Efficiency of a Two-Stage Market for a Fixed-Capacity Divisible Resource

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***Abstract*—Two stage markets allow participants to trade resources like power both in a forward market (so consumption or production can be planned in advance), and in a spot market (allowing adjustments to be made contemporaneously with consumption and production.) However, two-stage markets introduce the possibility for a player to manipulate the market by creating an arbitrage between the two stages. We investigate the efficiency of two stage markets compared to that of a single stage market with strategic users. We show that the subgame perfect equilibrium efficiency of a two-stage market for a fixed, divisible resource and buyers with linear utility functions can be no worse than 82.8% compared to 75% for single stage markets. We also study the performance of two-stage market in the presence of uncertainty.**

I. Introduction

Forward markets allow buyers and sellers of a resource delivered over time to learn the trading outcome in advance so that they can plan their consumption or production of the resource respectively. However, random shocks may happen between the time the forward market is run and when service is delivered that can affect both the cost of supplying and utility of consuming a resource. Therefore it is useful to have a spot market, so that traders can act after the realization of these random shocks are known. However, a spot market alone may lead to inadequate time for planning resource production and consumption. To get the benefits of both forward and spot markets, two stage markets are used in domains such as electricity and also have potential for allocating other constrained resources such as network capacity.

However, two-stage markets possess a potential weakness – the possibility that players can manipulate the market by creating an arbitrage opportunity. If a buyer can manipulate the prices so that the spot market price is higher than the forward market price, that buyer can buy in the forward market and sell in the spot market and earn a trading profit. In contrast, if all users were "price takers and bought up to the point that marginal benefit equals the market price, the market would achieve an efficient allocation. Thus the incentive buyers have to manipulate prices which can potentially lead to inefficiency. In this work, we focus our investigation around the fundamental question of how inefficient a two-stage market can be, and whether the inefficiency is better or worse than a single stage market for a fixed capacity resource.

Capacity allocation using single stage markets in communication networks has been widely studied and there is a very large literature. Two particularly related works to ours are [1] and [2] . In [1], it is shown that allocation can be inefficient with price anticipating users: users that can anticipate the effects of their actions (then the model becomes a game). Later on, Johari et. al. [2] showed that the inefficiency can be no worse than 3/4, which is a tight and achievable bound when utility functions are linear.

Two-stage markets are routinely used in electricity markets and have been widely studied. In the most common setting, energy retailers use forward markets to acquire part of their energy requirements. The spot market allows retailers and generators to adjust to fluctuations in consumption and production. A large number of theoretical studies are available in the literature, and we refer to some of the work that is most related. Allaz et. al. [3] investigate the forward-spot market as a bilevel game with a Nash-Cournot model and show that the existence of the contract market increases the efficiency of spot markets. Moreover, Yao and Oren [4] incorporate these bilevel games into some spatial electricity markets with transmission constraints. More recently, Zhang et. al. [5] studied two-stage markets in a stochastic equilibrium setting. In the context of cognitive radios, some recent studies [6], [7] explore the use of primary and secondary contract pricing to achieve the optimal allocation in spectrum markets.

However, most of the studies consider supply fluctuations while variations in the demand are just in response to the changing market price. To the best of our knowledge, we are the first to derive an efficiency bound on a two-stage sum-bid style market for a divisible, fixed capacity resource.

In our model, we suppose that users can resell capacity acquired in the forward market in the spot market. Our analysis approach is to identify conditions for sub-game perfect Nash equilibria. For simplicity of analysis we restrict buyers to have linear utility functions. Our main result shows that the worst case efficiency is bounded by $2\sqrt{2}-2$ as compared to 75% for a single stage market [2].

In section II, we describe the two-stage market model and equilibrium concept. We illustrate some key behaviour of a two stage market with an example in section refs:exa. In section IV, we derive the bound on efficiency. We then describe the uncertainty model and investigate the arbitrage related aspects in section VI via example. Finally, we summarize and conclude in section VII.

