# The Price of Anarchy in Competing Differentiated Services Networks

John Musacchio and Shuang Wu

***Abstract*— We investigate competition between network providers that offer service to two types of traffic differing in their sensitivity to delay. We first consider competition amongst network providers who offer differentiated services by providing a priority queue for the delay sensitive traffic. We compare this to a situation in which all the competing network providers have network architectures that treat traffic of both types the same way. Our model of competition is Cournot in that service providers choose a rate to offer traffic of each type, and in-turn the total rate offered to each type of traffic determines the price of each traffic type. We are interested in the price of anarchy in these games of competition, which is defined as the ratio of the maximum achievable social utility versus the social utility attained when service providers selfishly maximize profits and reach a Nash equilibrium. We find that the price of anarchy is no more than $\frac{4}{3}$ in our model of competing providers who offer differentiated services. In competition with providers that do not offer preferential service to delay sensitive traffic, we find the price of anarchy can be higher than $\frac{4}{3}$, and we derive bounds for a number of important cases.**

## I. Introduction

Real time applications become unusable when the latency or the loss rate exceeds certain levels. In contrast, a file transfer can tolerate a fair amount of delay and loss without much degradation of perceived performance. If these two share a congested network, the delay sensitive applications may not be able to be utilized. Differentiated services have the potential to increase the social welfare without increasing its capacity by protecting the delay sensitive traffic and increasing the utility of the delay sensitive traffic at a modest cost in utility to other traffic. By social welfare we mean the utility gathered by the traffic, or equivalently, the utility gathered by the traffic net the price paid by the traffic, plus provider profits.

When providers compete, they seek to maximize their individual profits rather than the overall social welfare of the system. Therefore when a system of competing providers reaches an equilibrium, a Nash equilibrium, the social welfare achieved by that equilibrium is generally less than the maximum social welfare achievable if a perfect social planner were to select the provider strategies. A metric for quantifying this efficiency loss is called the *price of anarchy*, which is simply the ratio of the maximum achievable social welfare to the welfare achieved in Nash equilibrium.

In this work we investigate whether the price of anarchy for competing network providers who offer priority service to delay sensitive traffic is different than the price of anarchy for competing network providers whose networks treat delay sensitive and delay insensitive traffic the same way.

Our model of competition is closely related to a class of models first proposed by Acemoglu and Ozdaglar [1]. However the model of [1] studies a game where service providers choose prices, not rates, and also have only one type of user. Furthermore, the model of [1] models latency as a cost, whereas in the model we develop here we model the effect of latency by imposing a constraint on the providers' strategy spaces.The authors of [2] extend the model of [1] to consider elastic demand, and Ozdaglar found a tight bound on the price of anarchy for this class of models [3]. The same bound is derived in a different way in [4]. Our model is also very similar in nature to a model studied by Johari and Tsitsiklis [5]. In [5], the strategic agents select rates as in our model, though in that model the agents are said to be users selecting rates rather than providers. Our model considers traffic of two different types, which makes the analysis of our model different.

Our work is part of a larger class of work studying the efficiency loss in network games that involve pricing and/or routing. Examples include [6] which studies the efficiency loss from selfish routing on a passive network, and [7], which studies the efficiency loss from a network pricing mechanism with strategic users. An earlier work that helped motivate our current work is by MacKie-Mason and Varian [8]. It explores the implications of congestion pricing for capacity expansion in centrally planned, competitive, and monopolistic environments. A number of other works have investigated economic issues of Internet providers offering differentiated services. Some examples include [9] and [10].

Another body of literature related to our work investigates multiproduct Cournot competition – competition between producers of multiple goods that might be imperfect substitutes for each other. The literature investigates how factors such as: costs to firm entry, capacity investments, the degree of substitutability of the products, and heterogeneity in consumers affect the outcomes of such competition. Examples of this work include [11], [12],and [13].

A particularly closely related work in the multiproduct competition literature is recent work by Perakis and Kluberg that investigates the efficiency loss of oligopolistic competition [14]. It turns out that the analysis of the price of anarchy of our priority architecture game reduces to a special case of a model studied by Perakis [14], though we derived our result independently of this work. The queueing model we use to develop our economic models is not related to [14], nor does the analysis of our shared architecture game reduce to a case of the model of [14].

Research supported in part by NSF grant CNS-0626397.

J. Musacchio and S. Wu are with the Technology and Information Management Program, University of California, Santa Cruz, Santa Cruz, CA 95064, USA {johnm,swu}@soe.ucsc.com

## II. Model Justification

We are interested in competition amongst providers who serve two types of traffic differing in their sensitivity to delay. Furthermore, our goal is to compare competition amongst providers who offer priority queueing to the delay sensitive traffic to competition amongst providers who treat both types of traffic the same in their network. For short, we refer to less delay-sensitive traffic such as web browsing as "web traffic" and traffic from all real-time applications like voice over IP as "voice traffic". Real-time applications usually require packet delays of less than 100ms while web browsing can tolerate larger delays as long as the throughput is adequate.

In what follows, we explore a simple, stylized queuing model to capture qualitatively how the delays in a network are affected by the amount of voice traffic, web traffic, and whether the provider uses a priority or shared queue architecture. This queueing model will not appear directly in our final models of provider competition, but instead we will use it to determine structural properties of our competition model – specifically the shapes of the feasible regions of voice and web traffic each provider can offer in either the shared or priority architecture models.

We seek to capture the differences in the traffic characteristics in our queueing model, because these characteristics influence how well the traffic of each type mixes. While the characteristics of "voice" traffic depend on the codec and voice traffic does not always arrive as a constant rate, voice traffic generally does arrive in a more regular fashion than web traffic. We will assume that the voice traffic has much smaller packets than web traffic, which would be close to the case if what we call "voice" were truly just voice. Of course if it were video, the small packet assumption would be a stronger simplification of reality. These simplifying assumptions allow us to suppose that voice traffic arrives as a constant bit fluid of rate $y$ and the web traffic arrives as a Poisson process of rate $x$. This assumption captures the fact that the web traffic has a less regular arrival pattern than voice. Using these assumptions, we develop M/M/1 based models for the priority and shared architecture cases as we detail in the next sections. Although M/M/1 is far from being an exact model of queueing delays in the Internet, it does have the qualitative feature that the delays grow very large as the traffic approaches the capacity of the link. For this reason, it is a useful model to analyze in order to develop a qualitative understanding of a network where users can select their provider on the basis of price and delay, and providers select their service rates in order to maximize profits.

