The Price of Anarchy in a Network Pricing Game.

Sept 27, 2007
Allerton Conference

John Musacchio
Assistant Professor
Technology and Information Management
University of California, Santa Cruz
johnm@soe.ucsc.edu

Joint work with:

Shuang Wu
University of California, Santa Cruz

Overview

- Model
  - Single source-destination pair
  - Competing providers
  - Non-atomic users
  - Traffic dependent latency
  - Elastic user demand
- Model due to
  - Acemoglu and Ozdaglar [1]
- Elastic user demand extension:
  - Hayrapetyan, Tardos, Wexler [3]

[1] D. Acemoglu and A. Ozdaglar, "Competition and Efficiency in Congested Markets," *Math. of OR*, Feb. 2007.

[3] A. Hayrapetyan, E. Tardos and T. Wexler, "A Network Pricing Game for Selfish Traffic," *Distributed Computing,* March *2007.*

Overview

- Price of anarchy:

$$\frac{\text{Social Welfare with Optimal Prices}}{\text{Social Welfare Nash with Nash prices}} = \frac{3}{2}$$

- Result due to Ozdaglar [2]
- We prove the same result a different way
  - Ozdaglar proof: mathematical programming argument
  - Our proof: circuit analogy, linear algebra

[2] A. Ozdaglar, ``Price Competition with Elastic Traffic,'' to appear in *Networks*

Some Other Related Work

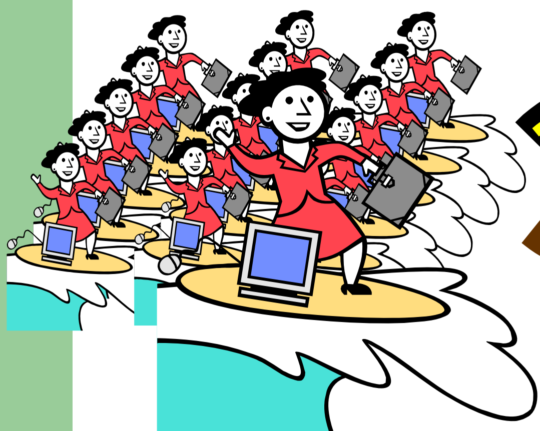
- Roughgarden 02, 03
  - Selfish routing games
  - Taxes to induce optimal routing
- Johari and Tsitsiklis 05
  - Cournot rate allocation mechanisms
  - Different situation
  - Very similar structure

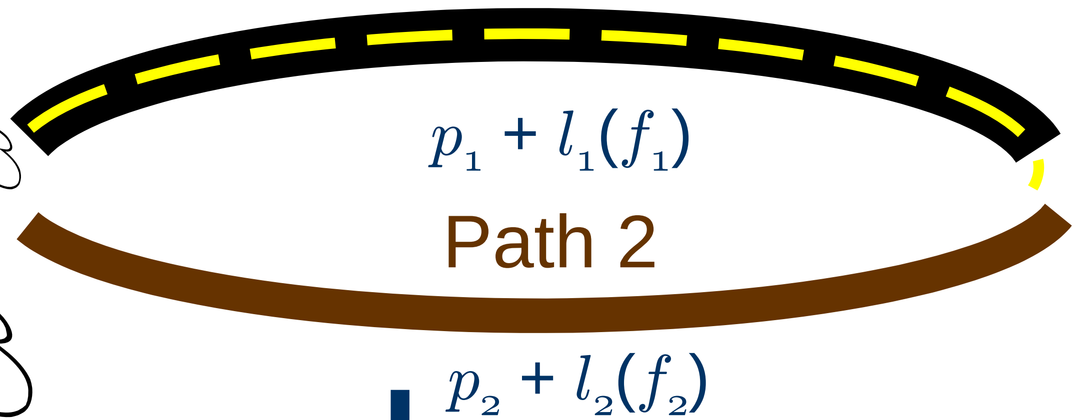
Organization

- Overview
- Model Description
- Nash and Social Optimum Characterization
- Circuit Analogy
- PoA Proof Overview

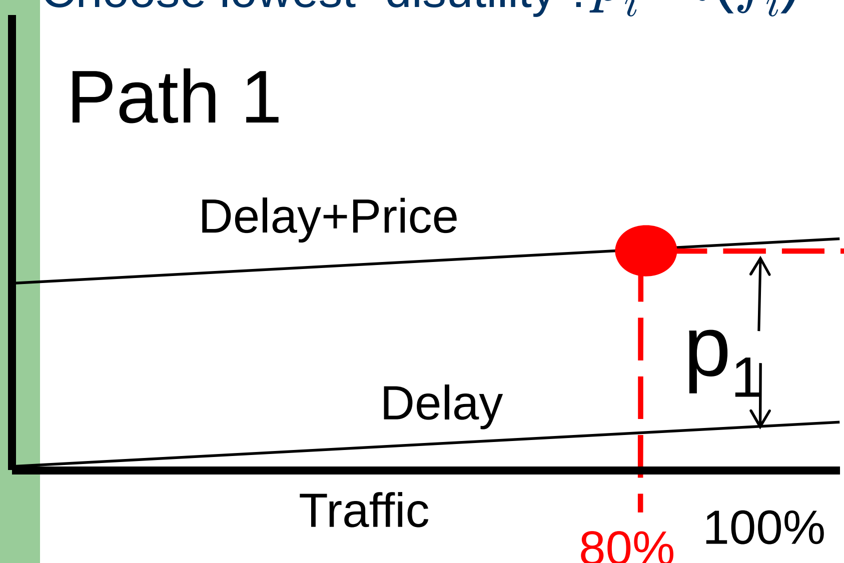
Wardrop Equilibrium for given prices

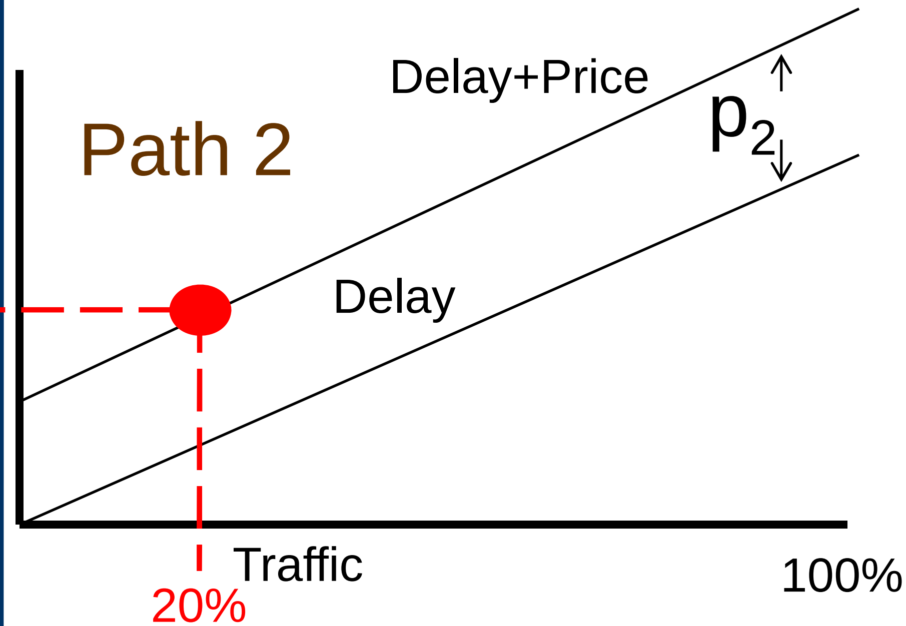
Non-Atomic Users





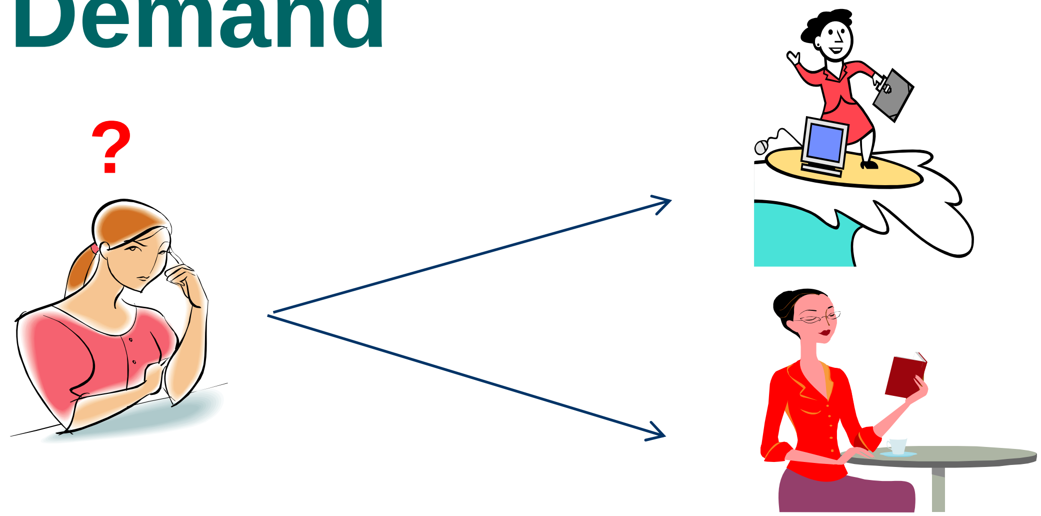
Choose lowest "disutility": $p_i + l(f_i)$