II. System Model

We consider a two-stage market in which n users trade for the capacity c of a divisible resource such as the capacity of communication link. Without loss of generality, we assume $c = 1$. Users are assumed to be strategic. As in [2], [1], each player selects a bid and then each player pays their bid

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and is given an allocation proportional to their bid. Unlike [2], there are two rounds of bidding. In the "forward" round, which we also call the first stage market, each player i of n players chooses bid w_i and receives an initial allocation of $x_i \triangleq \frac{w_i}{\sum_{j=1}^n w_j}$. Since each player pays the full amount of their first stage bid w_i to the network manager, this is equivalent to charging each user a per-unit price of $\mu \triangleq c^{-1}\sum_{j=1}^n w_j = \sum_{j=1}^n w_j$. In the second round, the entire link capacity is again "on the market" and each user selects a bid v_i. The second stage price is then set to be $\rho \triangleq c^{-1}\sum_{j=1}^n v_j = \sum_{j=1}^n v_j$, and each user i gets a final allocation $y_i \triangleq v_i/\rho$. The net payment each user makes in the second round is $v_i - \rho x_i = \rho(y_i - x_i)$. The reasoning is that i's initial allocation x_i is treated as an asset in the second stage, with a value determined by the second stage price. Thus each user i's net payment is the second stage price ρ times the net change in their allocation. Since the capacity remains fixed between rounds, the sum of all these net payments is 0. Consequently, the money in the second round is exchanged between the players, with no net money received by the network manager.

We assume players have perfect information, i.e., each player knows his own utility function and that of his opponents, and each player can see all the actions that were taken in the first stage before he selects a second stage action.

Each user i's utility depends on his final allocation y_i, and thus his utility is $U_i(y_i)$ where $U_i(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly concave increasing twice differentiable function.

His net payoff for the game, including first stage payment, and second stage net payment is

$$J_i(\mathbf{w}, \mathbf{v}) = U_i(y_i) - v_i + \rho x_i - w_i.$$

A. Equilibrium concept

Let $\mathbf{w}$ and $\mathbf{v}$ be the action profile in the first and second stages respectively. A pure strategy for player i is a pair $(w_i, v_i(\cdot) : \mathbf{w} \to v_i)$ that specifies a first stage bid and a function mapping the first stage action profile to a second stage bid. Later, we will suppose that observable random events or "shocks" can happen between stages. When we add that feature, a player will be able to make his second stage bid a function of these observations as well.

1) Sub-game perfect equilibrium: A sub-game perfect equilibrium (SPE) strategy profile specifies a Nash equilibrium strategy for each substage of the original multistage game. The standard way to construct an SPE is backwards induction [8, chap 3].

Definition 1: $(\mathbf{w}^*, \mathbf{v}^*(\cdot))$ is a subgame perfect equilibrium for the two stage market, if for each i

$$v_i^*(\mathbf{w}^*) \in \arg\max_{v_i \in \mathbb{R}^+} J_i(\mathbf{w}^*, \{\mathbf{v}_{-i}^*, v_i\}), \text{ and}$$
$$w_i^* \in \arg\max_{w_i \in \mathbb{R}^+} J_i(\{\mathbf{w}_{-i}^*, w_i\}, \mathbf{v}^*(\{\mathbf{w}_{-i}^*, w_i\})).$$

B. Allocation efficiency

The social optimal allocation is defined by:

SO: Maximize $\sum_i U_i(y_i)$, subject to $\sum_i y_i \le 1; y_i \ge 0, \ \forall i.$

Because the objective function is continuous and the feasible reason is compact, an optimal solution exists. Since the feasible reason is convex and functions $U_i(.)$ are strictly concave, the optimal solution is unique.

We define the efficiency ratio of an SPE of the game to social optimum as

$$E = \frac{\text{Non-cooperative welfare}}{\text{Social optimal welfare}} = \frac{\sum_i U_i(y_i)}{\sum_i U_i(y_i^{SO})}. \quad (1)$$

where $\{y_i\}$ is assumed to be an SPE allocation vector and $\{y_i^{SO}\}$ the socially optimal solution. Later in the paper, we will compare this ratio of the two stage game to the same ratio for a single stage market. We denote the single stage Nash equilibrium allocation as y^{SS}. The worst case efficiency ratio is widely known as "Price of Anarchy" (PoA).

