A Game Theoretic Model for Network Upgrade Decisions

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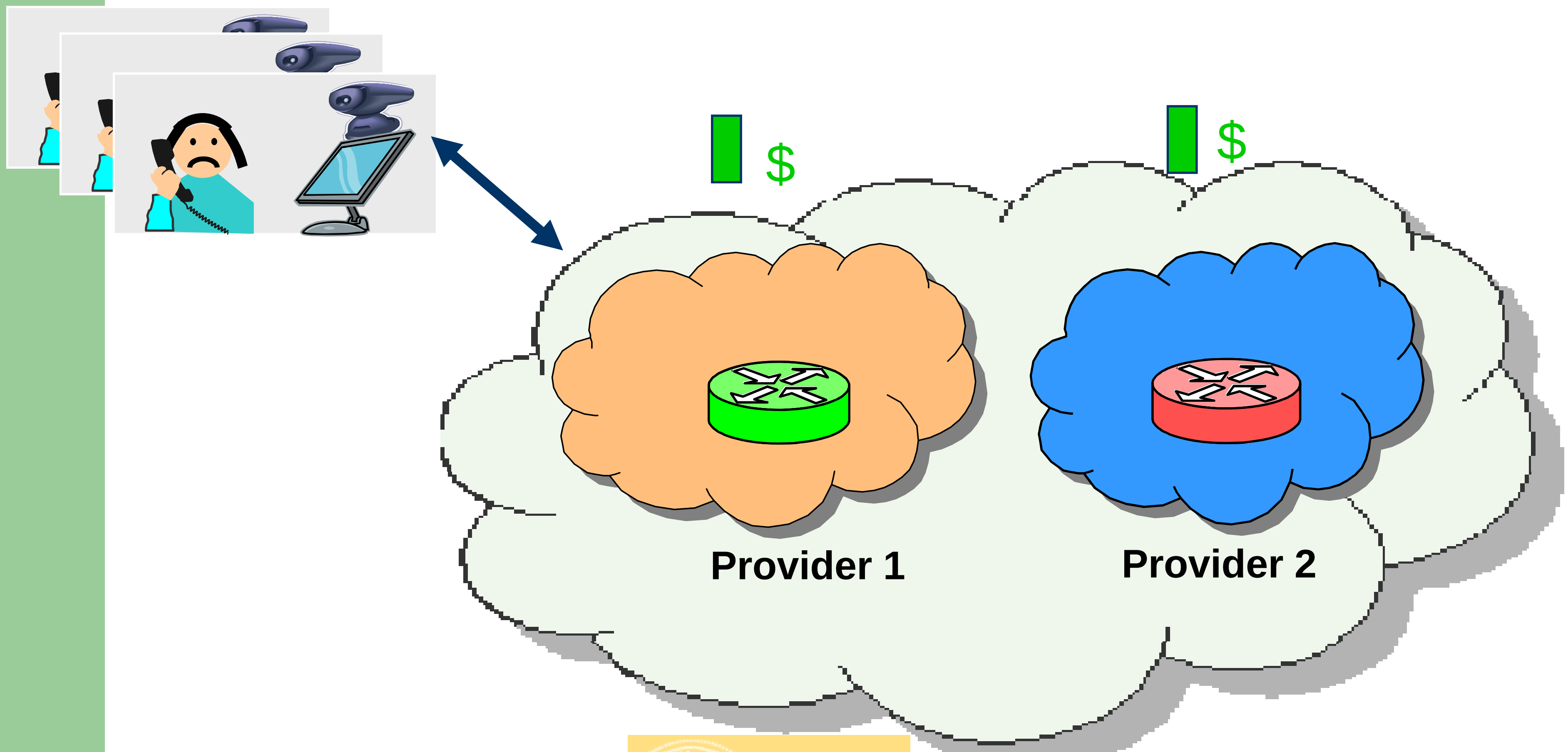
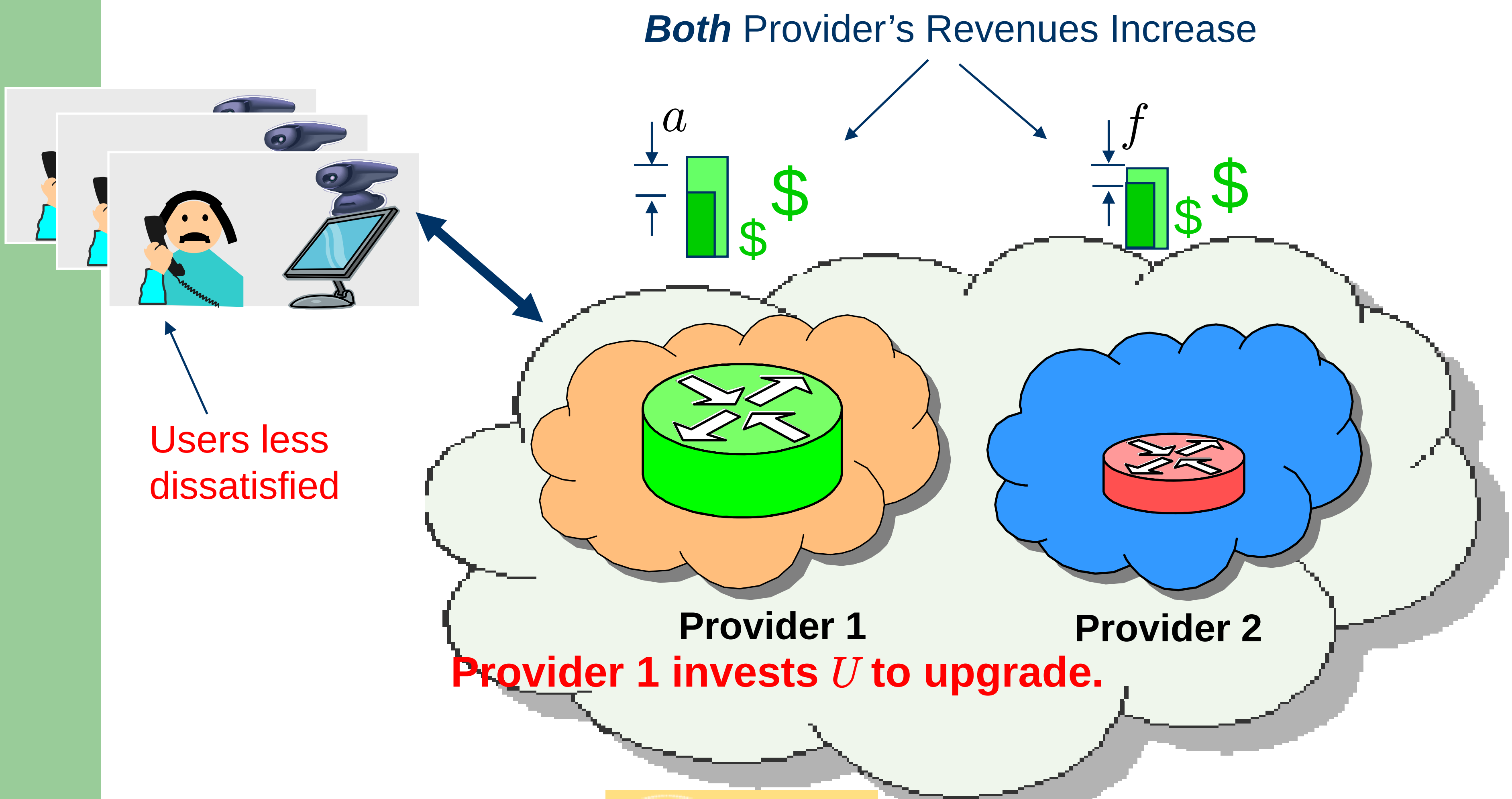
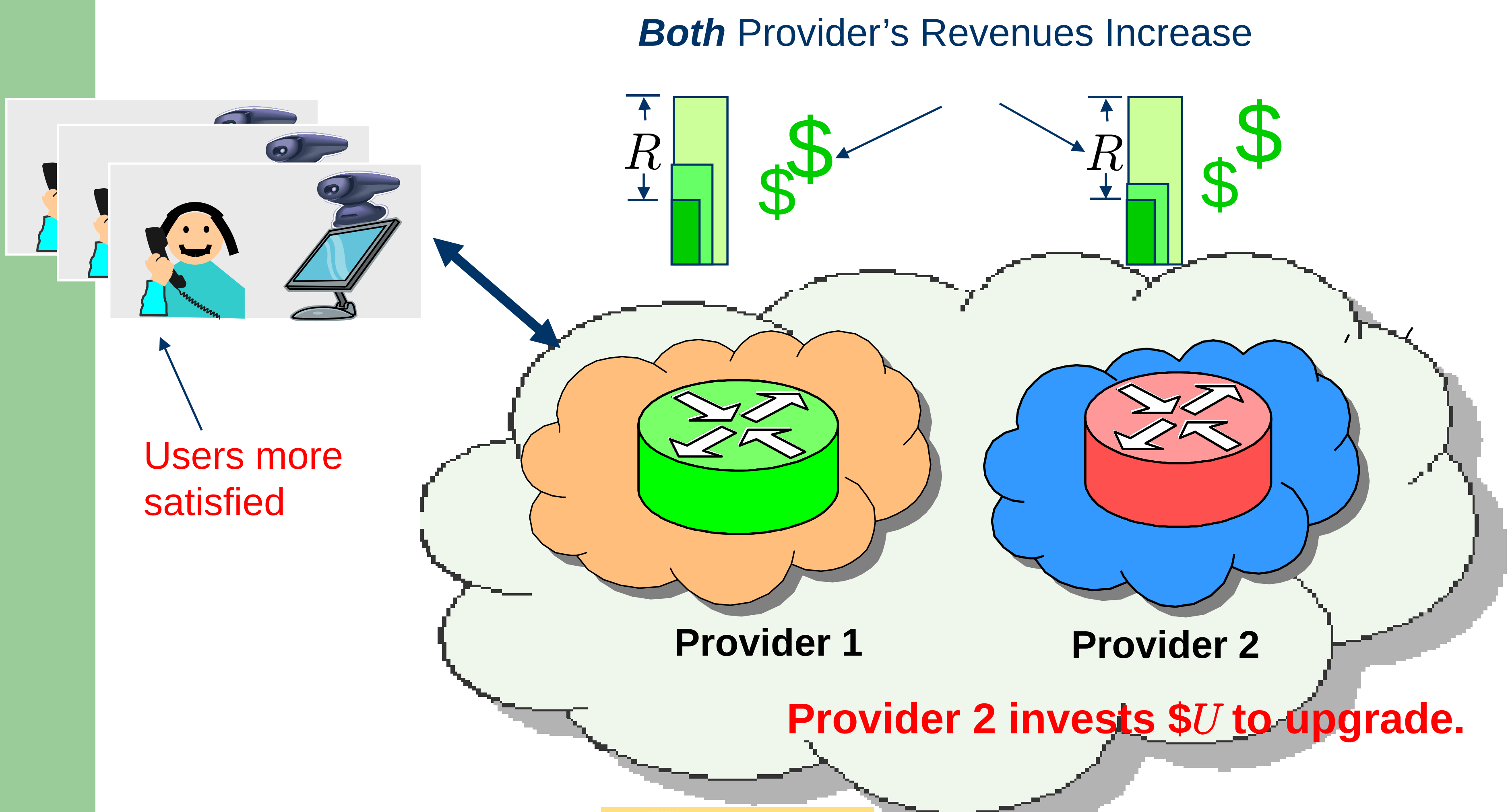
Starting Network



