_{k,n-k}A^{-1}_{k+1,n-k-1}, A_{k,n-k}]$
negation	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$[S^{-1}_{-k-1,n}, A_{-k,n}]$
dual	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$[A_{n,k}, S_{n,k}]$
double	$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$	$[S_{2k,n}S_{2k+1,n}, A_{2k,n}]$
ndnd	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$[S^{-1}_{-k-1,-n}, A^{-1}_{-k,-n-1}]$

It might be instructive to derive R-notation form for one of the transformals. Dual is a good transformal to demonstrate, as it has a very nice form in R-notation. If S and A are chosen to be $\begin{pmatrix} u_{k,n} & s_{k,n} \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} r_{k,n} & 1 \\ 0 & 1 \end{pmatrix}$ respectively, then the defintion of Dual gives us the splitting pair $\left[\begin{pmatrix} r_{n,k} & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} u_{n,k} & s_{n,k} \\ 0 & 1 \end{pmatrix}\right]$, to which we can apply a splitformal with $B_{k,n} = \begin{pmatrix} s_{n,k} & 0 \\ 0 & 1 \end{pmatrix}$. This gives us a nice splitting in R-notation:

$$r_{k,n} \leftarrow \frac{s_{n+1,k}}{s_{n,k}} u_{n,k}$$

$$s_{k,n} \leftarrow \frac{1}{s_{n,k}}$$

$$u_{k,n} \leftarrow \frac{s_{n,k+1}}{s_{n,k}} r_{n,k}.$$

All the examples so far have been over the monoid consisting of 2×2 matrices with (0, 1) as the second row. This a particularly rich area, because of the correspondence with R-notation. Another promising line of inquiry is to consider general 2×2 matrices. Then the product $\prod_n \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ corresponds to composition of the functions $\frac{a_n x + b_n}{c_n x + d_n}$. Although it is not possible for continued fractions $\begin{pmatrix} 1 & 0 \\ c_n & 1 \end{pmatrix}$ to form splitting pairs with other continued fractions or with series, they could split with the general functions corresponding to $\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$.

Another possible place for research, is to generalize $wxyz$-transformals to real values of w, x, y, and z, by appropriate definitions for the products.

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