III. Motivating example: One user vs. Many Symmetric Users

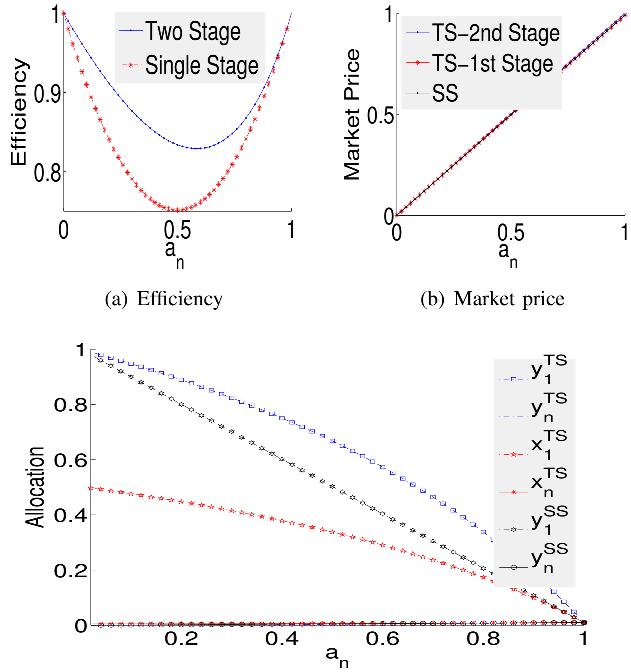
Before we begin the analysis, we attempt to unveil some important behaviour of a two stage market with the help of a simple example. We consider two types of strategic users with linear utility: a single user having utility $U_1(x) = a_1 x$, and N symmetric users with utility $U_i(x) = a_i x$. (Notation N is chosen to be distinct from n, the total number of players.) We choose this example because it is known to have the worst-case efficiency for a single stage market [2]. For comparison, we numerically analyze the two-stage market and the analogous single stage market side by side.

Numerical analysis suggests that there exists a unique SPE for this example. In Fig.1, we depict the equilibrium behaviour for the $a_1 = 1$ and letting a_n vary in the range $[0, 1]$. When $a_n \approx 0$, user 1 has a much larger utility function and the other users can be thought of as "small." As a_n approaches 1 it becomes more and more like having $(N + 1)$ symmetric users. We pick N to be 100. We make the following important observations from Fig. 1:

i. ***Efficiency:*** The two-stage market appears more efficient compared to the single-stage market in Fig. 1(a). Further, observe that the user with larger slope (user 1) acquires more resources (in 1(c)) in the 2 stage market than in the one-stage market and hence the system is more efficient. (Note the most efficient allocation would be to give the player with the highest slope the entire resource.)

ii. ***Non-equal stage prices:*** The spot market price is just slightly higher than the forward price in Fig. 1(b). Therefore, since users are strategic, users might want to buy at a lower price in forward market and sell in spot market to make profit. However, users know that if they try to exploit this arbitrage opportunity by buying more in the forward market, the forward market price would grow and the arbitrage opportunity would lessen. This tradeoff keeps the two prices from becoming exactly equal.

iii. ***Price manipulation by the "big" player:*** In a single stage market, the player with the largest slope ("big" player) bids so that the price is less than the slope of their utility function so that they get a positive payoff. This behavior keeps the price low enough so that users



(a) Efficiency (b) Market price

(c) Rate allocation

Fig. 1. Comparison: Single stage market vs Two stage market; SO: Social optimal, SS: Single stage market, TS: Two stage market.

with lesser slopes buy as well, even though it is not socially optimal for them to have any allocation. A similar effect appears to happen in the two-stage market, with a notable difference. In the two-stage market the small users buy some of the resource in the first stage, but in the second stage they sell some of it back to the "big" player. This trade amongst the players results in the "big" player getting a larger allocation than he would in single stage market.

iv. ***Arbitrage:*** Clearly, the possibility of arbitrage in such a simple example, opens up an avenue of arbitrage related investigation. Can one construct an example for which the two-stage market performs worse than the one-stage market? We investigate this key question in this work.

In the following sections, we discuss the existence of equilibria and their efficiency. For the simplicity of analysis, we restrict our analysis to the linear utility functions of the form $U_i(y_i) = a_i y_i$ with the assumption $a_i > 0$ for all i. The treatment of nonlinear, concave utility functions is a topic of future work.

IV. Equilibrium Analysis

A. Existence of equilibrium in the spot market game

We first show the spot market game has a unique equilibrium, which enables us to explicitly express the user's payoff in terms of user's forward stage actions.

*Definition 2 (**Spot equilibrium**):* Let $\mathcal{G}_s(\mathbf{x})$ denote the sub-game that follows a first stage that results in an initial allocation vector $\mathbf{x}$. In this sub-game each user chooses v_i to maximize $J_i(\mathbf{w}, \{v_i, \mathbf{v}_{-i}\}) = U_i(v_i/\rho) - v_i + \rho x_i - w_i$ where recall $\rho = \sum_{j=1}^{N} v_j$. Note that the spot market game is strategically equivalent to a game in which each user chooses his bid v_i to maximize $r_i(\mathbf{x}, \{v_i, \mathbf{v}_{-i}\}) \triangleq U_i(v_i/\rho) - v_i + \rho x_i$ over nonnegative v_i. Thus a Nash equilibrium of the spot market sub-game is a vector $\mathbf{v} \geq 0$ such that for all i

$$r_i(\mathbf{x}, \{v_i, \mathbf{v}_{-i}\}) \geq r_i(\mathbf{x}, \{\tilde{v}_i, \mathbf{v}_{-i}\}) \quad \text{for all } \tilde{v}_i \geq 0.$$

Notice that the spot market game is similar to the single stage allocation game in [2]. The key difference is that users have additional value because of carried over resource from the first stage. The following lemma notes that given a vector $\mathbf{x}$ there exist a unique equilibrium in the spot market sub-game.