Network After One Upgrade



Network After Both Upgrade



Outline

- Two Providers
- N providers
- Upgrade costs that decline

Multi-Stage Game

- Payoffs n periods in future discounted by δ^n.
- In each period, players simultaneously decide whether to upgrade.
- A player sees what the other player did in previous periods.
- A strategy $\underline{s}_i$ is a plan of action for each period k as a function of the history h_k.

Per-Period Revenue Matrix:

	Not Upgrade	Upgrade
Not Upgrade	0,0	f, a
Upgrade	a, f	R, R

$$f < a < R$$

- The provider incurs a *one time* cost of U when upgrading.
- Upgrading is *irreversible*.

Solution Concepts

- An ordinary Subgame Perfect Nash Equilibrium (SPE) is
  - a strategy profile
    - (assignment of strategies to all players)
  - such that in any subgame,
  - no player can improve her payoff by unilaterally deviating from the strategy profile.
- SPE Condition:

$$p|_{h_k}(s_i^*, s_{-i}^*) \geq p|_{h_k}(s_i, s_{-i}^*)$$

- For all i , h_k

SPE by Iterated Strict dominance

- Strategy s_i is *dominated* if

$$p(s_i, s_{-i}) < p(s_j, s_{-i})$$

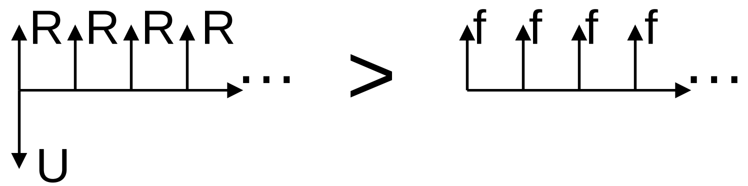
For some j and all s_{-i}

Idea of Iterated Strict Dominance:

- Make G' by removing dominated strategies from original game G.
- Repeat…
- If after this process only one strategy profile remains, we say it is an *SPE by iterated strict dominance*.
  - Players can deduce what opponents will do, without "guessing."