### A. Priority Architecture

In this setting we suppose that the network provider offers a priority queue to the voice traffic. We suppose that the delay effects are dominated by a single "bottleneck" queue, and that this queue has a single server that serves both web and voice. We make the further simplifying assumption that the server serves voice with strict priority, and is preemptive. We suppose that the capacity of the network (bottleneck) is $c$, the arrival rate of web traffic is $x$, and the arrival rate of voice traffic is $y$ and $x + y < c$. With the simplifying assumptions we discussed earlier, namely that voice traffic arrives as a constant fluid at rate $y$, and that web traffic arrives as a Poisson process, web traffic is served as if it were in a single server $M/M/1$ queue of capacity $c - y$. Thus, the average delay experienced by web traffic is $\frac{1}{c-x-y}$. Our simplifying assumptions are also such that as long as $x+y < c$ the delay seen by voice packets should be close to $0$.

### B. Shared Architecture

In this scenario, traffic of both types is queued together. We observe that the queuing systems of both architectures (priority and shared) are work conserving. Thus, when both architectures are offered identical incoming traffic, the two systems must have the same average delay by Little's result:

$$D = \frac{(\frac{x}{c-x-y} + y \cdot 0)}{x + y}.$$

With the shared architecture, the queue length just before the arrival of a voice or a web packets has the same distribution. However, an arriving web packet is delayed by the service times of the packets in the queue plus its own service time, whereas the service time of a single voice packet is negligible. Therefore, $D_x = \frac{1}{c} + D_y$, where $D_x$ is the average delay for a web packet, and $D_y$ is the average delay for a voice packet. The average delay across all packets satisfies

$$D = \frac{yD_y + xD_x}{y + x}.$$

Combining the above three equations we find the average delays are:

$$D_x = \frac{c - y}{(c - x - y)c}, \qquad D_y = \frac{x}{(c - x - y)c}.$$

We suppose that the voice traffic requires that the delay be no more than $D_{max}$. The delay requirement for voice reduces to the following restriction on $x$ and $y$.

$$\begin{array}{ll} (1 + \frac{1}{D_{max}c})x + y < c & \text{if } x > 0 \\ x < c & \text{if } y = 0. \end{array} \quad (1)$$

The last condition is due to the fact that if the network does not try to carry any voice traffic, the delay requirement on voice no longer needs to be met.

## III. Model

### A. Demand

We suppose that there are separate demand curves for web and voice. The price for web services is a function of the total rate $x = \sum_i x_i$ offered to web traffic where $x_i$ is the rate offered by provider $i$. The price is affine in $x$ and takes the form $p_w(x) = k - x$. The demand for voice has a similar form – the price for voice traffic is $p_v(y) = ak - by$. Note that there is no loss of generality in supposing the slope of the web demand function is $-1$ because this can be made to be by a change of units.

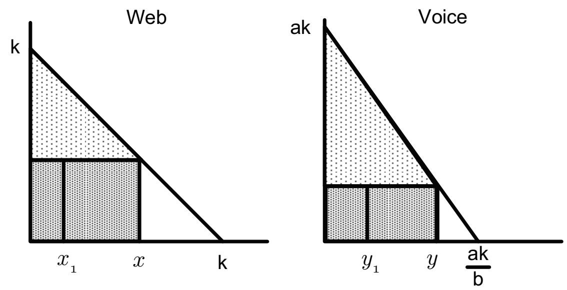


Fig. 1. The relationships between the demand functions, and flows. The dark shaded areas represent provider profits while the lighter shaded areas represent consumer surplus.

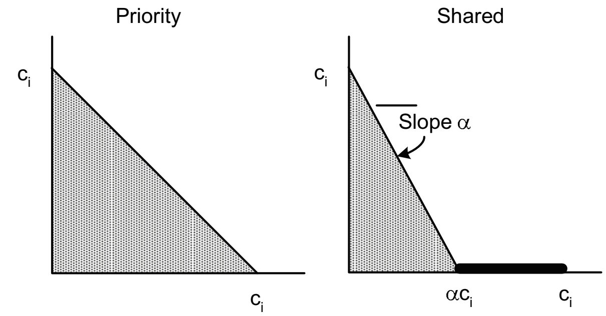


Fig. 2. The strategy spaces of the priority architecture game and shared architecture game.

### *B. User Welfare*

We use consumer surplus as our metric of user welfare. If the total rate offered web traffic is $x$, the consumer surplus is found simply by taking $\int_0^x p_w(z)dz - xp_w(x)$. The integral has the following interpretation. From the demand curve $p_w(\cdot)$, we see that the first $\epsilon$ units of traffic would be willing to pay a price as high as $k - \epsilon$ per unit, the next $\epsilon$ units of traffic would be willing to pay a price of $k - 2\epsilon$ per unit, and so on. Thus integrating the demand curve from 0 to $x$ captures the total amount the traffic is willing to pay, and then subtracting the amount it actually pays $xp_w(x)$, yields the surplus enjoyed by the traffic (users). We compute the consumer surplus of the voice traffic in exactly the same way.

### *C. Provider Profit*

The payoff or profit $\pi_i$ of each provider $i$ is simply

$$\begin{aligned}\pi_i(x_i, y_i; x_{-i}, y_{-i}) &= p_w(x)x_i + p_v(y)y_i \\ &= (k - x_{-i} - x_i)x_i + (ak - by_{-i} - by_i)y_i \qquad (2)\end{aligned}$$

where $x_{-i} = \sum_{j \neq i} x_j$ and $y_{-i} = \sum_{j \neq i} y_j$.