Lemma 4.1: Assume $n > 1$, and for each i, the utility function $U_i(.)$ is concave, increasing and continuously differentiable. Then there exist a unique Nash equilibrium $\mathbf{v} \geq 0$ of the spot sub-game $\mathcal{G}_s(\mathbf{x})$ which satisfies $\sum_s v_s > 0$.

The proof is just a minor adaptation of the result in [2], so we omit it to save space. Moreover, by evaluating the first order conditions of each player, the spot equilibrium can be shown to satisfy the following:

$$\begin{aligned} U_i'(y_i)\, y_{-i} &= \rho\, x_{-i} && \text{if } y_i > 0, \\ U_i'(0) &\leq \rho\, x_{-i} && \text{if } y_i = 0, \end{aligned} \tag{2}$$

and $\sum_i y_i = 1$ where the quantity $U_i'(y_i) = a_i$ since we are only considering linear utility functions at this point.

B. On SPE Existence in the Forward-Spot Game

Exploiting the fact that a first stage allocation vector $\mathbf{x}$ leads to a unique sub-game equilibrium in the second stage, we may define the equilibrium of the overall two stage game in the following way.

*Definition 3 (**Forward-Spot SPE**):* Let $\mathcal{G}$ be the forward-spot game. A Forward-Spot SPE is a strategy profile $(\mathbf{w}^*, \mathbf{v}^*(\cdot))$, $\mathbf{v}^*(\cdot) : \mathbf{x} \to \mathbb{R}^+$ such that $\mathbf{v}^*(\mathbf{x})$ is a spot-market equilibrium for each $\mathbf{x} \in \{\chi \in \mathbb{R}^n_+ : \sum_{i=1}^n \chi_i = 1\}$ and for each i, w_i maximizes

$$J_i(\{\mathbf{w}^*_{-i}, w_i\}, \mathbf{v}^*(\mathbf{w}^*/||\mathbf{w}^*||_1))$$

over nonnegative real numbers.

Since the equilibrium in the spot subgame is unique, it is convenient to use the simplified notation $J(\mathbf{w})$ to be the payoff vector when the action profile $\mathbf{w}$ is played in the first stage and the unique $\mathbf{v}^*(\mathbf{w}^*/||\mathbf{w}^*||_1)$ is played in the second stage.

While $J(\mathbf{w})$ is continuous (except at $\mathbf{w} = 0$) its derivatives with respect to w_i are not continuous. This is because the derivatives depend on which subset of players finish the final stage with a positive allocation. This makes it difficult to prove that an SPE always exists. To make progress, we look for a weaker result. We consider a modified game in which we essentially "know" ahead of time which players will finish with a positive allocation and which will not. This is achieved by imposing a set of constraints on the players first stage bids that cause a particular set of players to finish the final stage with a positive allocation. From (2), we know that $a_i y_{-i} = \rho x_{-i}$ for any player i that finishes with positive allocation. By adding these identities, we get the relation

$$\rho = \frac{n_p - 1}{\sum_{j \in \mathcal{P}} a_j^{-1} x_{-j}} = H^{\mathcal{P}} \frac{n_p - 1}{n_p - x_p} = \frac{(n_p - 1)||\mathbf{w}||_1}{\sum_{j \in \mathcal{P}} a_j^{-1} w_{-j}} \tag{3}$$

where $\mathcal{P}$ be the set of players with positive final allocation, and $n_p = |\mathcal{P}|$, $H^{\mathcal{P}}$ is the weighted hyperbolic mean of the numbers $\{a_j\}_{j\in\mathcal{P}}$ with weights $\{x_{-j}\}_{j\in\mathcal{P}}$, $H^{\mathcal{P}}_{-i}$ is the unweighted hyperbolic mean of the numbers $\{a_j\}_{,j\in\mathcal{P},j\neq i}$, and $x_p = \sum_{j\in\mathcal{P}} x_j$. (See Lemma A.1 in the appendix for details.)

Since $a_i^{-1}\rho x_{-i} < 1$ for players that finish with a positive allocation, and $a_i^{-1}\rho x_{-i} \geq 1$ for those that finish with a zero allocation, we can write a set of inequality constraints on $\mathbf{w}$ that enforce which players end up finishing with a positive allocation. That is what we do in the following definition.