Theorem 1

Suppose $U < \frac{R-f}{1-\delta}$ Equivalently:



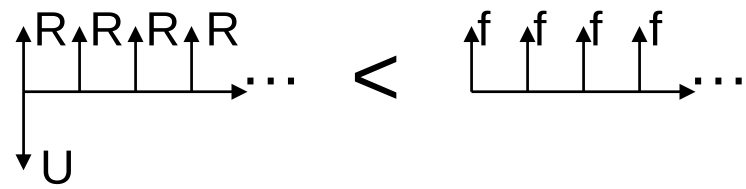
Then an SPE is:

- Upgrade Immediately – Both providers choose to upgrade in the current period

If Also	Then
$U < \frac{a}{1-\delta}$	• Upgrade Immediately is a unique SPE by iterated strict dominance
$U \geq a + \delta \frac{R}{1-\delta}$	The following are also SPE: • No First Upgrade – No provider willing to upgrade first • Delayed Upgrade – Each provider waits until period n to upgrade • Mixed Upgrade – Each provider upgrades with probability α in each period.

Theorem 1 Continued

Suppose: $U > \frac{R-f}{1-\delta}$ Equivalently:

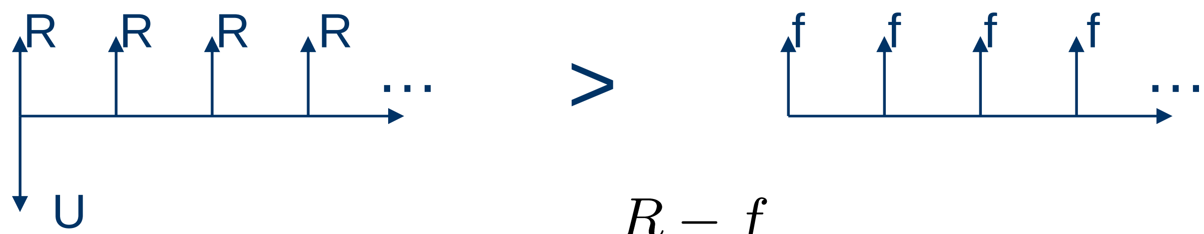


If Also	Then an SPE is:
$U < \frac{a}{1-\delta}$	• Never upgrade – Do not upgrade no matter what
$U > \frac{a}{1-\delta}$	• Asymmetric Freeride – One player upgrades the other freerides • Mixed Freeride – Players upgrade with probability α, until one upgrades. Then other one freerides

Theorem 1

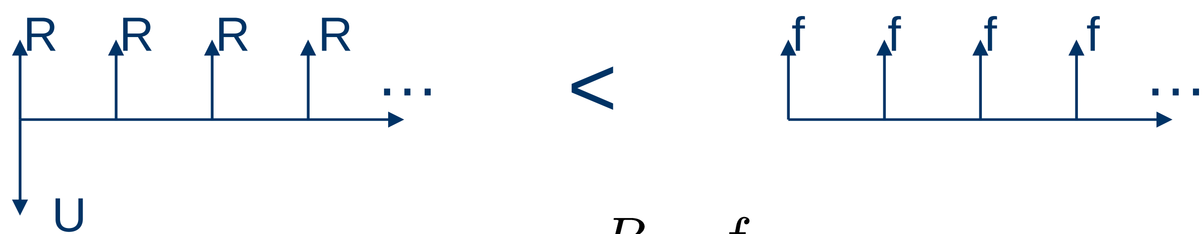
- Two Groupings of Cases:

Group 1:



$$\rightarrow U < \frac{R-f}{1-\delta}$$

Group 2:



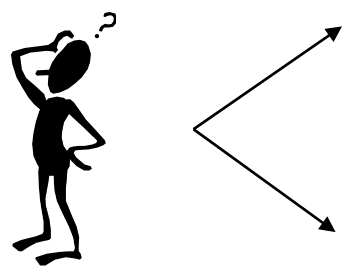
$$\rightarrow U > \frac{R-f}{1-\delta}$$

“Upgrade Immediately”

- Both providers Upgrade in current period.
- SPE if

$$U \leq \frac{R-f}{1-\delta}$$

- “Proof”