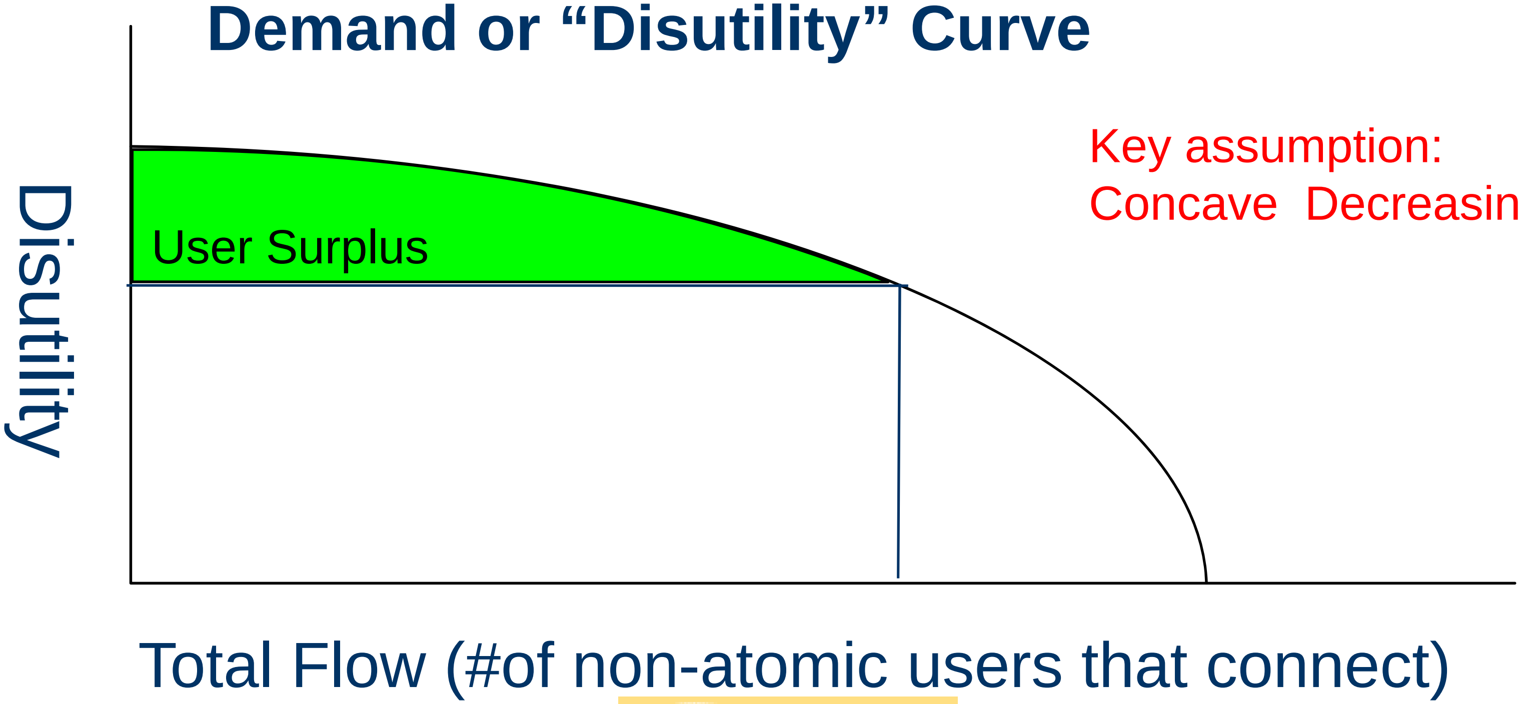




UC SANTA CRUZ

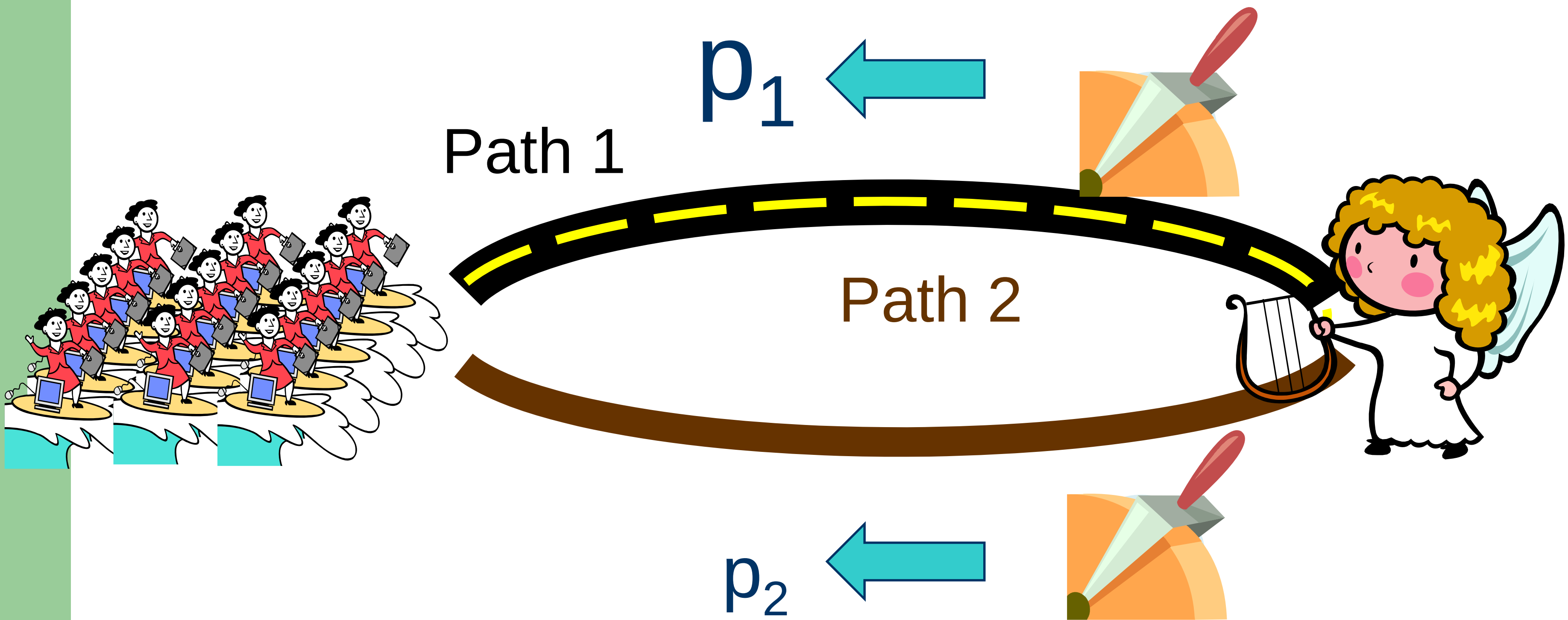
Elastic Demand



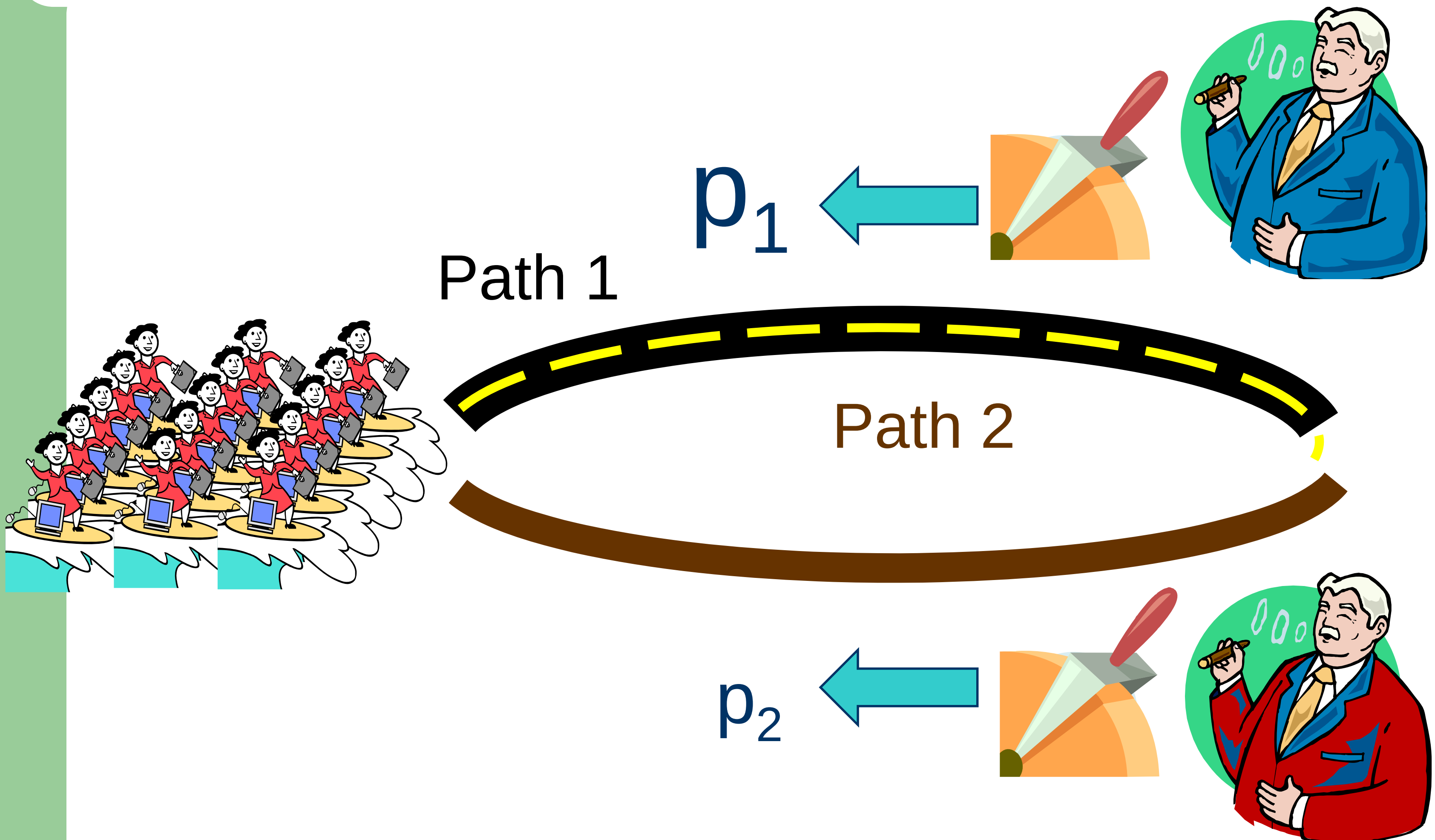


Key assumption:
Concave Decreasing

Social Optimum Pricing