Definition 4: The *coupled constraint forward-spot (CCFS) game* $\mathcal{G}^c(\mathcal{P})$ is defined for any set $\mathcal{P} \subseteq 2^{\{1\ldots n\}}$ such that $|\mathcal{P}| \geq 2$. $\mathcal{G}^c(\mathcal{P})$ is a coupled constraint game with identical payoff functions and player-move structure as the original forward-spot game $\mathcal{G}$ with the following added constraints on the first stage bids:

for $i \in \mathcal{P}$:

$$[H^{\mathcal{P}}_{-i}]^{-1} w_i \geq \sum_{j\in\mathcal{P},j\neq i} (a_i^{-1} - [H^{\mathcal{P}}_{-j}]^{-1}) w_j + a_i^{-1} \sum_{j\in\mathcal{P}^c} w_j, \quad (4)$$

for $i \in \mathcal{P}^c$:

$$0 \leq \sum_{j\in\mathcal{P},j\neq i} (a_i^{-1} - [H^{\mathcal{P}}_{-j}]^{-1}) w_j + a_i^{-1} \sum_{j\in\mathcal{P}^c} w_j, \quad (5)$$

$$w_i \geq w_j \text{ for all } i,\ j \in \{1...n\} \text{ such that } a_i > a_j, \quad (6)$$

$$\sum_{j\in\mathcal{P}} [H^{\mathcal{P}}_{-j}]^{-1} w_j \leq 1. \quad (7)$$

The set of vectors $\mathbf{w}$ satisfying the above constraints we denote as $\mathcal{S}(\mathcal{P}) \subseteq \mathbb{R}^n$.

Note that (6) requires that players with steeper utility functions bid more in the first stage and turns out to be needed to show our equilibrium existence result for the CCFS game. Condition (7) can be shown to be equivalent to enforcing that $\rho \geq \mu$.

One question that immediately arises is whether one can choose the set of positive players $\mathcal{P}$ so that $\mathcal{S}(\mathcal{P})$ is nonempty. It turns out one always can as stated in the lemma that follows.

Lemma 4.2: There exists a discrete set $\mathcal{P}$ with $|\mathcal{P}| \geq 2$ such that $\mathcal{S}(\mathcal{P})\backslash 0$ is nonempty.

The proof is by construction and found in the appendix.

The following result establishes the existence of an equilibrium in the CCFS game.

Theorem 4.1 (Equilibrium existence in the CCFS game): Consider the *CCFS* game $\mathcal{G}^c(\mathcal{P})$. If feasible region $\mathcal{S}(\mathcal{P})\backslash 0$ is nonempty, then $\mathcal{G}^c(\mathcal{P})$ has a pure subgame perfect (forward-spot) equilibrium.

V. Efficiency of a two stage market

We are now ready to state our main result on equilibrium efficiency. The result applies to any SPE of the original "unconstrained" game (provided such an equilibrium exists) as well as to SPE of the CCFS game, provided the equilibrium meets the condition in the theorem statement.

Theorem 5.1: [Price of Anarchy] The price of anarchy is $(2\sqrt{2}-2) \approx 82.8\%$ in (i) any SPE of the game $\mathcal{G}$ or (ii) any SPE of the the CCFS game $\mathcal{G}^c$ for which players in $\mathcal{P}$ choose first stage bids satisfying $\frac{dJ_s}{dw_s}(\mathbf{w}) \geq 0$.

The last condition in the theorem statement says roughly that players with positive allocation would not benefit by lowering their bids slightly if they were allowed to by the constraints. The proof of the theorem follows.

Proof: Without loss of generality we assume player 1 has the largest slope a_1. We analyze the efficiency ratio of an arbitrary Subgame Perfect Nash Equilibrium (SPE) with equilibrium first stage bid vector $\mathbf{w}$.

Let $\mathcal{P}$ be the set of players with positive final allocation in SPE, and $n_p = |\mathcal{P}|$. From Lemma A.3 in the appendix, the right derivative of J_1 with respect to w_1 exists and must be nonpositive in any SPE of the unconstrained game. If we are dealing with the CCFS game there is no upper bound in force for player 1 in equilibrium, and hence again the right derivative of J_1 with respect to w_1 exists and must be nonpositive. Moreover, there must be a well defined set $\mathcal{P}+$ that is the set of players with positive final allocation when the first stage bid vector is $(\mathbf{w_{-1}}, w_1+\epsilon)$ for a sufficiently small and positive ϵ. Let $n_{p+} = |\mathcal{P}+|$. Using the FOCs from Lemma A.3 we have