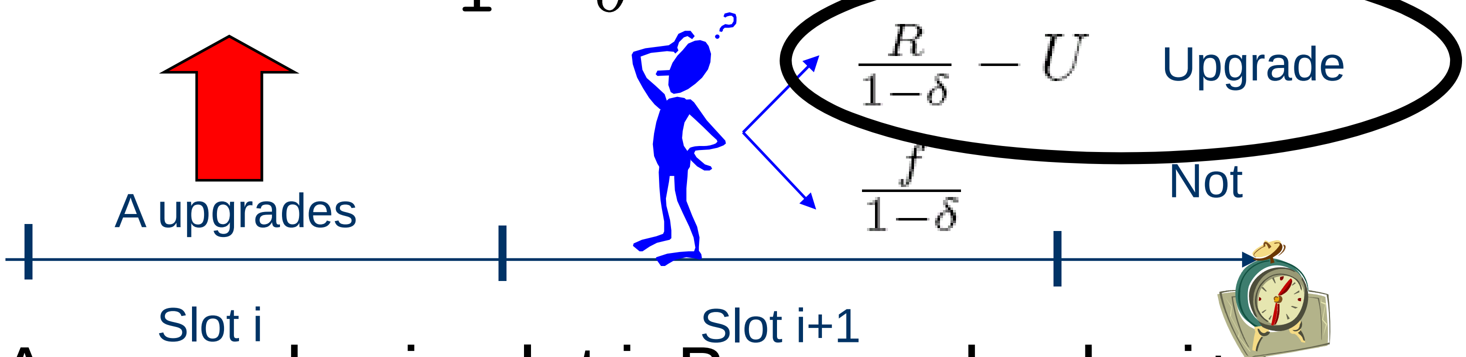


$$\frac{R}{1-\delta} - U$$ Stick

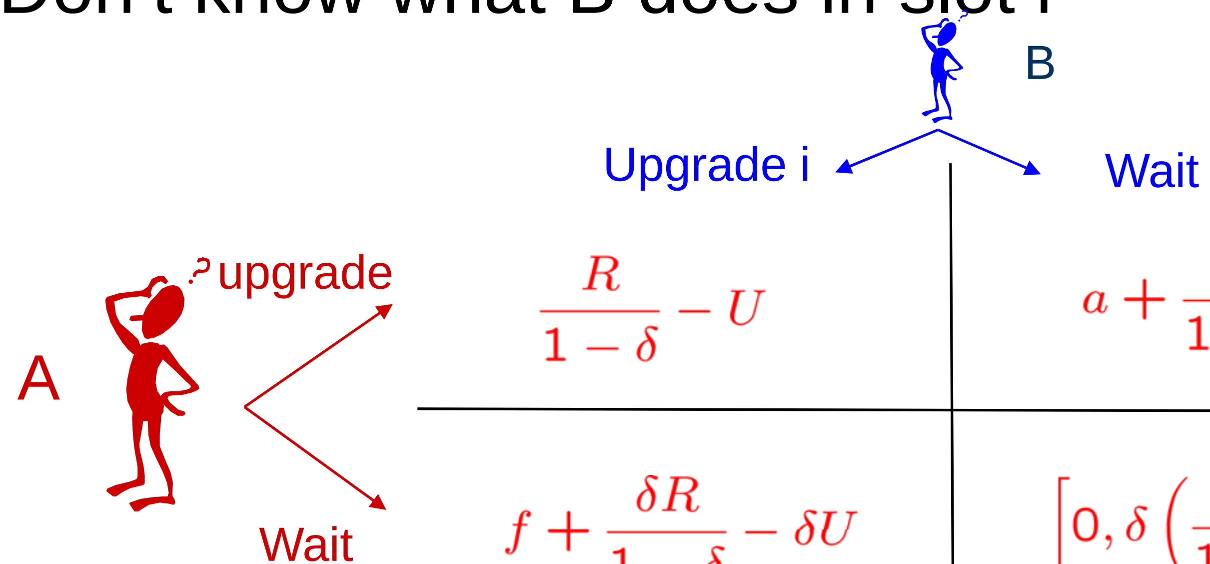
$$\frac{f}{1-\delta}$$ Deviate

Upgrade Immediately (Cond. For Uniqueness)

Suppose: $U < \dfrac{R - f}{1 - \delta}$



- If A upgrades in slot i, B upgrades by i+1
- Don't know what B does in slot i



	Upgrade i	Wait
upgrade	$\frac{R}{1-\delta} - U$	$a + \frac{\delta R}{1-\delta} - U$
Wait	$f + \frac{\delta R}{1-\delta} - \delta U$	$\left[0, \delta\left(\frac{R}{1-\delta} - U\right)\right]$

Upgrade Immediately (Uniqueness)

- Upgrade is better for both conditional payoff functions if

$$U < \min\left(\frac{R-f}{1-\delta}, \frac{a}{1-\delta}\right)$$

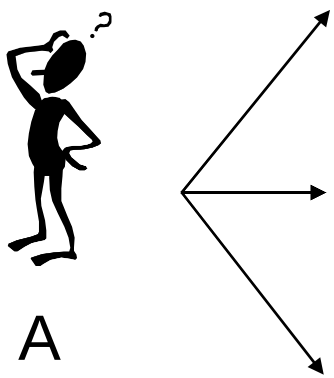
Delayed Upgrade

- Each provider expects the other to upgrade in period n.
- Srtategy
  - Upgrade in period n.
  - But, If opponent upgrades in period i<n, upgrade in i+1.
- n=∞ corresponds to "no first upgrade."
- Conditions:

$$a + \delta\frac{R}{1-\delta} \leq U \leq \frac{R-f}{1-\delta}$$

Delayed Upgrade Proof

- Suppose player B plays "upgrade in period n"



$$(a - U) + \frac{\delta R}{1-\delta}$$ Upgrade at 0

$$\delta^n \left[\frac{R}{1-\delta} - U \right]$$ Upgrade at n

$$\delta^n \left[\frac{f}{1-\delta} \right]$$ Freeride

- Upgrade at n best if:

$$\frac{a}{1-\delta^n} + \frac{(\delta - \delta^n)R}{(1-\delta)(1-\delta^n)} < U < \frac{R-f}{1-\delta}$$