### *D. Social Welfare*

The social welfare is the sum of provider profits and consumer surplus. It reduces to the following expression which is only a function of the total flow $x$ and $y$ offered to web and voice respectively.

$$W(x, y) = kx - \frac{1}{2}x^2 + aky - \frac{b}{2}y^2. \qquad (3)$$

### *E. Priority Architecture Game*

The payoff function of each player $i$ is as defined by (2), and each player's strategy space is a web flow - voice flow pair in the set

$$\mathcal{S}_p(c_i) = \{(x_i, y_i) : x_i + y_i \leq c_i, \;\; x_i \geq 0, \;\; y_i \geq 0\}.$$

This set reflects that for a priority architecture, the voice traffic's requirement for delay can be met as long as the total traffic carried is not more than capacity $c_i$. The strategy space is illustrated in figure 2.

### *F. Shared Architecture Game*

Again, the payoff function of each player $i$ is as defined by (2), and each player's strategy space is a web flow - voice flow pair in the set

$$\mathcal{S}_s(c_i, \alpha_i) = \left\{(x_i, y_i) : \;\; x_i \geq 0, y_i \geq 0, \;\; \begin{array}{ll} \alpha_i x_i + y_i \leq c_i & \text{if } y_i > 0 \\ x_i \leq c_i & \text{if } y_i = 0 \end{array}\right\}$$

where $\alpha_i > 1$ stands in for the quantity $(1 + \frac{1}{c_i D_{max}})$ in expression (1). Recall that expression (1) gives the constraints on $(x_i, y_i)$ to meet the delay requirement for voice. As we discussed earlier, the constraint on web traffic relaxes to $x_i \leq c_i$ when $y_i = 0$ because the delay requirement for voice no longer needs to be met. The strategy space is illustrated in Figure 2. Note that the strategy space is not convex.

### *G. Price of Anarchy*

A Nash equilibrium of either game is defined as a vector of web flows $\mathbf{x}$ and voice flows $\mathbf{y}$ such that for each player $i$

$$\pi_i(x_i, y_i; x_{-i}, y_{-i}) \geq \pi_i(\tilde{x}_i, \tilde{y}_i; x_{-i}, y_{-i})$$

for any $(\tilde{x}_i, \tilde{y}_i) \in \mathcal{S}_p(c_i)$ in the priority architecture game and $(\tilde{x}_i, \tilde{y}_i) \in \mathcal{S}_s(c_i, \alpha_i)$ in the shared architecture game. Note that we do not consider mixed strategy equilibria in this paper. These are equilibria in which players (providers) choose a probability distribution across their possible strategies rather than picking one strategy.

The social optimum welfare is defined by the problem

$$\begin{array}{lll} \max & W\left(\sum x_i, \sum y_i\right) & \\ \text{s.t.} & (x_i, y_i) \in \mathcal{S}_p(c_i) \; \forall i & \text{Priority Game} \\ & (x_i, y_i) \in \mathcal{S}_s(c_i, \alpha_i) \; \forall i & \text{Shared Game.} \end{array}$$

Let $x^*$, $y^*$ be the total web and voice flow respectively in a solution to the above problem. The *price of anarchy* is defined as the ratio

$$\frac{W(x^*, y^*)}{W(x, y)}$$

where $(x, y)$ are the total web and voice flows in Nash equilibrium. If the Nash equilibrium is not unique, the price of anarchy is taken to be the maximum of the above ratio taken across the set of Nash equilibria. If a (pure strategy) Nash equilibrium does not exist, we take the price of anarchy to be undefined.

## IV. Priority Architecture Game

In this section we develop a bound on the price of anarchy in the priority architecture game. We begin our analysis by characterizing the social optimum welfare. The social optimum welfare is found by solving the following optimization problem

$$\begin{aligned}&\max \quad W(x,y)\\&\text{s.t.} \;\; x+y\le c,\;\; x\ge 0,\;\; y\ge 0\end{aligned}$$

where $c=\sum c_i$. Note that once optimal total flows $x^*$ and $y^*$ are found with the above problem, they can feasibly be partitioned into flows $(x_i^*, y_i^*)$ on each provider's network. This problem is a maximization of a concave function with convex feasibility constraints, and Slater's condition holds, so we may use Lagrangian techniques to solve the problem. Using basic Lagrangian techniques we find that the optimal solution satisfies

$$\begin{aligned}&k-x^*=ak-by^* && \text{if } x^*>0 \text{ and } y^*>0,\\&k-x^*=ak-by^*=0 && \text{if } x^*+y^*<c,\\&k-x^*\ge 0, \quad ak-by\ge 0.\end{aligned}$$

These conditions have the geometric interpretation of filling the area under both the voice and web disutility curve such that the highest areas are filled first until either the entire capacity is exhausted or the demand on both curves is exhausted. An upper bound on the optimal social welfare can be found by solving the following problem

$$\max \quad W(x,y) \;\;\text{s.t.}\;\; x+y\le c.$$

Note that the solution of the above problem is an upper bound on achievable social welfare, because the above problem does not enforce a positivity constraint on $x$ or $y$.

If $c\le k(1+a/b)$, meaning that sum of available capacity is less than the maximum possible demand of voice and web, the solution to the above problem occurs on the constraint $x+y\le c$. By writing the Lagrangian of the above problem, solving for $x$ and $y$ and substituting into the expression for $W(x,y)$ we find that the solution to this problem is

$$\tilde{S}:=\frac{2k(a+b)c+(a-1)^2k^2-bc^2}{2(b+1)}. \tag{4}$$

The maximum social welfare is also bounded by the area under both demand curves which is

$$\hat{S}:=\frac{1}{2}k^2(1+\frac{a^2}{b}). \tag{5}$$

We now turn to analyzing the Nash equilibrium of the priority architecture game. One question is whether the Nash equilibrium exists and if it is unique. It is straightforward to prove that a Nash equilibrium exists by using the results of Rosen [15]. The strategy spaces are convex, and each player's payoff function is concave with respect to his strategies. This is sufficient to show the existence of a Nash equilibrium using the results of [15]. We can also prove uniqueness by showing that the sum of the players' payoff functions is diagonally strictly concave as defined in [15]. We omit this proof so that we can focus on the price of anarchy.