$$\begin{aligned}\frac{\mu}{\rho} &\geq 2\rho[H^{\mathcal{P}}_{-1}]^{-1}(x_{-1} - y_{-1}) + y_{-1} \\ &\geq 2\frac{n_{p+}-1}{n_{p+}-x_p} H^{\mathcal{P}+}[H^{\mathcal{P}+}_{-1}]^{-1}(x_{-1} - y_{-1}) + y_{-1} \quad (8)\end{aligned}$$

where $H^{\mathcal{P}+}$ is the weighted hyperbolic mean of the numbers $\{a_j\}_{j\in\mathcal{P}+}$ with weights $\{x_{-j}\}_{j\in\mathcal{P}+}$, $H^{\mathcal{P}+}_{-i}$ is the unweighted hyperbolic mean of the numbers $\{a_j\}_{,j\in\mathcal{P}+,j\neq i}$, and $x_p = \sum_{j\in\mathcal{P}+} x_j$. Note that the weighted arithmetic mean of the numbers $\{H^{\mathcal{P}+^{-1}}_{-i}\}$ with weights $\{\frac{x_i}{x_p}\}$ is equal to $\frac{n_{p+}-x_p}{(n_{p+}-1)x_p} H^{\mathcal{P}+^{-1}}$, and $H^{\mathcal{P}+^{-1}}_{-1}$ is the largest of the numbers in $\{H^{\mathcal{P}+^{-1}}_{-i}\}$. Thus $H^{\mathcal{P}+} H^{\mathcal{P}+^{-1}}_{-1} \geq \frac{n_{p+}-x_p}{x_p(n_{+p}-1)}$. For more details on the algebra behind these identities see Lemma A.1 of the appendix. This leads to the relation

$$x_{-1} \leq \frac{1}{2}\frac{\mu}{\rho} + \frac{1}{2}y_{-1}. \quad (9)$$

The sum of the players' utility functions satisfies

$$\begin{aligned}\sum_{i\in\mathcal{P}} a_i y_i &= \sum_{i\in\mathcal{P}} a_i(1-y_{-i}) = \sum_{i\in\mathcal{P}} a_i - \sum_{i\in\mathcal{P}} \rho x_{-i} \\ &= \sum_{i\in\mathcal{P}} a_i - \rho(n_p - x_p) = a_1 + \sum_{i\in\mathcal{P},i\neq 1} (a_i - \rho) - \rho x_z.\end{aligned}$$

where $x_z = 1 - x_p$ is the total first stage allocation to players that finish the second stage with zero allocation. The efficiency ratio satisfies

$$E = \frac{\sum a_i y_i}{a_1 \times 1} = \frac{a_1 + \sum_{i\in\mathcal{P},i\neq 1}(a_i - \rho) - \rho x_z}{a_1} =$$
$$\frac{x_{-1}y_{-1}^{-1} + \sum_{i\in\mathcal{B}} y_{-i}^{-1}(y_i - x_i) + \sum_{i\in\mathcal{S}} y_{-i}^{-1}(y_i - x_i) + 1(0 - x_z)}{x_{-1}y_{-1}^{-1}}.$$

where $\mathcal{B} = \{i : y_i > x_i, i \in \mathcal{P}\}$ and $\mathcal{S} = \{i : y_i \leq x_i, i \in \mathcal{P}\}$ Let y_s be the largest of y_i's with $i \in \mathcal{S}$. Replacing each y_{-i}^{-1} in the summation over S with y_{-s}^{-1} makes the expression smaller (larger magnitude negative) since each term in the summation is negative and $y_{-s}^{-1} \geq y_{-i}^{-1}$ for each $i \in \mathcal{S}$. Similarly replacing each y_{-i}^{-1} in the summation over $\mathcal{B}$ with y_{-s}^{-1} makes the expression smaller since each term in the summation is positive and $y_{-s}^{-1} < y_{-i}^{-1}$ for each $i \in \mathcal{B}$. This gives

$$E \geq \frac{x_{-1}y_{-1}^{-1} + y_{-s}^{-1}(y_{-1} - x_{-1})}{x_{-1}y_{-1}^{-1}}.$$

Note we collected in the $(0-x_z)$ term when writing the above by noting that $y_{-s}^{-1} \geq 1$. The above is decreasing in x_{-1}. Now consider player s. This player has a positive allocation and thus is in $\mathcal{P}$. The derivative $\frac{dJ_s}{dw_s}(\mathbf{w}) \geq 0$ by the assumptions of the theorem for the CCFS case while this follows from the first order optimality condition for the unconstrained game since $x_s > 0$ by Lemma A.7 in the appendix.