“Mixed Upgrade”

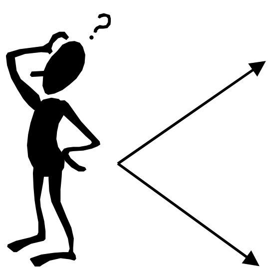
- Subgame starting with no-one upgraded:
  - Each upgrades with probability α.
- Subgame starting with one-upgraded:
  - The not-upgraded upgrades immediately.
- SPE if

$$a + \delta \frac{R}{1-\delta} \leq U \leq \frac{R-f}{1-\delta}$$

Mixed Upgrade

- Suppose B follows mixed upgrade and that

$$U \leq \frac{R-f}{1-\delta}$$



$$\alpha R + (1-\alpha)a + \delta\frac{R}{1-\delta} - U.$$ Upgrade

$$J = (1-\alpha)\delta J + \alpha\left(f + \delta\frac{R}{1-\delta} - \delta U\right)$$ Wait

Mixed Upgrade

- Difference

$$U = \frac{1-\delta+\alpha\delta}{1-\delta}\left(\alpha R + (1-\alpha)a + \frac{\delta R}{1-\delta}\right) - \frac{\alpha}{1-\delta}\left(f + \frac{\delta R}{1-\delta}\right)$$

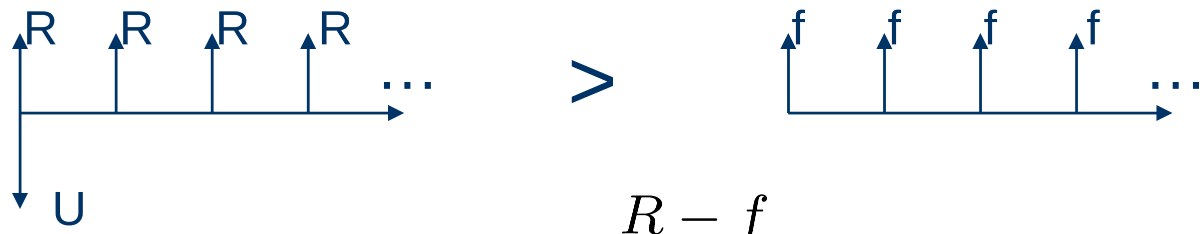
- Sufficient condition for existence of a zero

$$a + \delta\frac{R}{1-\delta} \leq U \leq \frac{R-f}{1-\delta}$$

Theorem 1 proof

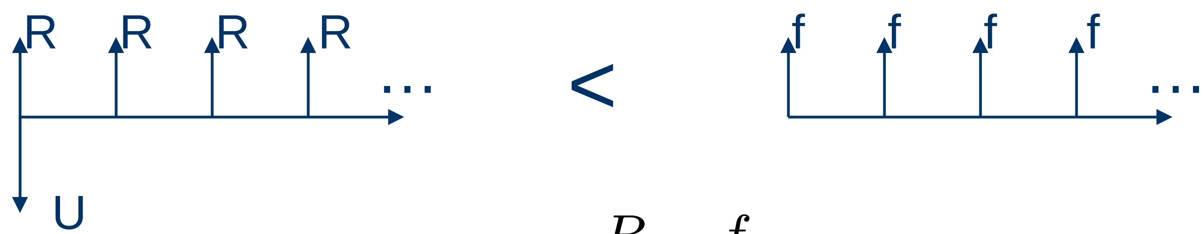
- Two Groupings of Cases:

Group 1:



$$\rightarrow U < \frac{R - f}{1 - \delta}$$

Group 2:



$$\rightarrow U > \frac{R - f}{1 - \delta}$$

Asymmetric Free-ride

- Player A upgrades slot 1.
- Player B never upgrades.

- SPE if

$$\frac{R-f}{1-\delta} \leq U \leq \frac{a}{1-\delta}$$

Other Cases for $\frac{R-f}{1-\delta} < U$

- "Mixed Freeride"
  - Condition: $U < \frac{a}{1-\delta}$
  - Each upgrades with probability α; if one upgrades first the other freerides.
  - Proof: Similar to "Mixed Upgrade."
- "Never Upgrade"
  - Condition: $U \geq \frac{a}{1-\delta}$
  - No one upgrades, even if the other one were to.
  - Proof: easy.