Before proving our result on the price of anarchy in the priority architecture game, we characterize the best response of the player. Given the flows of the other players, $x_{-i}$ and $y_{-i}$, player $i$ chooses his strategy $(x_i, y_i)$ to satisfy the following problem

$$\begin{aligned}&\max \quad (k-\sum_i x_i)x_i+(ak-b\sum_i y_i)y_i\\&\text{s.t} \quad\; c_i-x_i-y_i\ge 0,\;\; x_i\ge 0,\;\; y_i\ge 0.\end{aligned}$$

The Lagrangian of player $i$'s profit is

$$\begin{aligned}L(x_i,y_i,\lambda_i,\mu_i,\nu_i;x_{-i},y_{-i})&=(k-x_{-i}-x_i)x_i+\\&(ak-by_{-i}-by_i)y_i+\lambda_i(c_i-x_i-y_i)+\mu_i x_i+\nu_i y_i.\end{aligned}$$

Setting the derivatives of Lagrangian with respect to $x_i$ and $y_i$ respectively we have:

$$\begin{aligned}\frac{d}{dx_i}L&=(k-x)-x_i-\lambda_i+\mu_i=0,\\\frac{d}{dy_i}L&=(ak-by)-by_i-\lambda_i+\nu_i=0.\end{aligned}$$

Furthermore, the remaining KKT conditions are:

$$\begin{aligned}\mu_i x_i=0,\quad \nu_i y_i=0,\quad \lambda_i(c_i-x_i-y_i)=0,\\ \mu_i\ge 0,\quad \nu_i\ge 0,\quad \lambda_i\ge 0.\end{aligned}$$

We are now ready to state and prove the main result of this section. It turns out that this Theorem is a special case of results of some recent work by Perakis and Kluberg [14], though we found our results independently. We give the full proof here for completeness, especially since we will use this result to prove later results.

*Theorem 1:* The price of anarchy for the priority architecture game is no more than $\frac{4}{3}$ for any $N\ge 1$, and any positive $k$, $a$, $b$, and $\{c_i\}_{i=1\dots N}$.

*Proof:* Let $\mathbf{x}$ and $\mathbf{y}$ be a Nash equilibrium. We consider the following three cases.

**Case 1:** Suppose there is at least one player $i$, with $x_i+y_i<c_i$.

For such a player, the KKT conditions require that

$$x_i=\frac{1}{2}(k-x_{-i})^+ \;\text{ and }\; y_i=\frac{1}{2b}(ak-y_{-i})^+ \tag{6}$$

where $x_{-i}=\sum_{j\ne i}x_j$ and $y_{-i}=\sum_{j\ne i}y_j$. Thus we have that

$$\begin{aligned}x&=x_{-i}+\frac{1}{2}(k-x_{-i})^+\ge\frac{k}{2},\\y&=y_{-i}+\frac{1}{2b}(ak-y_{-i})^+\ge\frac{ak}{2b}.\end{aligned}$$

Thus at least half the maximum demand (maximum demand being the flow at which the price becomes 0) is met for both voice and web. By substituting these lower bounds on voice and web flow into (3), one sees that the welfare achieved is at least $3/4$ of the total area under both demand curves, given by (5).

**Case 2:** Suppose that each player $i \in \{1, ..., N\}$ is at its capacity constraint $x_i + y_i = c_i$ and furthermore $x_i > 0$, and $y_i > 0$.

Under the conditions of this case, the KKT conditions require that

$$(ak - by) - by_i = \lambda_i, \quad (k - x) - x_i = \lambda_i.$$

By summing each of the above equations respectively across $i = 1...N$ we find that

$$N(ak - by) - by = \lambda, \quad N(k - x) - x = \lambda.$$

where $\lambda = \sum_i \lambda_i$. From the above expressions we can isolate $x$ and $y$ to find

$$y = \frac{Nak - \lambda}{b(N+1)}, \quad x = \frac{Nk - \lambda}{N+1}.$$

Since all players are on the capacity boundary, it must be that $x + y = c$. Thus

$$\frac{Nk(a+b) - (b+1)\lambda}{b(N+1)} = c$$

from which we find that

$$x = \frac{(1-a)Nk + (N+1)bc}{(b+1)(N+1)}, y = \frac{(a-1)Nk + (N+1)c}{(b+1)(N+1)}.$$

We now write an expression for four times the Nash social welfare minus three times the upper bound on optimal social welfare. By showing that this expression is positive, we will be able to show that the price of anarchy is bounded by 4/3. We find that $(N+1)^2(b+1)(4W(x,y) - 3\tilde{S})$ evaluates to

$$-\left[\frac{1}{2}b(N+1)^2\right]c^2 + \left[k(N+1)^2(b+a)\right]c + \frac{1}{2}k^2(a-1)^2(N+3)(N-1). \quad (7)$$

The derivative w.r.t. c is

$$-\left[b(N+1)^2\right]c + \left[k(N+1)^2(b+a)\right]$$

and this is positive on $[0, (b+a)k/b]$, which is the full range of possible $c$ by the following reasoning. If $c$ were larger than $(b+a)k/b$, under the assumption of case 2 that all capacity is offered, the price of either voice or web would be negative. A selfish provider would reduce the flow offered if the price were negative. Because the derivative is positive on the full range of possible $c$, the choice of $c = 0$ minimizes expression (7). Upon substitution of $c = 0$, expression (7) becomes

$$\frac{1}{2}k^2(a-1)^2(N+3)(N-1)$$

which is easily seen to be nonnegative for all $N \geq 1$.

**Case 3:** Suppose that each player $i \in \{1, ..., N\}$ is at its capacity constraint $x_i + y_i = c_i$. Furthermore, for a nonempty subset of players $\mathcal{S}$, either $y_j = c_j$ or $x_j = c_j$ for each $j \in \mathcal{S}$.