Rearranging the expression for $\frac{dJ_s}{dw_s}$ from Lemma A.3 yields

$$y_{-s} \geq \frac{\mu}{\rho} - 2\rho[\bar{H}_{-i}^{\mathcal{P}-}]^{-1}(y_s - x_s) \geq \frac{\mu}{\rho}$$

where the last inequality follows since $y_s \leq x_s$. Using this result and (9) we have $x_{-1} \leq \frac{1}{2}y_{-s} + \frac{1}{2}y_{-1}$. Substituting this upper bound we get

$$E \geq \frac{y_{-s} + y_{-1}^2 y_{-s}^{-1}}{y_{-s} + y_{-1}}.$$

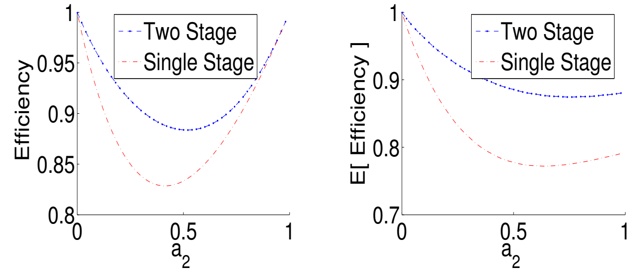
The derivative of the above with respect to y_{-s} can be shown to be negative. Since $y_{-s} \leq 1$, we have $E \geq \frac{1+y_{-1}^2}{1+y_{-1}}$. This expression is minimized when $y_{-1} = \sqrt{2}-1$ and at that point it has a value of $2\sqrt{2} - 2$. ■

VI. Demand Uncertainty

A key feature of a two-stage market is its ability to respond to a demand shock. In this section, we compare a two-stage market to a forward only market in the presence of random events or "shocks" that effect demand. The most interesting types of shocks are the shocks that effect different users differently, since these shocks change the optimal allocation of the resource amongst the users.

We consider a simple noise model: the so called "differential mode noise" – the effect of noise is different on each user's demand. In the following we briefly define the equilibrium model for each market type with noise. We then consider a set of parameter to perform evaluation. We suppose that not all the users are exposed to shock– the so called "differential shock".

Let the random noise or shock be denoted by the random vector $\boldsymbol{\beta}$. The utility of user i takes the form $U^{\beta}(.)_i = \boldsymbol{\beta}_i(\omega)U_i(.)$. The random shock $\boldsymbol{\beta} : \Omega \to \Xi$ is a continuous random vector defined on the probability space $(\Omega, \mathcal{F}, P)$ with known distribution. To simplify the notation, we write $\boldsymbol{\beta}(\omega)$ as $\boldsymbol{\beta}$. We suppose that players know the distribution of $\boldsymbol{\beta}$ when they choose their forward market bid, and they know its realization when choosing their spot market bid. Hence this information structure leads to the second stage action profile $\mathbf{v}(\cdot,\cdot)$ being a function of both the first stage action profile $\mathbf{w}$ and the realization of $\boldsymbol{\beta}$.



(a) No Shock (b) With Shock

Fig. 2. Comparison: Impact of shock in a Single and Two-stage market.

A natural equilibrium extension for the model with random shock is the subgame perfect Bayesian equilibrium (PBE) defined as follows.

Definition 5: $(\mathbf{w}^*, \mathbf{v}^*(\cdot,\cdot))$ is a PBE for the two stage market, if for each i and each β in the support of the distribution of $\boldsymbol{\beta}$,

$$v_i^*(\mathbf{w}^*, \beta) \in \arg\max_{v_i \in \mathbb{R}^+} J_i^{\beta}(\mathbf{w}^*, \{\mathbf{v}_{-i}^*, v_i\})$$

and also for each i,

$$w_i^* = \arg\max_{w_i \in \mathbb{R}^+} \mathbf{E}_{\boldsymbol{\beta}}[J_i^{\boldsymbol{\beta}}(\{\mathbf{w}_{-i}^*, w_i\}, \mathbf{v}^*(\{\mathbf{w}_{-i}^*, w_i\}, \boldsymbol{\beta}))$$

where $J_i^{\boldsymbol{\beta}}(\mathbf{w}, \mathbf{v}) = \boldsymbol{\beta}_i U_i(y_i) - v_i + \rho x_i - w_i$.

We illustrate the performance by considering a simple example with two users, competing for unit capacity. Let the shock $\boldsymbol{\beta}_1$ be uniformly distributed in the range $[0, 2]$ and $\boldsymbol{\beta}_2 == 1$, hence only user 1 faces a random shock.

For the sake of exposition, we fix $a_1 = 1$ and compute the efficiency at equilibrium for different choices a_2 from 0 to 1.

