HW 5 Answer Key

Section 3.2

4. Base case: $n = 0$

$2 = \frac{1-(-7)}{4}$ is true.

We assume that

$$2 - 2.7 + 2.7^2 - 2.7^3 ... 2.(-7)^n = \frac{1-(-7)^{n+1}}{4}.$$

We want to show that

$$2 - 2.7 + 2.7^2 - 2.7^3 ... 2.(-7)^{n+1} = \frac{1-(-7)^{n+1+1}}{4}.$$

Using the inductive hypothesis we are able to re-write the left side of the equation

$$\frac{1-(-7)^{n+1}}{4} + 2.(-7)^{n+1}$$
$$= \frac{1-(-7)^{n+1} + 8.(-7)^{n+1}}{4}$$
$$= \frac{1+7.(-7)^{n+1}}{4}$$
$$= \frac{1-(-7).(-7)^{n+1}}{4}$$
$$= \frac{1-(-7)^{n+1+1}}{4}$$

QED

6 By looking at the first few sums, we guess that the sum is $n/n+1$ Prove this by induction.

Base case: $n = 1$

$$\frac{1}{1+1} = \frac{1}{2}$$

We assume that

$$\frac{1}{1*2} + \frac{1}{2*3} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

We want to show that

$$\frac{1}{1*2} + \frac{1}{2*3} + ... + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2}$$

Using the inductive hypothesis we are able to re-write the left side of the equation

$$\frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{n+1}{n+2}$$

QED

8. Inductive Hypothesis:

$\sum_1^n j^3 = [n(n+1)/2]^2$ whenever n is a positive integer

Base case:

$$n = 1$$

$$\sum_1^1 j^3 = 1^3 = 1$$

$$[1(1+1)/2]^2 = 1^2 = 1$$

$$1 = 1$$

Inductive Step:

$$\sum_1^{n+1} j^3 = [(n+1)(n+2)/2]^2$$

$\sum_1^{n+1} j^3$	$= [(n+1)(n+2)/2]^2$	What you're trying to prove
$\sum_1^{n+1} j^3$	$= \sum_1^n j^3 + (n+1)^3$	math
	$= [(n+1)(n+2)/2]^2 + (n+1)^3$	Inductive hypothesis
	$= (n+1)^2(n^2/4 + n + 1)$	math
	$= (n+1)^2((n^2 + 4n + 4)/4)$	math
	$= ((n+1)(n+2)/2)^2$	math

QED

10 The base case is clear, since $1 \cdot 1! = 2! - 1$
Assuming the inductive hypotheses, we have
$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! + (n+1) \cdot (n+1)!$
$= (n+1)! - 1 + (n+1) \cdot (n+1)! = (n+1)!(1 + n + 1) - 1 = (n+2)! - 1$
QED

11. Let P(n) be $1 + nh \leq (1+h)^n, h \geq 0$.
Basis Step: It is true for $n = 0, 1 + 0.h = 1 \leq 1 = (1+h)^0$.
Inductive Step: Assume $1 + nh \leq (1+h)^n$.Then since $(1+h) \geq 0, (1+h)^{n+1} \geq (1+h)(1+h)^n \geq (1+h)(1+nh) = 1 + (n+1)h + nh^2 \geq 1 + (n+1)h$.

14. Base case is $n = 2.2! < 2^2$.
Inductive Step: $(n+1)! < (n+1)^{n+1}$

(n+1)!	= (n+1)n!	math	
(n+1)n!	$< (n+1)n^n$	Inductive Hypothesis	QED
$(n+1)n^n$	$< (n+1)(n+1)^n$	math	
$(n+1)(n+1)^n$	$= (n+1)^{n+1}$	math	

15 This problem basically asks you to prove
$\sum\limits_1^n j \cdot (j+1) = \frac{n(n+1)(n+2)}{3}$ Base Case $n = 1$:
$1 \cdot (1+1) = 1 \cdot 2 = 2$
$\frac{1(1+1)(1+2)}{3} = \frac{1 \cdot 2 \cdot 3}{3} = \frac{6}{3} = 2$
$2 = 2$ QED
Assuming the inductive hypothesis prove: $\sum\limits_1^{n+1} j \cdot (j+1) = \frac{(n+1)(n+2)(n+3)}{3}$
$\sum\limits_1^{n+1} j \cdot (j+1)$
$= \sum\limits_1^n j \cdot (j+1) + (n+1)(n+2)$
$= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) = \frac{n(n+1)(n+2)}{3} + \frac{3(n+1)(n+2)}{3}$
$= \frac{(3+n)(n+1)(n+2)}{3} = \frac{(n+1)(n+2)(n+3)}{3}$ QED