Network Pricing



Wardrop Equilibrium for given (p_1, p_2, p_3)

For now consider linear latency case:

$$l_i(f_i) = a_i f_i + b_i$$

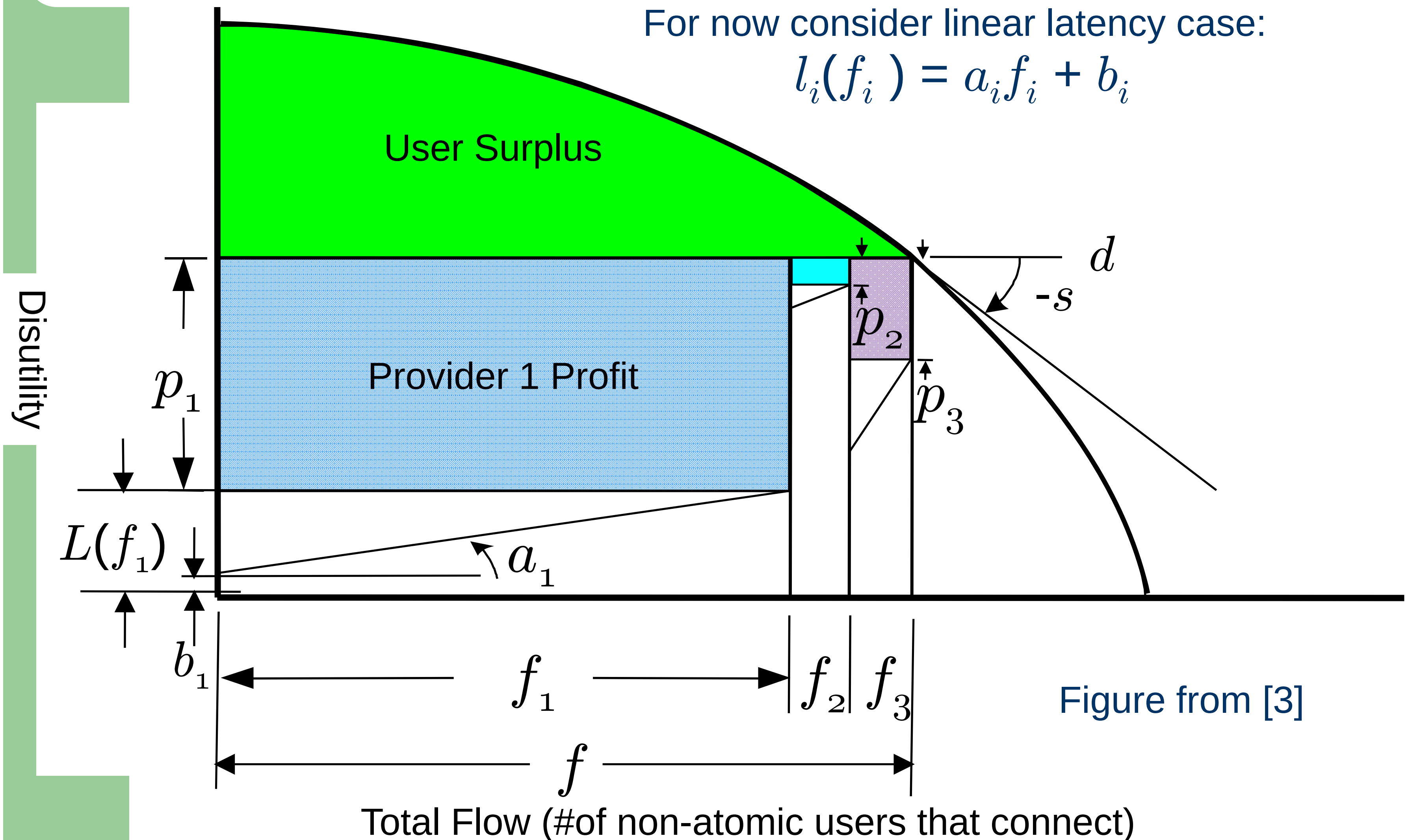
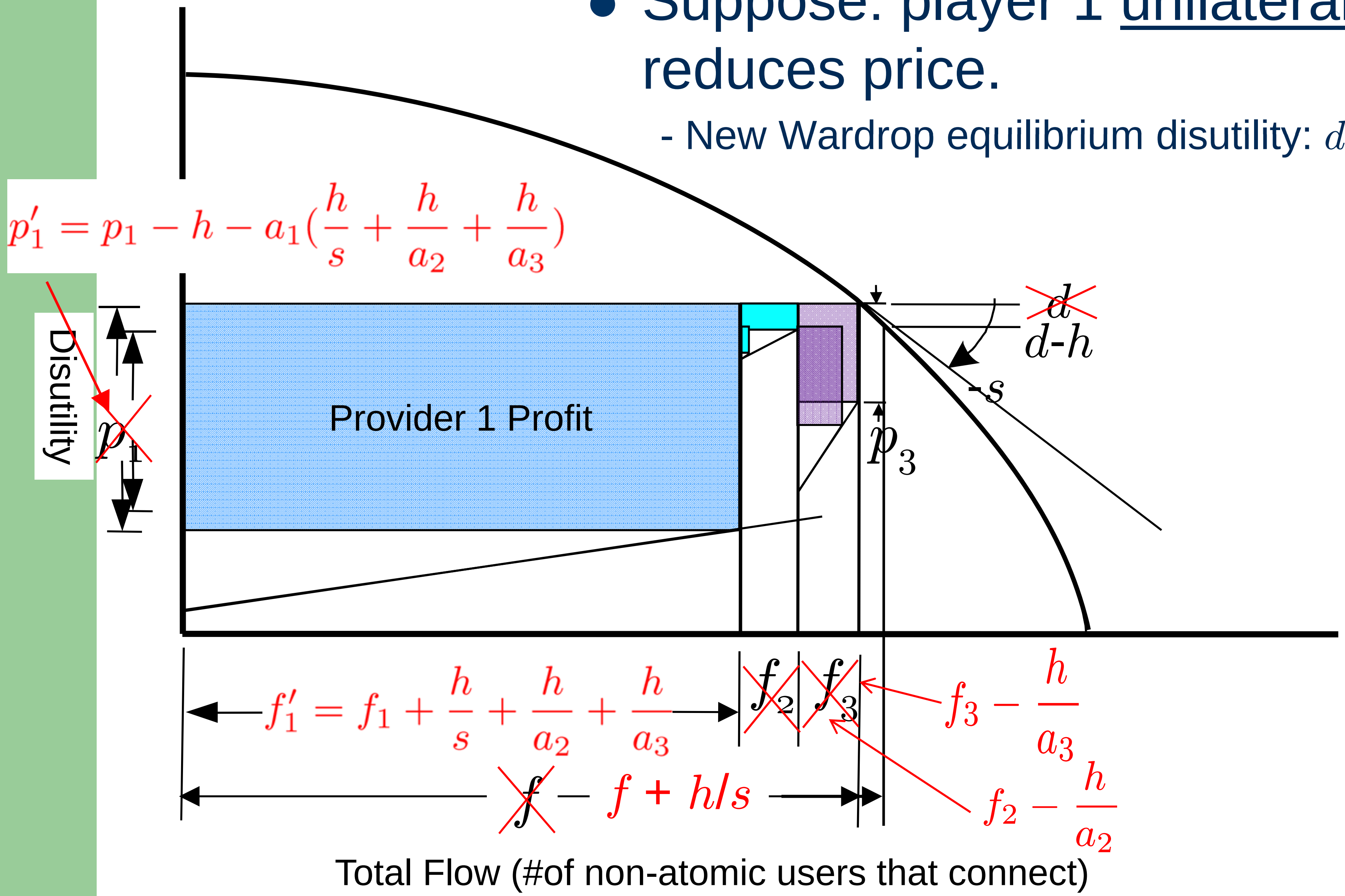