Suppose that for two players $\{j, k\} \in \mathcal{S}$, $y_j = c_j$ and $x_k = c_k$. The KKT conditions require that $(k - x) - c_k \geq (ak - by)$ and $(k - x) \leq (ak - by) - c_j$ which is not possible. Therefore either all players in $\mathcal{S}$ carry voice traffic exclusively or all the players in $\mathcal{S}$ carry web traffic exclusively.

Suppose that all players in $\mathcal{S}$ carry voice exclusively. Note that because the model is symmetric between voice and web, the case for which all players in $\mathcal{S}$ carry web is isomorphic to the all players in $\mathcal{S}$ carrying voice case. Let $\tilde{c} = \sum_{\mathcal{S}} c_j$. By the KKT conditions

$$(ak - b\tilde{c}) \geq (ak - by) - c_j \geq k - x.$$

This relationship implies that

$$(ak - b\tilde{c}) \geq \min(ak - bt, k - s) \text{ for any } t + s = c. \quad (8)$$

Now consider a new game for which all players in $\mathcal{S}$ are removed, and the demand curves are modified by the following rules

$$k' = k, a'k' = ak - b\tilde{c}, b' = b$$

where $k'$, $a'$, and $b'$ are the new demand curve parameters. Consider the players that remain for the new game, the set of which we denote as $\bar{\mathcal{S}}$. We argue that it is a Nash equilibrium for these players to carry the same flow vector that they carried in the Nash equilibrium of the original game. This is because the following equalities show that the first order condition is satisfied for each $j \in \bar{\mathcal{S}}$:

$$\begin{aligned}(a'k' - b\sum_{l \in \bar{\mathcal{S}}} y_l) - by_j &= (ak - b\tilde{c} - b\sum_{l \in \bar{\mathcal{S}}} y_l) - by_j \\ &= (ak - b\sum_{l=1..N} y_l) - by_j \\ &= (k - \sum_{l=1...N} x_l) - x_j \\ &= (k' - \sum_{l \in \bar{\mathcal{S}}} x_l) - x_j.\end{aligned}$$

The social optimal welfare of the original game can be found by solving the following problem which we give the name *ORIGINAL*:

$$\begin{aligned}\max \quad & kx - \frac{1}{2}x^2 + aky - \frac{b}{2}y^2 \\ \text{s.t.} \quad & x + y = c, \ x \geq 0, \ y \geq 0.\end{aligned}$$

Note that the constraint that $x + y$ must equal $c$ because it must be that $c \leq k(1 + a/b)$ for if otherwise, the case assumption that all providers offer their full capacity would not hold. (All providers offering their full capacity if $c > k(1 + a/b)$ would cause the price of either web and voice to go negative.) Now consider the following related problem we give the name *MODIFIED*:

$$\begin{aligned}\max \quad & k\tilde{x} - \frac{1}{2}\tilde{x}^2 + (ak - b\tilde{c})\tilde{y} - \frac{b}{2}\tilde{y}^2 + akr - \frac{b}{2}r^2 \\ \text{s.t.} \quad & \tilde{x} + \tilde{y} + r = c, \ r \leq \tilde{c}, \ \tilde{x} \geq 0, \ \tilde{y} \geq 0, \ r \geq 0.\end{aligned}$$

Consider a feasible vector $[x, y]$ to the problem *ORIGINAL*. If $y \leq \tilde{c}$ then the assignment $[r, \tilde{y}, \tilde{x}] = [y, 0, x]$ is a feasible point for problem *MODIFIED* and gives the same objective value as the vector $[x, y]$ does for *ORIGINAL*. Similarly, if $y > \tilde{c}$, then the assignment $[r, \tilde{y}, \tilde{x}] = [\tilde{c}, y - \tilde{c}, x]$ is a feasible point for problem *MODIFIED* and gives the same objective value as the vector $[x, y]$ does for *ORIGINAL*. In reverse, a feasible point $[r, \tilde{y}, \tilde{x}]$ of *MODIFIED* maps to the feasible point $[x, y] = [r + \tilde{y}, \tilde{x}]$ of problem *ORIGINAL*, and the value of the objective $[x, y]$ achieves in *MODIFIED* is at least as large as $[r + \tilde{y}, \tilde{x}]$ achieves in *ORIGINAL*. Therefore the optimal objective of *MODIFIED* and *ORIGINAL* are the same.

From evaluating the KKT conditions of *MODIFIED* we find the following condition:

$$r = \tilde{c} \quad \text{if} \quad (ak - b\tilde{c}) \geq \min((k - \tilde{x}), (ak - b\tilde{c} - b\tilde{y}))$$

the latter condition must be true because of property (8). Therefore an optimal solution to *MODIFIED* must have $r = \tilde{c}$. The remaining variables of *MODIFIED* must solve the problem:

$$\begin{aligned} \max \quad & k\tilde{x} - \frac{1}{2}\tilde{x}^2 + (ak - b\tilde{c})\tilde{y} - \frac{b}{2}y^2 \\ \text{s.t.} \quad & \tilde{x} + \tilde{y} = c - \tilde{c}, \;\; \tilde{x} \geq 0, \;\; \tilde{y} \geq 0. \end{aligned}$$

We call this problem *NEW*, because it finds the optimal welfare that can be achieved in the new game that we defined earlier. Let $J_N^*$ and $J_O^*$ be the optimal welfare from *NEW* and *ORIGINAL* respectively. By comparing problems *NEW* to *MODIFIED*, and our result that $r = \tilde{c}$ we have that

$$J_O^* = J_N^* + (ak - b\tilde{c})\tilde{c} - \frac{b}{2}\tilde{c}^2$$

From the analysis of **Case** 2, we have that the welfare achieved from the capacity offered by the players of the NEW game must be at least $\frac{3}{4}J_N^*$. ■

## V. Shared Architecture Game

Since the strategy space of the shared architecture game is not convex, it is difficult to prove whether or not Nash equilibria always exist. Our focus in this paper is to study price of anarchy of such equilibria, rather than their existence. Therefore, all the results on the price of anarchy in this section only apply provided a pure strategy Nash equilibrium exists.

