Assignment 1 ISM 207, Random Process Models in Engineering

Instructor: John Musacchio ,
Due: April 17, 2012

1. Suppose some rolls a fair 6 sided die over and over again until they get a 6. What are Ω, $\mathcal{F}$, and P? Let X be the sum of all the rolls made, including the final 6. What is $E[X]$?

2. Suppose that X and Y are independent random variables with densities $f_X(x)$ and $f_Y(y)$ respectively. Let $Z := aX + bY$ where a and b are scalars. What is the density of Z?

3. Suppose that X and Y are independent uniformly distributed random variables on $[0, 1]$. Let $Z := X - Y$. What is $E[X|Z]$? What is $E[X|Y]$?

4. Suppose that X is a random variable that takes values in the set $\{1, 2, ..., N\}$. Show that

$$E[X] = \sum_{n=1}^{N} P(X \geq n).$$

   Note: this fact is true even if X takes values in the set $\{0, 1, 2, ...\}$. A related useful fact is that if X is a non-negative real valued random variable, i.e. it takes values on $[0, \infty)$, then $E[X] = \int_0^\infty P(X > x)dx$.

5. (a) Let $X := \sum_{i=1}^{n} 1_{A_i}$ where each A_i is an event, and 1_{A_i} is the indicator random variable that takes on the value 1 if A_i happens and 0 otherwise. Derive a formula for the variance of X in terms of $P(A_i)$ and $P(A_i \cap A_j), i \neq j$.

   (b) If k balls are put at random into n boxes, what is the variance of X, the number of non-empty boxes?

6. (a) Let $\phi : \mathbb{R} \to [0, \infty)$ be an increasing function, let x be any arbitrary real number and suppose $\phi(x) \neq 0$. Show that

$$P(X \geq x) \leq \frac{E[\phi(x)]}{\phi(x)}.$$

   Hint: fix x, and define

$$h(y) = \begin{cases} 0 & y < x \\ \phi(x) & y \geq x. \end{cases}$$

   Note that $h(y) \leq \phi(y)$ for all y. Use the integral definition of expected value to show that $E[h(X)] \leq E[\phi(X)]$. For this problem, you may assume that X has a density $f_X(x)$, but the result you are proving is actually true even if X does not have a density.

   (b) Use the fact you proved in part (a) to show that if X is a non-negative random variable with mean $E[X]$,

$$P(X \geq x) \leq \frac{E[X]}{x}.$$

   This is known as the Markov Inequality.

   (c) Use the fact that you proved in part (a) to show if X is any real valued random variable with mean $E[X]$ and variance $\mathrm{var}(X)$,

$$P((X - E[X])^2 \geq x) \leq \frac{\mathrm{var}(X)}{x}.$$

   This is known as the Chebyshev Inequality.

7. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a convex function, that is,

$$\alpha g(x) + (1 - \alpha)g(y) \geq g(\alpha x + (1 - \alpha)y)$$

   for all $\alpha \in (0, 1)$ and $x, y \in \mathbb{R}$. Also suppose that X is a random variable with density $f_X(x)$. Show that

$$E[g(X)] \geq g(E[X]).$$

   This useful fact is known as Jensen's inequality.

   Hint: Let $m = E[X]$. Observe that $\frac{1}{2}g(m + \epsilon) + \frac{1}{2}g(m - \epsilon) \geq g(m)$ and this in turn implies that $g(m) - g(m - \epsilon) \leq g(m + \epsilon) - g(m)$. Therefore

$$\lim_{\epsilon \downarrow 0} \frac{g(m) - g(m - \epsilon)}{\epsilon} \leq \lim_{\epsilon \downarrow 0} \frac{g(m + \epsilon) - g(m)}{\epsilon}.$$

   Pick a to be any number between the two limits define the affine function $l : \mathbb{R} \to \mathbb{R}$ such that $l(x) = a(x - c) + g(c)$. Show that $g(x) \geq l(x)$ for all x.