Figure from [3]

[3] A. Hayrapetyan, E. Tardos and T. Wexler, "A Network Pricing Game for Selfish Traffic," *Distributed Computing,* March *2007.*

Consequence of price change

- Suppose: player 1 unilaterally reduces price.
  - New Wardrop equilibrium disutility: d-h.



Nash Equilibrium Analysis

Convenient definition:

$$\delta_i \triangleq \left[\frac{1}{s} + \sum_{j \neq i} \frac{1}{a_j}\right]^{-1}$$

New profit – old profit:

$$\pi_i' - \pi = h\delta_i^{-1}(p_i - (\delta_i + a_i)f_i) - o(h^2)$$

=0

Nash equilibrium condition:

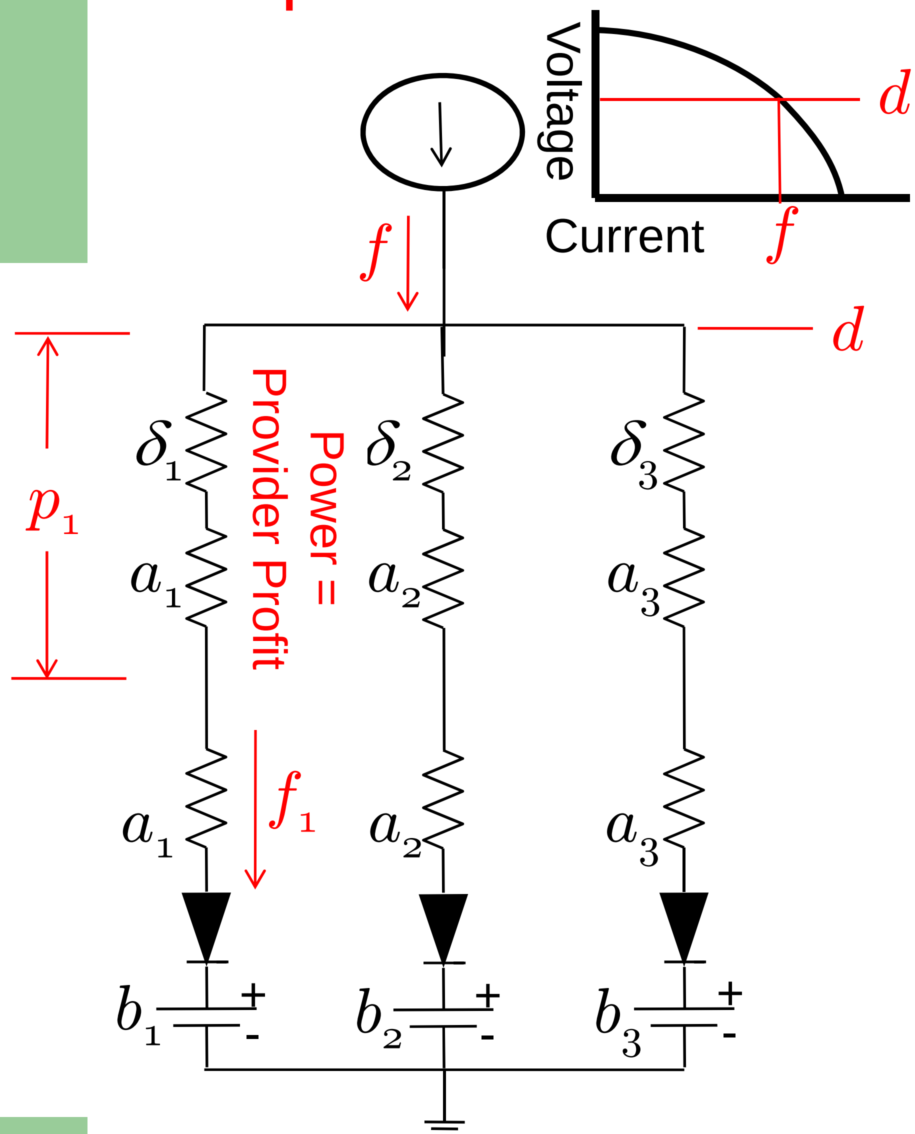
$$\frac{p_i}{f_i} = a_i + \delta_i$$

Social optimum pricing

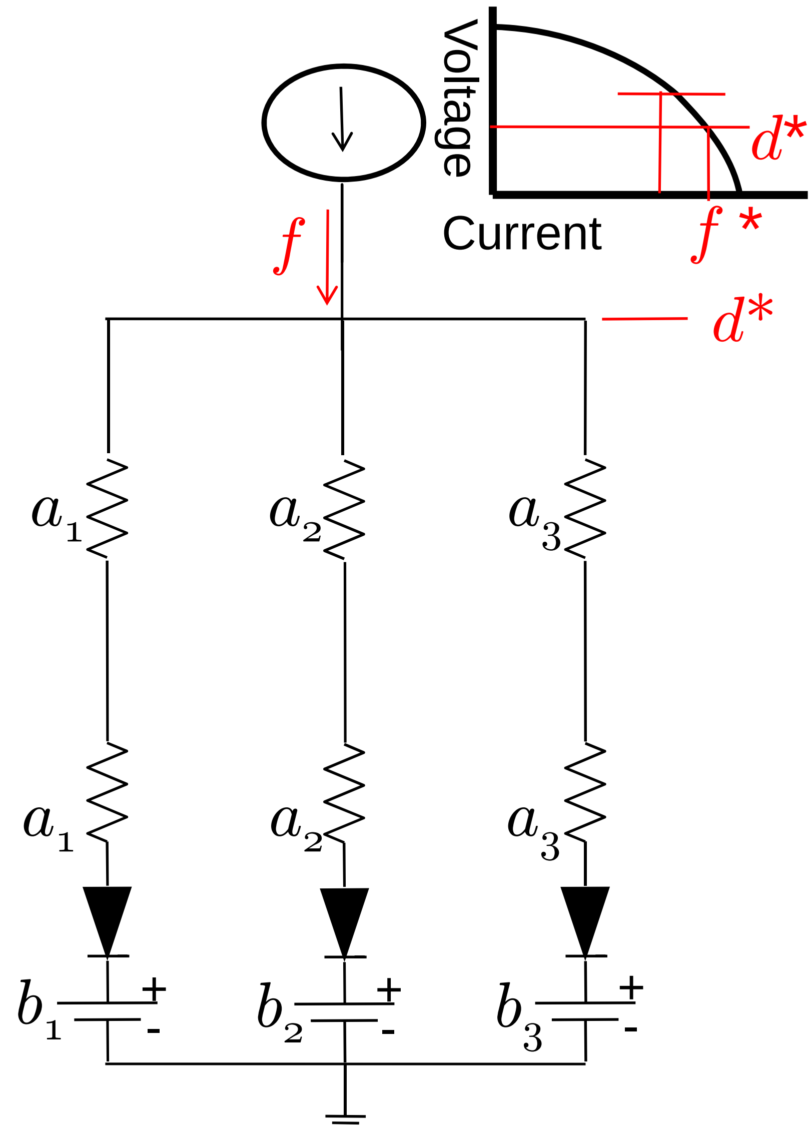