Before proceeding with our analysis of the shared case, we make the following definitions. For each player $i$ we divide the strategy space into the following union (not-disjoint) of two convex regions:

- **Region A:** $\alpha x_i + y_i \leq c_i$, $x_i \geq 0$, $y_i > 0$.
- **Region B:** $y_i = 0$, $x_i \in [0, c_i]$

We first consider the single player case. We state and prove the following theorem.

*Theorem 2:* The price of anarchy is no more than 2 in the single player case of the shared architecture game for any positive $k$, $a$, $b$, $c_1$.

*Proof:* We proceed by considering the possibilities of which regions the Nash equilibrium and social optimum flows may lie in.

**Case 1:** *Social optimum in region A, Nash equilibrium in region A*

We can view this as a modified game where region A is the only feasible region. We argue that this modified game can be mapped to the $N = 1$ case of the priority game by the following reasoning. By a change of units in how we measure web traffic, we can let $x' = \alpha x$ where $x'$ is the amount of web traffic in smaller units. In the new units, the web demand function becomes $k' - x'/(\alpha^2)$ where $k' = k/\alpha$, and the capacity constraint becomes $x_i' + y_i = c_i$. Another change of units, this time in the units of money makes the slope of the new web demand function 1. By applying the results of Theorem 1, we find that the price of anarchy in this case is no more than $4/3$.

**Case 2:** *Social optimum in region B, Nash equilibrium in region B*

We can view this as a game where region B is the only feasible region. This modified game can be mapped to a priority game where the voice traffic market is negligible small. Again, by Theorem 1, the price of anarchy in this case is no more than $4/3$.

**Case 3:** *Social optimum in region B, Nash equilibrium in region A*

Player 1 has the option to switch to region B, and maximize his profit across points in region B, but he chooses not to do so. Therefore it must be that $\pi_1 \geq \frac{k^2}{4}$ if $c_1 \geq \frac{k}{2}$, or otherwise $\pi_1 \geq c_1(k - c_1)$. The optimum social welfare satisfies $S = \frac{k^2}{2}$ if $c_1 \geq k$, or otherwise $S = kc_1 - \frac{1}{2}c_1^2$. Clearly $2\pi_1 - S \geq 0$ when $c_1 \geq \frac{k}{2}$. When $c_1 < \frac{k}{2}$, $2\pi_1 - S = kc_1 - \frac{3}{2}c_1^2$ which is nonnegative for $c_1 \in [0, \frac{k}{2}]$.

**Case 4:** *Social optimum in region A, Nash equilibrium in region B*

Player 1 has the option to switch to the most profitable point in region A. Suppose he were to make this switch. The social welfare achieved, which we call $W'$ would be at least $\frac{3}{4}S$, where $S$ is the optimum social welfare, by Theorem 1. It is straightforward geometric exercise to show that in both the web and voice markets a monopolist never offers more than half of the amount of traffic that would make the price 0. Using this fact, it follows that the profit is never less than $\frac{2}{3}$ the social welfare achieved. Therefore the profit player 1 would earn by switching to an optimal strategy in region A is at least $\frac{2}{3} \times \frac{3}{4}S = \frac{1}{2}S$. It must be that the profit he earns in region B must be at least as high, or else he would have switched. Thus $\pi_1 \geq \frac{1}{2}S$. ■

The following example shows that the bound found by Theorem 2 is tight.

*Example 1:* Consider the shared game with $N = 1$ provider, $k = 2$, $ak = \frac{1}{2} + \epsilon$, $c_1 = 2$, $b \to 0$, and $\alpha \to \infty$. These demand functions are illustrated by Figure 3. Provider 1's best strategy in region A is $y_1 = 2$ and $x_1 = 0$ which provides a profit of $\pi_1 \to 1 + 2\epsilon$. Provider 1's best strategy in region B is $x_1 = 1$ which yields a profit of 1. Therefore provider 1 chooses the best strategy in region A, which yields a consumer surplus that vanishes to 0 as $b \to 0$. The social optimum is to allocate all of capacity $c_1 = 2$ to the voice market, which achieves a social surplus of 2. Thus the price of anarchy of this example approaches 2 as $\epsilon \to 0$.

We now turn to studying the shared game for more than 1 provider. We begin by studying the following example.

*Example 2:* Consider the share game with $N = 2$

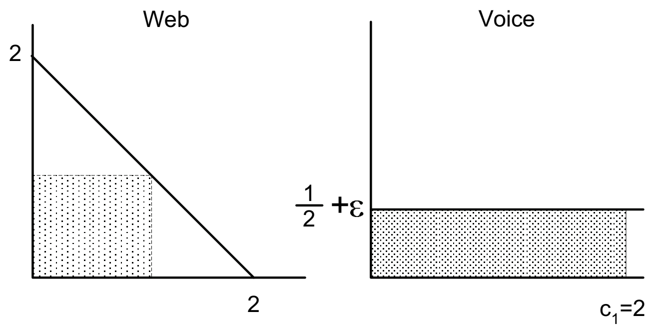


Fig. 3. The demand functions of Example 1. Note that the provider earns a larger profit serving voice traffic than he could earn serving web traffic.

providers, $k = 1$, $ak = 1$, $b = 1$, $c_1 = c_2 = 1$, $\alpha_1 = \alpha_2 = \alpha$. Using Lagrangian optimization techniques it is straightforward algebraic exercise to show that any Nash equilibrium for which both providers choose a strategy in region A (an A-A equilibrium for short), it must be that

$$\begin{array}{ll} x_1 = x_2 = \frac{2\alpha+1}{3(\alpha^2+1)} & \text{if } \alpha \geq 2, \\ y_1 = y_2 = \frac{\alpha^2-\alpha+3}{3(\alpha^2+1)} & \text{if } \alpha \geq 2, \\ x_1 = x = 2 = y_1 = y_2 = \frac{1}{3} & \text{if } \alpha < 2. \end{array}$$