Outline

- Two Providers
- N providers
- Upgrade costs that decline

N-Providers in Tandem

- Let
  - f(j) = **f**ree-rider benefit if j providers upgrade.
    - f(0) == 0
  - a(j) = early **a**dopter benefit if j providers upgrade.
    - a(N) == R
- All Upgrade a SPE if:

$$U < \frac{R - f(N-1)}{1-\delta}$$

Upgrade Immediately (Dominance)

- Suppose we enter a subgame with 2 not-upgraded

	upgrade	freeride
upgrade	$\frac{a(N)}{1-\delta} - U$	$\frac{a(N-1)}{1-\delta} - U$
freeride	$\frac{f(N-1)}{1-\delta}$	$\frac{f(N-2)}{1-\delta}$

- Upgrading Dominant if:

$$U < \min\left(\frac{a(N) - f(N-1)}{1-\delta}, \frac{a(N-1) - f(N-2)}{1-\delta}\right)$$

Upgrade Immediately (Dominance)

- Induction argument shows that everyone upgrading is unique SPE by iterated deletion of dominated strategies if:

$$U < \min_{j=0,\ldots,N-1} \left[\frac{a(j+1) - f(j)}{1-\delta} \right]$$

Outline

- Two Providers
- N providers
- Upgrade costs that decline

Declining Upgrade Cost Model

- Continuous Time
- Future Revenues are discounted at rate δ
  - Example if both providers upgrade at time 0, the P.V. of each's revenues is:

$$\int_0^\infty e^{-\delta t} R = \frac{R}{\delta}$$

- Upgrade costs decline at rate γ
  - γ includes declining costs + discounting so $\gamma > \delta$.
  - P.V. of upgrade cost at time t' is:

$$e^{-\gamma t'} U$$

Critical Times

- Time when upgrading looks better than freeriding:

$$t_f^* = \arg\max_{t\in\mathbb{R}^+}\left[e^{-\delta t}\frac{R-f}{\delta} - Ue^{-\gamma t}\right] = \left[\frac{1}{\gamma-\delta}\log\left(\frac{\gamma U}{R-f}\right)\right]^+$$

- Time when being a first adopter looks attractive:

$$t_a^* = \arg\max_{t\in\mathbb{R}^+}\left[e^{-\delta t}\frac{a}{\delta} - Ue^{-\gamma t}\right] = \left[\frac{1}{\gamma-\delta}\log\left(\frac{\gamma U}{a}\right)\right]^+$$

Theorem 3

Suppose $t_a^* \geq t_f^*$

- Then the only SPE is:
  - Both providers upgrade at time t_f^*
- Proof outline:
  - Step 1: A upgrades before t_f^*
    - → B upgrades at t_f^* (dominant)
  - Step 2: A upgrades after t_f^* and B not yet upgraded
    - → B Upgrades immediately after A (dominant).
  - Step 3: Because A can induce B to upgrade at t_f^*, A best option is to upgrade at t_f^*

Theorem 3 Proof – Step 1

- Suppose A upgrades at time t' < t_f^*
- B's best response in ensuing subgame:

Rev. after upgrade — Freeride revenue before upgrade — Upgrade cost

$$t = \arg\max_{t \geq t'} \left[e^{-\delta t} \frac{R}{\delta} - (e^{-\delta t'} - e^{-\delta t}) \frac{f}{\delta} - U e^{-\gamma t} \right]$$

$$= \arg\max_{t \geq \tau} \left[e^{-\delta t} \frac{R - f}{\delta} - U e^{-\gamma t} \right]$$

$$= t_f^*$$

Theorem 3 Proof – Step 2

- Suppose A upgrades at time t' > t_f^*
- B's best response in ensuing subgame:

$$t = \arg\max_{t \geq t'} \left[e^{-\delta t} \frac{R}{\delta} - (e^{-\delta t'} - e^{-\delta t}) \frac{f}{\delta} - U e^{-\gamma t} \right]$$

$$= \arg\max_{t \geq t'} \left[e^{-\delta t} \frac{R - f}{\delta} - U e^{-\gamma t} \right]$$

$$= t'$$

Theorem 3 Proof – Step 3

- Suppose A considers upgrading before t_f^*.
- Considering induced behavior on B, A's best response is:

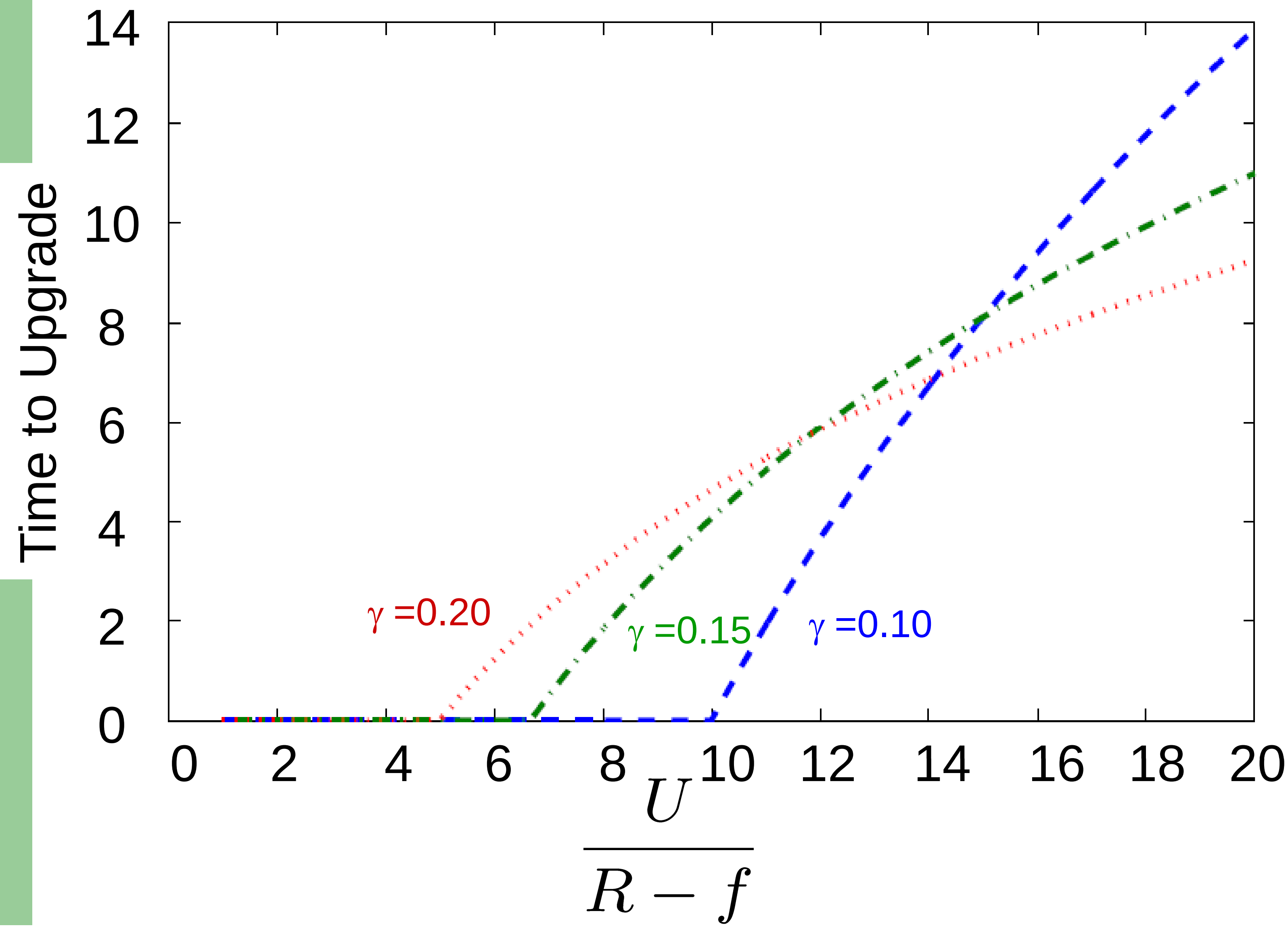
$$\arg\max_{t \le t_f^*} \left[e^{-\delta t_f^*} \frac{R}{\delta} + (e^{-\delta t} - e^{-\delta t_f^*}) \frac{a}{\delta} - U e^{-\gamma t} \right]$$
$$= \arg\max_{t \le t_f^*} \left[e^{-\delta t} \frac{a}{\delta} - U e^{-\gamma t} \right]$$
$$= \min(t_a^*, t_f^*) = t_f^*$$

- Similarly, can show that A upgrading after t_f^* gives lower payoff.

Theorem 4

- Suppose $t_a^* < t_f^*$
- It is a SPE for
  - One provider to upgrade at time t_a^*
  - The other provider to upgrade at time t_f^*
- Proof:
  - Similar as Theorem 3's proof.

Time to Upgrade, δ = 0.05



Conclusions

- Discrete Time Model
  - Strong conditions required to have unique SPE.
    - Otherwise many strange SPE possible.
  - Freeriding effect may prevent upgrades even when it is socially optimal to do so.
- Continuous Time with Declining Costs
  - *Unique* SPE if $\tau_f^* \leq \tau_a^*$
  - Asymmetric SPE otherwise.
  - Freeriding delays upgrades.
  - More rapidly declining upgrade costs may *increase* the time until networks upgrade.