- Price so that users see the cost they impose on society.
- Latency cost on link i: $(a_i f_i^* + b_i) f_i^*$
- Marginal cost: $2a_i f_i^* + b_i$
- Latency seen by user: $a_i f_i^* + b_i$
- Difference: $a_i f_i^*$
- Conclusion: $p_i^* = a_i f_i^*$ achieves social optimum

Circuit analogy

Nash Equilibrium:

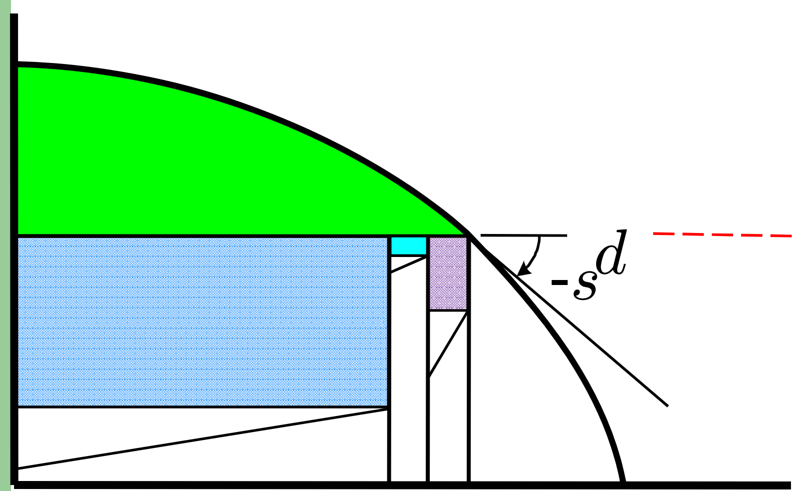


Social Optimum:

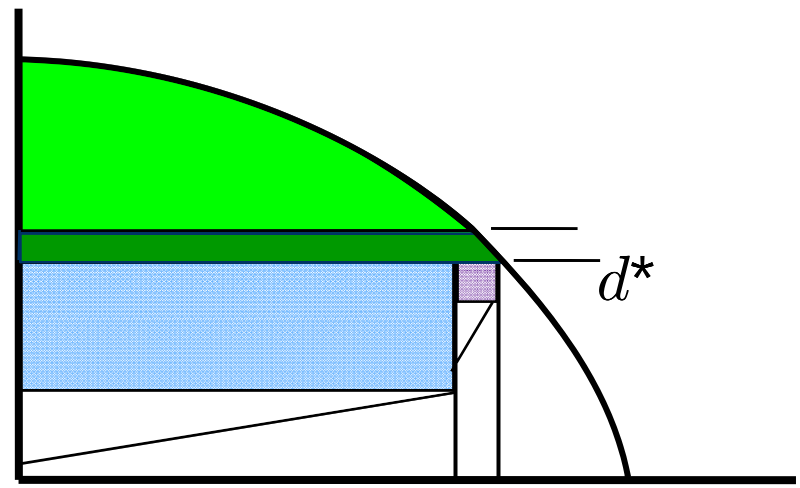


Nash vs. Social Opt. – Original Game

Nash Equilibrium:

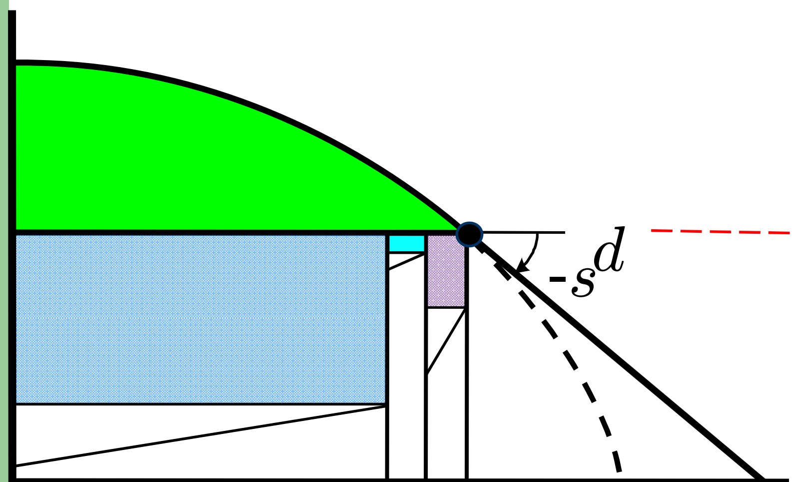


Social Optimum:



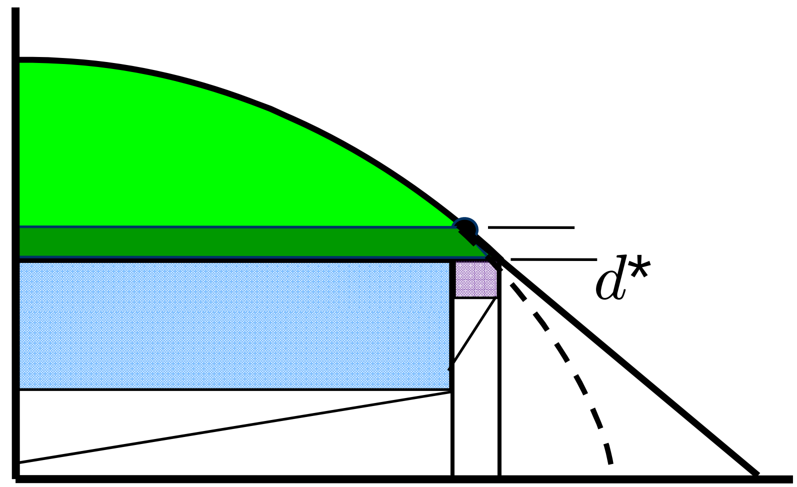
Nash vs. Social Opt. – Modification 1

Nash Equilibrium:



Flow & Social Welfare Unchanged

Social Optimum:

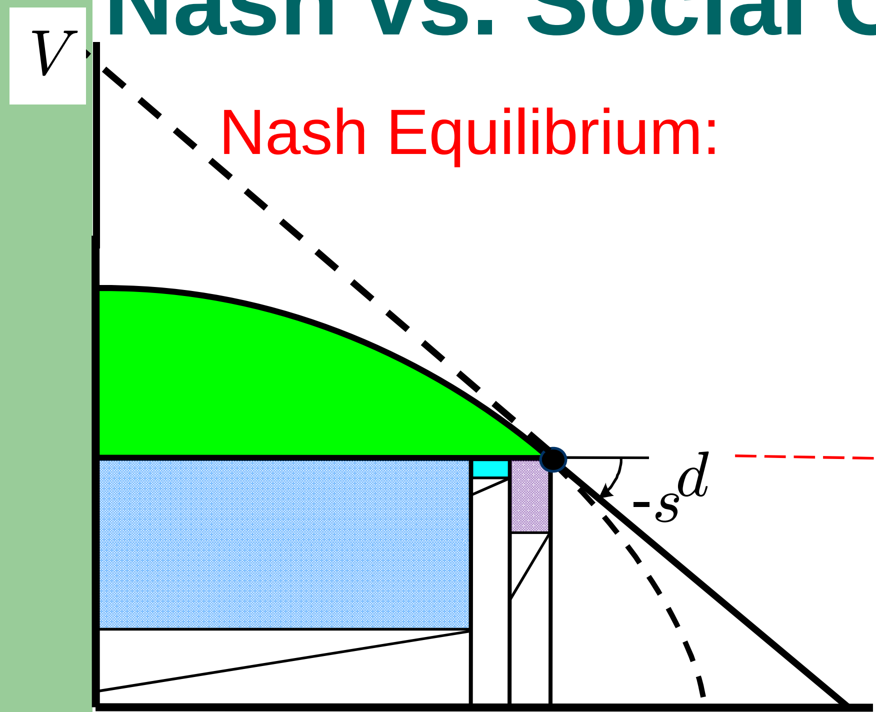


Flow & Social Welfare Not Reduced

Price of Anarchy Not Reduced

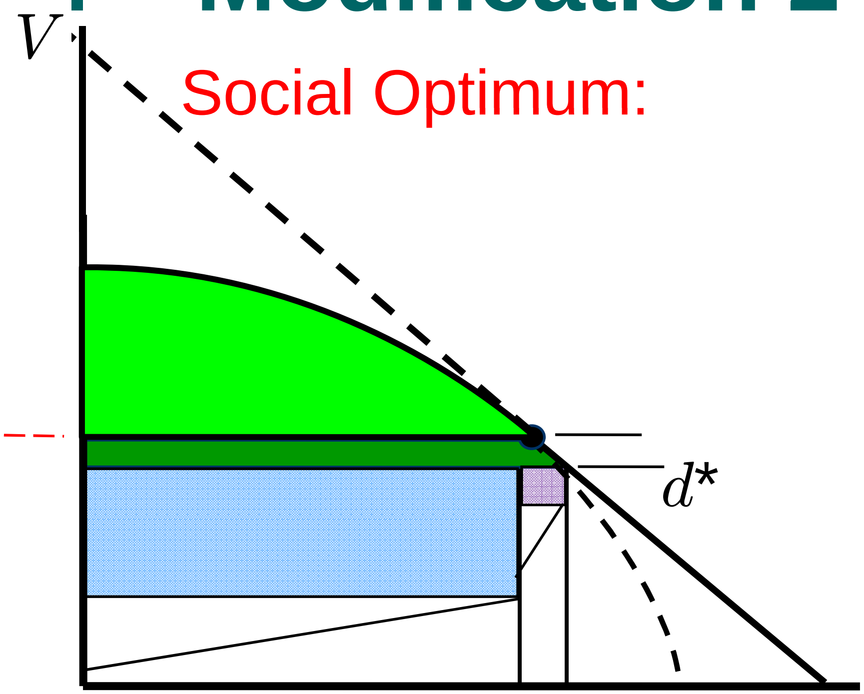
Nash vs. Social Opt. – Modification 2

Nash Equilibrium:



- Flow Unchanged
- Social Welfare reduced by light green area

Social Optimum:

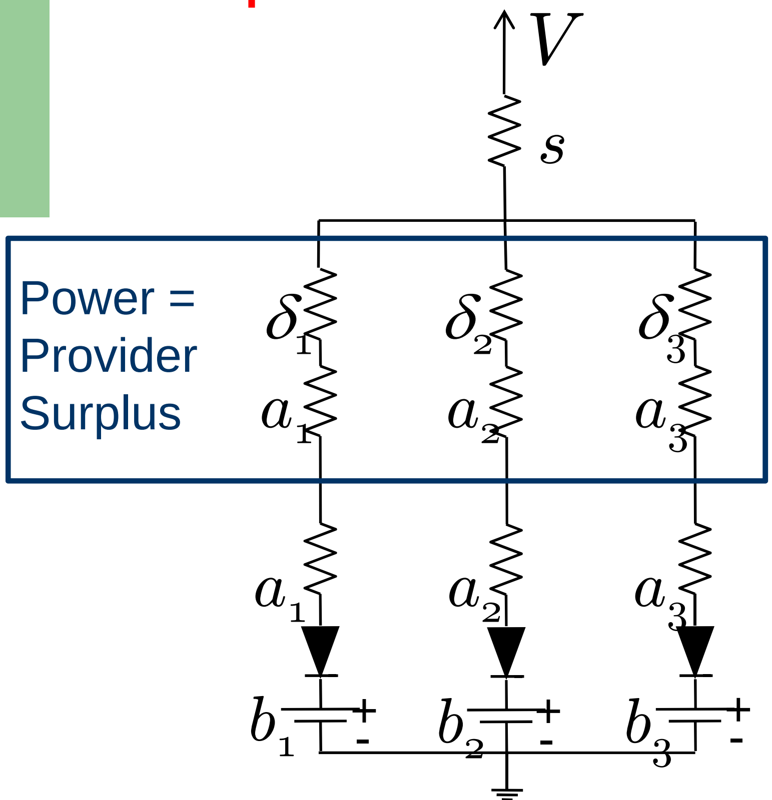


- Flow Unchanged
- Social Welfare reduced by light green area

Price of Anarchy Not Reduced

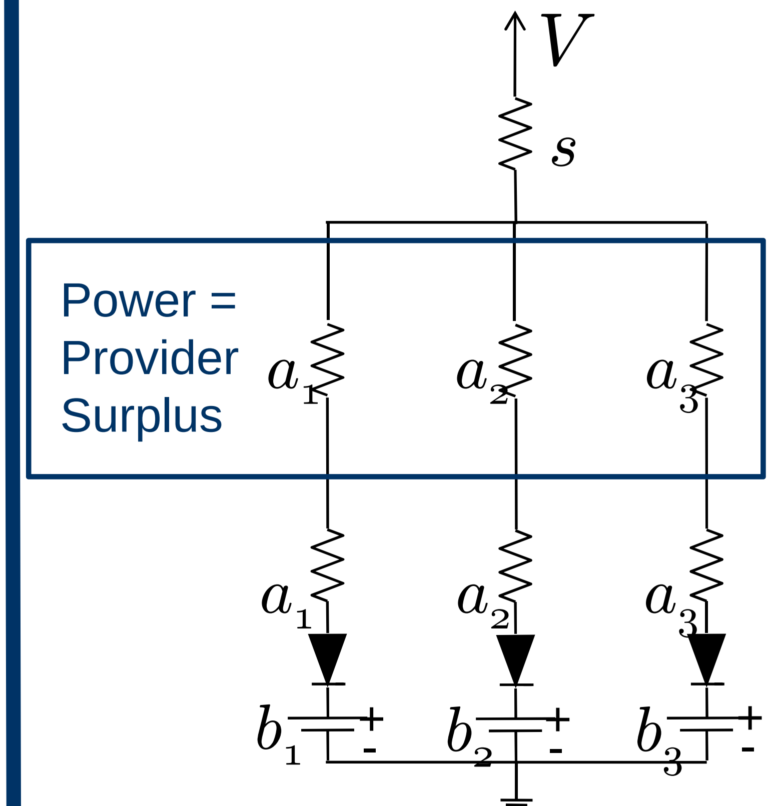
Circuit analogy

Nash Equilibrium:



$$W_l = \sum_i f_i^2(a_i + \delta_i) + \frac{1}{2} f^2 s$$

Social Optimum:



$$W_l^* = \sum_i f_i^{*2} a_i + \frac{1}{2} f^{*2} s$$

$$W_t = \sum_i f_i^2(a_i + \delta_i)$$

$$W_t^* = \sum_i f_i^{*2} a_i + \frac{1}{2}(f^{*2} - f^2) s$$

Matrix – Vector Notation

$$F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots f_n \end{bmatrix} \quad F^* = \begin{bmatrix} f_1^* \\ f_2^* \\ \vdots f_n^* \end{bmatrix} \quad M = \begin{bmatrix} 1 & 1 & \ldots \\ 1 & 1 & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

$$\Delta = \begin{bmatrix} \delta_1 & 0 & \ldots \\ 0 & \delta_2 & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix} \qquad A = \begin{bmatrix} a_1 & 0 & \ldots \\ 0 & a_2 & \ldots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