Similarly we can find the required conditions for an A-B equilibrium as:

$$x_1 = \frac{2\alpha - 1}{3 + 4\alpha^2}, \quad y_1 = \frac{2\alpha^2 - \alpha + 3}{3 + 4\alpha^2},$$
$$x_2 = \frac{1 - x_1}{2}, \quad y_2 = 0.$$

It is straightforward to see that a B-B equilibrium is never possible for this example because one provider will always have an incentive to unilaterally switch to region A. We can write algebraic expressions based on the fact that an A-A equilibrium is valid only if no provider has an incentive to unilaterally deviate to his best strategy in region B. This condition is that

$$(k - x_1 - x_2)x_1 + (ak - by_1 - by_2)y_1 \geq (k - x_2)^2/4.$$

Likewise an A-B equilibrium is only valid if no provider has an incentive to unilaterally deviate to his best strategy in the opposite region. This requires the above condition holding to keep the player 1 from switching to region B. By deriving player 2's best response in region A, one can show that for player 2 to not want to switch to region A, it must be that

$$(k - x_1 - x_2)x_2 > (k - x_1 - x_2')x_2' + (ak - by_1 - by_2')y_2'$$

where $x_2' = \frac{1+\alpha(1+y_1)-x1}{2(1+\alpha^2)}$ and $y_2' = c_2 - \alpha x_2'$ if $\alpha - 1 - y1 - \alpha x_1 > 0$, otherwise $x_2' = (k - x_1)/2$ and $y_2' = (ak - by_1)/(2b)$.

Figure 4 shows the social welfare achieved by the symmetric (A-A) equilibrium and the asymmetric equilibrium (A-B) as a function of $\alpha$. The traces of Figure 4 also indicate for which values of $\alpha$ each equilibrium type exists, because the trace disappears whenever there is no equilibrium of that

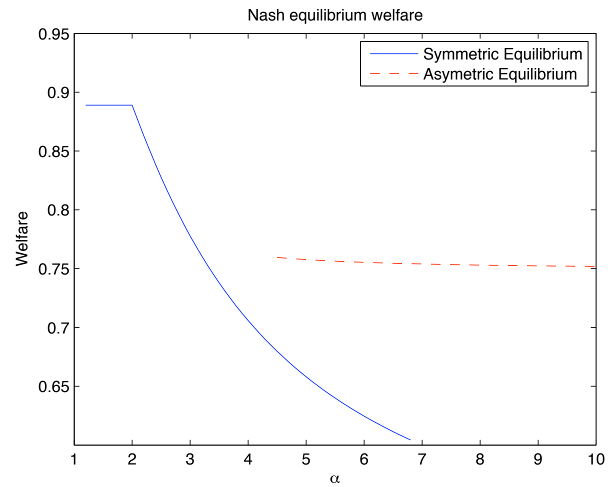


Fig. 4. The Nash equilibrium welfare of Example 2. Note that the symmetric equilibrium disappears for $\alpha$ larger than about 6.8, similarly the asymmetric equilibrium does not exist for $\alpha$ less than about 4.5. The social optimum welfare of the example is 1.

type. The optimum social welfare of this example is simply 1, independent of $\alpha$. This is because the entire area under both demand curves can be extracted by having one provider offer all their capacity to voice and the other offer all their capacity to web.

From Figure 4 we see that as $\alpha$ increases, the efficiency of the symmetric equilibrium drops, while from our analytical expressions for voice and web traffic in the A-A equilibrium, the proportion of rate offered to voice increases. This tends to lower the price of voice traffic and raise the price of web traffic. Eventually it becomes attractive for a provider to unilaterally switch to region B where he can specialize in web traffic and be free of the constraint $\alpha_i x_i + y_i \leq c_i$. It is at this critical level of $\alpha$ that the symmetric equilibrium disappears. At this point the symmetric equilibrium achieves a welfare of about 0.604, which corresponds to a price of anarchy of almost 5/3.

Example 2 demonstrates that a Nash equilibrium in which each provider offers both web and voice can be relatively inefficient. The following theorem bounds the price of anarchy of such equilibria in the case that the voice demand is larger (not smaller) than the web demand.

*Theorem 3:* Consider the shared game with $N$ providers, $a \geq 1$, $a/b \geq 1$, $\alpha_i = \alpha \geq 1$, for $i = 1...N$, any positive $k$, $\{c_i\}$. The price of anarchy of any Nash equilibrium in which each provider offers nonzero amounts of voice and web is no more than $\frac{8}{3}$.

*Proof:* Since the Nash equilibrium is such that all providers are offering flows in region A, by Theorem 1 it must be that the social welfare achieved is at least 3/4 the optimal social welfare of a priority game with web demand $p_w'(x) = \frac{k}{\alpha} - \frac{x}{\alpha^2}$ and voice demand $p_v'(y) = ak - by$. We call this modified game *NEW*. The prime symbols in the notation distinguishes the demand functions of game *NEW* from those of the original game. The optimum social welfare of the original game can be no larger than the optimum social welfare of a priority game with the same parameters, as the only difference would be a larger feasible set of flows. By the assumptions of this theorem the demand curve for voice has the property that it is larger than demand curve for web – specifically $p_v(y) > p_w(y)$ for all $y \in [0, ak/b]$. Thus in a social optimum allocation of flows, it will always be possible to extract at least half the total welfare from voice traffic. Game *NEW* has the same voice demand function, the same provider capacities, but a web demand function that has been modified. Therefore the social optimum welfare in game *NEW* should be at least half that of what is achievable in the original game. Thus the welfare achieved by the Nash equilibrium must be at least $3/8$ the social optimum welfare. ■

The following theorem places another bound on the price of anarchy, this time in the case that each player has sufficient capacity to carry the larger of half the web demand or half the voice demand.

*Theorem 4:* Consider the shared game with $N \geq 2$ providers, with $c_i \geq \max(\frac{k}{2}, \frac{ak}{2b})$ for $i = 1...N$, and arbitrary positive $k$, $a$, $b$, $\{\alpha_i\}_{i=1...N}$ . The price of anarchy in any Nash equilibrium, is no more than $\frac{(N+1)^2}{N^2} \leq \frac{9}{4}$.