18. The base case, $n=2$, is the true statement that $5/4$ is less than $6/4$. Assume the inductive hypothesis. Then we have
$1+\frac{1}{4}+\frac{1}{9}+...+\frac{1}{n^2}\leq 2-\frac{1}{n}$
$1+\frac{1}{4}+\frac{1}{9}+...+\frac{1}{n^2}+\frac{1}{n+1^2}\leq 2-\frac{1}{n}+\frac{1}{(n+1)^2}$
$=2-(\frac{1}{n}-\frac{1}{n+1^2})$
$=2-(\frac{n^2+2n+1-n}{n.(n+1)^2})$
$=2-\frac{n^2+n}{n.(n+1)^2}-\frac{1}{n.(n+1)^2}$
$=2-\frac{1}{n+1}-\frac{1}{n.(n+1)^2}\leq 2-\frac{1}{n+1}$

22.The statement is true for the base case, $n=0$, since $6|0$. Suppose that $6|(n^3-n)$. We must show that $6|((n+1)^3-(n+1))$ .If we expand the expression in question, we get $n^3+3n^2+3n+1-n-1=(n^3-n)+3n(n+1)$. By the inductive hypothesis 6 divides the first term. Furthermore clearly 3 divides the second term, and the second ternm is also even, since one of n and n+1 is even. Therefore 6 divides the second term as well. This tells us that 6 divides the given expression as desired.

36.We prove this by induction on k. The base case $k=0$, is the trivial statement that $1\equiv 1(modm)$. Suppose that the statement is true for k. We must show it for $k+1$. so let $a\equiv b(modm)$. By the inductive hypothesis we know that $a^k\equiv b^k(modm)$. Then we apply theorem 7 from section 2.3 to conclude that $a.a^k\equiv b.b^k(modm)$, which by definition says that $a^{k+1}\equiv b^{k+1}(modm)$, as desired.
This could also be done in other ways without using theorem by multiplying by a and b respectively and simplifying.

40. The base case is trivial since it merely says $\neg p_1$ is equivalent to itself. Assuming the inductive hypothesis, we look at $\neg(p_1\vee p_2\vee.....\vee p_n\vee p_{n+1})$. By DeMorgan's law (grouping all but the last term together) this is the same $\neg(p_1\vee p_2...\vee p_n)\wedge\neg p_{n+1}$ .But, by the inductive hypothesis, this equals, $\neg p_1\wedge\neg p_2...\neg p_n\wedge\neg p_{n+1}$, as desired.

43 The implication $P(1)\rightarrow P(2)$ is false.
The two sets do not overlap if $n+1=2$.

44 The flaw comes in the inductive step, where we are implicitly assuming
that $n\geq 1$ in order to talk about a^{n-1} in the denominator (otherwise the exponent is not a nonnegative integer, so we cannot apply the inductive hypothesis). We checked the base case only for $n=0$, so we are not justified in assuming that $n\geq 1$ when we try to prove the statment for $n+1$ in the inductive step. Indeed it is precisely at $n=1$ that the proposition breaks down.

46 Suppose that the statment $\forall nP(n)$ has been proved by this method. Let S be the set of counterexamples to P, i.e. let $S=\{n|\neg P(n)\}$. We will show that $S=\phi$. If $S\neq\phi$, then let n be the minimum element of S (which exists by the well-ordering property). Clearly $n\neq 1$ and $n\neq 2$, by the base cases of our proof method. But since n is the least element of S and $n\geq 3$, we know that $P(n-1)$ and $P(n-2)$ are true. Therefore by the inductive step of our proof method, we know that P(n) is also true. This contradicts the choice of n. Therefore $S=\phi$, as desired.

Base case:
$n=1$
$\sum_1^1 1/j(j+1)=1/1\cdot 2=1/2$
$n/(n+1)=1/(1+1)=1/2$
$1/2=1/2$
Inductive Step: $\sum_1^{n+1} 1/j(j+1)=(n+1)/(n+2)$

$\sum_1^{n+1} 1/j(j+1)$	$= (n+1)/(n+2)$	What you're trying to prove
$\sum_1^{n+1} 1/j(j+1)$	$= \sum_1^{n} 1/j(j+1) + 1/(n+1)(n+2)$	math
	= n/(n+1) +1/(n+1)(n+2)	inductive hypothesis
	$= n(n+2)+1)/(n+1)(n+2)$	math
	$= (n^2+2n+1)/(n+1)(n+2)$	math
	$= (n+1)/(n+2)$	math

QED