Providers used in Nash equilibrium can become "undercut" in social optimum

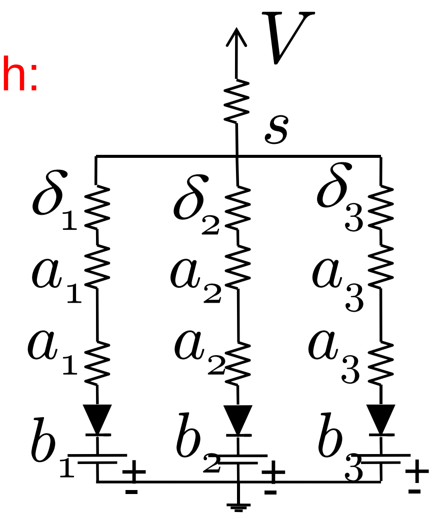
w.l.o.g. Providers: $1,\ldots,m$ not undercut

$m+1,\ldots,n$ undercut

$$F = \begin{bmatrix} \bar{F} \\ \underline{F} \end{bmatrix} \quad A = \begin{bmatrix} \bar{A}, & 0 \\ 0, & \underline{A} \end{bmatrix} \quad \Delta = \begin{bmatrix} \bar{\Delta}, & 0 \\ 0, & \underline{\Delta} \end{bmatrix}$$

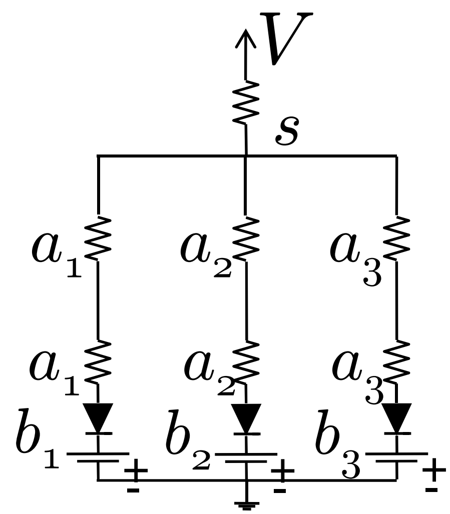
Relations between flow vectors

Nash:



$$(2A + sM + \Delta)F = \mathbf{V} - b.$$

Soc. Opt:



$$[2\bar{A} + s\bar{M}, 0_{m\times k}]F^* = [I_{m\times m}, 0_{m\times k}](\mathbf{V} - b)$$

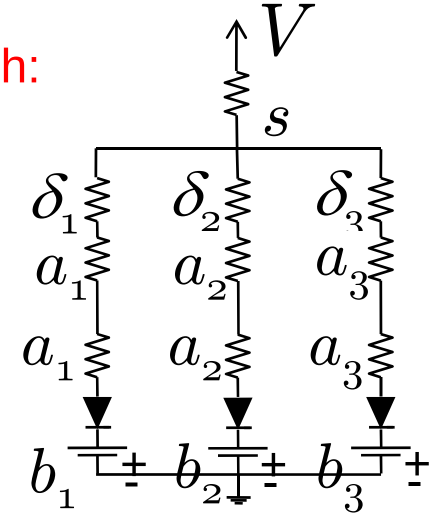
$$F^* = \begin{bmatrix} (2\bar{A} + s\bar{M})^{-1} & 0_{m\times k} \\ 0_{k\times m} & 0_{k\times k} \end{bmatrix} (\mathbf{V} - b).$$

$$F^* = \begin{bmatrix} (2\bar{A} + s\bar{M})^{-1} & 0_{m\times k} \\ 0_{k\times m} & 0_{k\times k} \end{bmatrix} \begin{bmatrix} 2\bar{A} + s\bar{M} + \bar{\Delta} & s1_{m\times k} \\ s1_{k\times m} & 2\underline{A} + s\underline{M} + \underline{\Delta} \end{bmatrix} F$$

$$F^* = \begin{bmatrix} I + (2\bar{A} + s\bar{M})^{-1}\bar{\Delta} & s(2\bar{A} + s\bar{M})^{-1}1_{m\times k} \\ 0_{k\times m} & 0_{k\times k} \end{bmatrix} F.$$

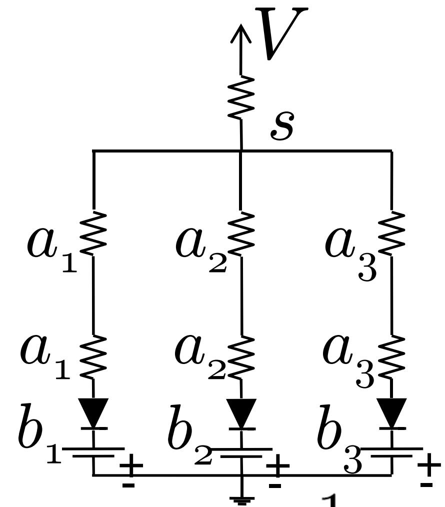
Social Welfare

Nash:



$$W_t = F^T(A + \Delta)F.$$

Soc. Opt:



$$W_t^* = \sum_i f_i^{*2} a_i + \frac{1}{2}(f^{*2} - f^2)s$$

$$W_t^* = \frac{1}{2} F^{*T}(2\bar{A} + sM)F^* - F^T(\frac{1}{2} sM)F$$

$$= \frac{1}{2} F^T \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} F$$

$$G_{11} = 2\bar{A} + 2\bar{\Delta} + \bar{\Delta}(2\bar{A} + sM)^{-1}\bar{\Delta},$$

$$G_{12} = s\bar{\Delta}(2\bar{A} + sM)^{-1} 1_{m \times k},$$

$$G_{21} = G_{12}^T, \text{ and}$$

$$G_{22} = s^2 1_{k \times m}(2\bar{A} + sM)^{-1} 1_{m \times k} - s 1_{k \times k}.$$

Social Welfare Comparison Metric

$$3W_t - 2W_t^* = [\bar{F}^T \underline{F}^T] \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \bar{F} \\ \underline{F} \end{bmatrix}$$

$$H_{11} = \bar{A} + \bar{\Delta} - \bar{\Delta}(2\bar{A} + sM)^{-1}\bar{\Delta},$$

$$H_{12} = -s\bar{\Delta}(2\bar{A} + sM)^{-1}1_{m\times k},$$

$$H_{21} = H_{12}^T, \text{ and}$$

$$H_{22} = 3\underline{A} + 3\underline{\Delta} + s1_{k\times k} - s^2 1_{k\times m}(2\bar{A} + sM)^{-1}1_{m\times k}.$$

Useful Algebraic Identities

Define:

$$\beta \triangleq 2/s + \mathrm{tr}(\bar{A}^{-1}), \quad \alpha \triangleq 1/s + \mathrm{tr}(A^{-1})$$

Then:

$$\bar{\Delta} = (\alpha \bar{A} - I)^{-1} \bar{A}$$

$$(2\bar{A} + s\bar{M})^{-1} = \frac{1}{2}\left[\bar{A}^{-1} - \frac{1}{\beta}\bar{A}^{-1} M \bar{A}^{-1}\right]$$

Proof:

- relation of δ_i's and A; matrix inversion lemma

UC SANTA CRUZ

Use Identities

$$3W_t - 2W_t^* = [\bar{F}^T \underline{F}^T] \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \bar{F} \\ \underline{F} \end{bmatrix}$$

$$H_{11} = (\alpha\bar{A} - I)^{-1}(\alpha^2\bar{A}^3 - \alpha\bar{A}^2 - \frac{1}{2}\bar{A} + \frac{1}{2\beta}\bar{M})(\alpha\bar{A} - I)^{-1}$$

$$H_{12} = \frac{1}{s\beta}(\alpha\bar{A} - I)^{-1}1_{m\times k}$$

$$H_{21} = H_{12}^T$$

$$H_{22} \geq \frac{2}{\beta}1_{k\times k}$$

Change Coordinates

$$Q \triangleq (\alpha \bar{A} - I)^{-1} \bar{F} \text{ and}$$

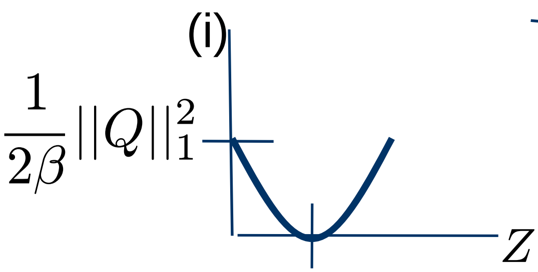
$$Z \triangleq ||\underline{F}||_1$$

$$3W_t - 2W_t^* \geq [Q^T, Z^T] \begin{bmatrix} \alpha^2 \bar{A}^3 - \alpha \bar{A}^2 - \frac{1}{2}\bar{A} + \frac{1}{2\beta}\bar{M} & -\frac{1}{\beta} 1_{m \times 1} \\ -\frac{1}{\beta} 1_{1 \times m} & \frac{2}{\beta} \end{bmatrix} \begin{bmatrix} Q \\ Z \end{bmatrix}$$

$$= \frac{1}{2\beta}||Q||_1^2 - \frac{2}{\beta}||Q||_1 Z + \frac{2}{\beta} Z^2 + Q^T(\alpha^2 \bar{A}^3 - \alpha \bar{A}^2 - \frac{1}{2}\bar{A})Q.$$

(i) Parabola in Z always $\geq$ 0.

(ii) This might be negative



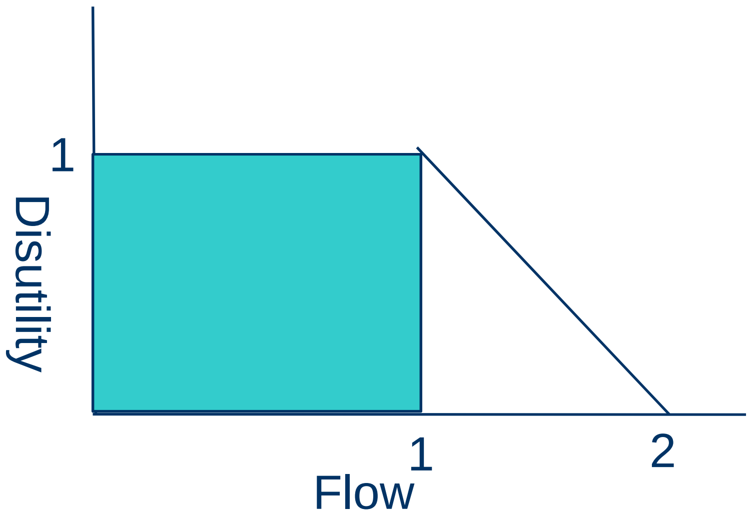
Case 1: $\min_j \bar{a}_j \geq \frac{1 + \sqrt{3}}{2} \alpha^{-1}$ (ii) Positive → done.

Case 2: Use existence of a "small" $\bar{a}_j$ to show that total "undercut flow" Z is small.

Tedious algebra → |(i)| > |(ii)|

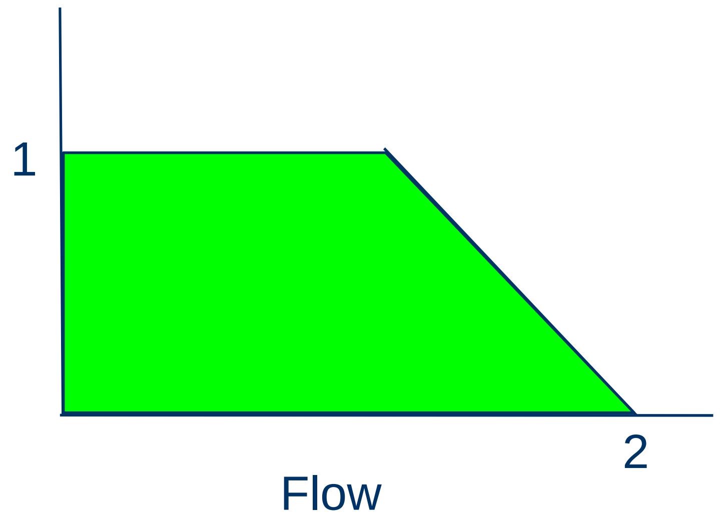
Worst Case

Nash:



Price = 1
Flow = 1
Social Welfare =1

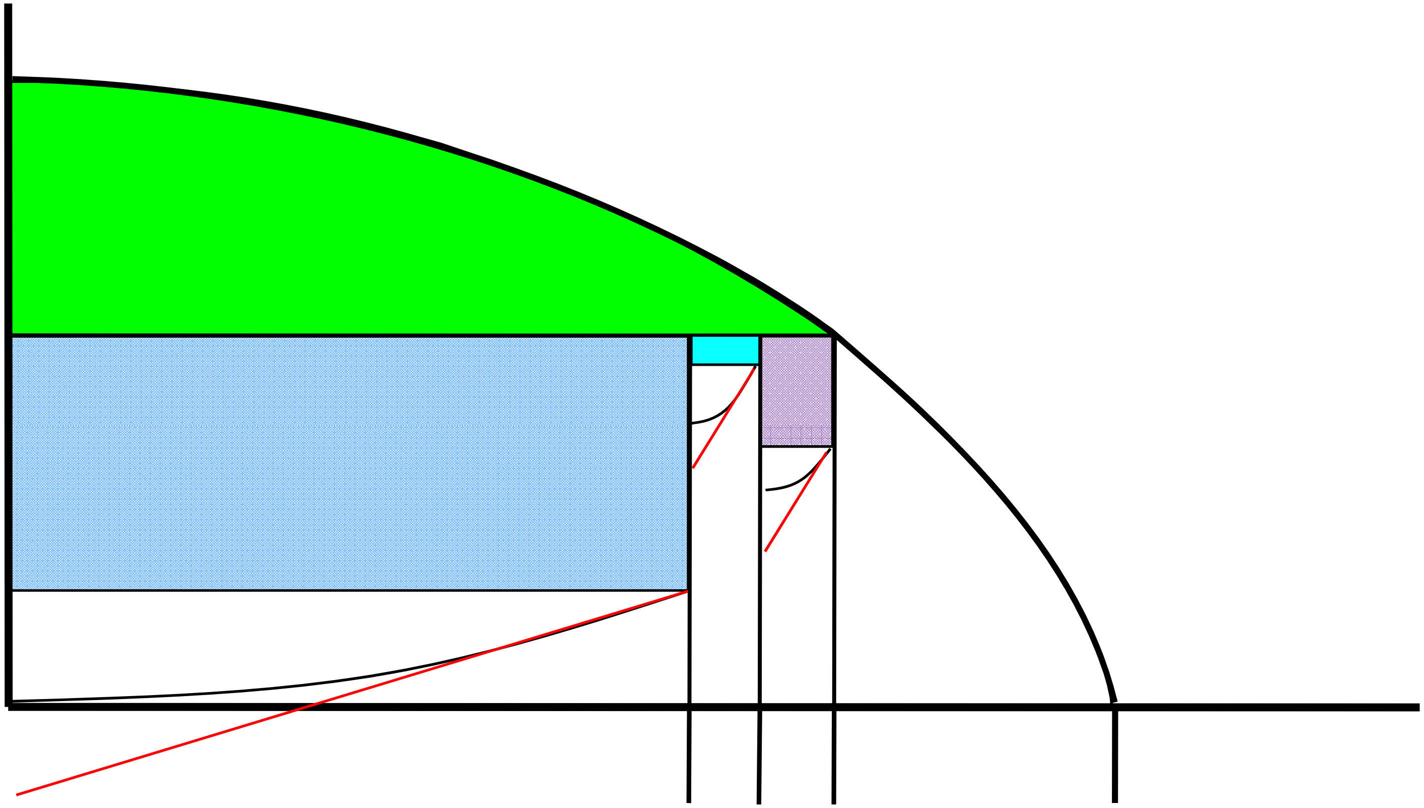
Social Optimum:



Price = 0
Flow = 2
Social Welfare = 3/2

Convex Latency

- Provided a pure strategy equilibrium exists...
- Linearize at equilibrium



Conclusions

- Analysis of network pricing game can be reduced to analysis of a circuit.
- Potential for using method for extended model.
- Analysis of circuit a bit more tedious than desired.