*Proof:* Each provider $i$ has the option to switch to a strategy of serving web only. The most profitable of such strategies is to serve web at rate $\frac{k-x_{-i}}{2}$ which is feasible by our assumptions on player capacities. Such a strategy would earn a profit of $\frac{(k-x_{-i})^2}{4}$, therefore the player's profit in Nash equilibrium must be at least that. By similar reasoning, his profit must exceed $\frac{(ak-by_{-i})^2}{4b}$. Thus player $i$'s profit is no smaller than

$$\bar{p}_i \triangleq \frac{(k-x_{-i})^2}{8} + \frac{(ak-by_{-i})^2}{8b}.$$

The total welfare of the system is no smaller than $\sum_i \bar{p}_i + \frac{1}{2}(\sum_i x_i)^2 + \frac{b}{2}(\sum_i y_i)^2$. With some algebra, we find that this reduces to

$$\frac{Nk^2}{8} - \frac{k(N-1)}{4}\sum_i x_i + \frac{N+3}{8}\sum_i x_i^2 + \frac{N+2}{4}\sum_j\sum_m^{j-1} x_j x_m$$

plus a parallel expression involving the voice flows $\{y_i\}$. The above is a quadratic form $\frac{1}{2}\mathbf{x}^T A\mathbf{x} + B\mathbf{x} + C$ with: $A = \frac{1}{4}I + \frac{N+2}{4}M$ where $M$ is a square matrix of all 1's, $B$ a row vector with all entries equal to $-\frac{(N-1)k}{4}$ and $C$ equal to $Nk^2/8$. $A$ is symmetric and positive definite, therefore we can minimize the quadratic form by setting $\mathbf{x} = -A^{-1}B'$. The inverse of $A$ is easily computable using the matrix inversion lemma. Substituting $x$ into the above expression we find that the minimum is $k^2\frac{N^2}{(N+1)^2}$. A similar procedure applied to the parallel expression involving the voice flows $\{y_i\}$ finds a minimum of $k^2\frac{(ak)^2 N^2}{2b(N+1)^2}$. By comparing this to the maximum achievable welfare of $\frac{1}{2}k^2 + \frac{(ak)^2}{2b}$, we obtain our result. ■

## VI. Conclusion and Future Work

In this paper we have formulated a model for studying competition amongst providers who either all have a priority queueing architecture or who all have architectures that treat delay sensitive and traffic sensitive the same. In the game amongst providers who have priority architectures – the "priority game" for short – we find that the price of anarchy is no more than $4/3$. We have also found that the price of anarchy can be larger than $4/3$ in the game amongst providers who queue traffic of both types together – the "shared game" for short. We have also derived bounds on the price of anarchy in a number of important cases of the shared game.

In future work we would like to prove a general, tight bound for all possible cases of the shared architecture game. We also would like to study competition between a provider that uses a shared architecture with one that uses a priority architecture, and also study the affect of provider capacity investment decisions.

## VII. Acknowledgement

The authors thank Jean Walrand for his useful suggestions and comments about this work.

## References

[1] D. Acemoglu and A. Ozdaglar, "Competition and Efficiency in Congested Markets", *Mathematics of Operations Research*, vol. 32, no. 1, pp. 1-31, Feb.2007.

[2] A. Hayrapetyan, E. Tardos and T. Wexler, "A Network Pricing Game for Selfish Traffic", ACM Symposium on Principles of Distributed Computing, pp. 284-291, Jul. 2005.

[3] A. Ozdaglar, "Price Competition with Elastic Traffic", vol. 52, no. 3, pp. 141-155, Oct. 2008.

[4] J. Musacchio, and S. Wu, "The Price of Anarchy in an Network Pricing Game", 45th Annual Allerton Conference on Communication and Control, Monticello, IL, Sept. 2007.

[5] R. Johari and J. Tsitsiklis, "A Scalable Network Resource Allocation Mechanism with Bounded Efficiency Loss", *IEEE Journal on Selected Areas in Communications*, vol. 24, no. 5, pp. 992-999, May 2006.

[6] T. Roughgarden and E. Tardos, "How Bad Is Selfish Routing?", *IEEE Symposium on Foundations of Computer Science*, pp. 93 - 102, vol. 49 , no. 2, pp. 236-259, Mar. 2000.

[7] R. Johari and J. N. Tsitsiklis, "Efficiency Loss in a Network Resource Allocation Game", *Mathematics of Operations Research*, vol. 29, no, 3, pp. 407-435, Aug. 2004.

[8] J. MacKie-Mason and H, Varian, "Pricing Congestible Network Resources", *IEEE Journal of Selected Areas in Communications*, vol. 13, no. 7, pp. 1141-49, Sept. 1995.

[9] A. Odlyzko: "Paris Metro Pricing for the Internet," *Proceedings of the 1st ACM Conference on Electronic Commerce*, pp. 140-147, 1999.

[10] R. Gibbens, R. Mason and R. Steinberg, "Internet Service Classes under Competition", *Selected Areas in Communications, IEEE Journal*, vol. 18, no. 12, pp. 2490-2498, Dec. 2000.

[11] P. Champsaur and J.-C. Rochet, "Multiproduct Duopolists," *Econometrica*, vol. 57, no. 3, pp. 533-557, May 1989

[12] U. Karmarkar and R. Pitbladdo , "Internal Pricing and Cost Allocation in a Model of Multiproduct Competition with Finite Capacity Increments," *Management Science*, vol. 39, no. 9, pp. 1039-1053, Sep. 1993.

[13] E. Gal-Or, "Quality and Quantity Competition," *Bell Journal of Economics*, pp. 590600, 1983.

[14] G. Perakis and J. Kluberg, "Loss of efficiency dure to Oligopolistic Competition in a Market for Differentiated Products," 46th Allerton Conference on Communication and Control, Monticello, IL, September 2008.

[15] J.B. Rosen "Existence and Uniqueness of Equilbirum Points for Concave N-Person Games," Econometrica, vol. 33, no. 3, pp 520-534, Jul. 1965.