A complete performance evaluation would also compare a two-stage market to a spot-only market. A spot-only market would be more responsive, but it would also have the disadvantage of not allowing the buyers to know anything in advance about their final allocation. Building a model to evaluate this tradeoff is a topic of future work.

VII. Remarks and Conclusion

In this work, we have studied the worst case efficiency when two stage market is deployed for the allocation of a divisible, constrained resource. We showed that the efficiency can be no worse than $2\sqrt{2} - 2$ for players with linear utility function. This indicates that the two-stage market has improved efficiency as compared to the analogous single stage market, which is known to have a worst case efficiency of $3/4$ (which is achievable with linear utility functions). With the help of a simple example, we have seen the potential improved responsiveness of a two-stage market vs a forward-only market. In continuing work, we intend to develop a more general analysis of the two-stage market's responsiveness to shocks, and a framework to compare two-stage markets to spot-only markets.

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Appendix

A. Preliminary Lemmas

Lemma A.1: In any equilibrium of the spot game with players in $\mathcal{P}$ having positive allocation, equation (3) and the following holds:

$$A\left(\{[H^{\mathcal{P}}_{-i}]^{-1}\}_{i\in\mathcal{P}}, \{\frac{x_i}{x_p}\}_{i\in\mathcal{P}}\right) = \frac{n_p - x_p}{x_p(n_p-1)}[H^{\mathcal{P}}]^{-1} \quad (10)$$

where $A\left(\{[H^{\mathcal{P}}_{-i}]^{-1}\}_{i\in\mathcal{P}}, \{\frac{x_i}{x_p}\}_{i\in\mathcal{P}}\right)$ is the weighted arithmetic mean of the numbers $\{[H^{\mathcal{P}}_{-i}]^{-1}\}_{i\in\mathcal{P}}$ with weights $\{\frac{x_i}{x_p}\}_{i\in\mathcal{P}}$, and $x_p = \sum_{j\in\mathcal{P}} x_j$.

Proof: The second stage FOCs require that $\rho x_{-j} = a_j y_{-j}$ if $y_j > 0$ and $\rho x_{-j} \geq a_j$ otherwise. The former relation is equivalent to $a_j^{-1} x_{-j} = \rho^{-1} y_{-j}$. Summing the equality conditions across $j \in \mathcal{P}$ gives the relation in the first part of (3). The weighted hyperbolic average of a set of numbers is found by summing the reciprocals times the weights, and dividing that total from the sum of the weights. Thus

$$H^{\mathcal{P}} = \frac{\sum_{j\in\mathcal{P}} x_{-j}}{\sum_{j\in\mathcal{P}} a_j^{-1} x_{-j}} = \frac{n_p - x_p}{\sum_{j\in\mathcal{P}} a_j^{-1} x_{-j}} = \rho\frac{n_p - x_p}{n_p - 1}.$$

Now turning to prove (10), we write the term on the right side of (10) as

$$\sum_{i\in\mathcal{P}} [H^{\mathcal{P}}_{-i}]^{-1}\frac{x_i}{x_p} = \frac{1}{(n_p-1)x_p}\sum_{i\in\mathcal{P}}\sum_{j\neq i} a_j^{-1} x_i$$
$$= \frac{1}{(n_p-1)x_p}\sum_{i\in\mathcal{P}} a_i^{-1} x_{-i} = \frac{n_p - x_p}{(n_p-1)x_p}[H^{\mathcal{P}}]^{-1}.$$

∎

Lemma A.2: The function $J_i(\mathbf{w})$ is continuous in $\mathbf{w}$ and almost everywhere differentiable.

Proof: We omit the full proof here due to space limitations. It is easy to verify that $\mathbf{x}$ is continuous in $\mathbf{w} > 0$. After this note that the spot market subgame has a unique equilibrium given an initial allocation vector $\mathbf{x}$. The solution satisfies a system of FOCs, and in turn the price ρ that satisfies this can be shown to satisfy a fixed point equation $\sum_{i=1}^n \min(1, \rho a_i^{-1} x_{-i}) = n - 1$. The left side is monotone increasing, Lipschitz continuous in ρ, and increases from 0 to n as ρ increases. Therefore there is a unique solution, and that solution is Lipschitz continuous in $\mathbf{x}$, and thus almost everywhere differentiable. From this, it is also easy to verify that $\mathbf{y}$ is continuous in $\mathbf{x}$ and thus in $\mathbf{w}$. From these facts the continuity of the function J_i can be established. ∎

Lemma A.3: If player i has a positive final allocation in an SPE of the forward-spot game, then the derivative of his payoff with respect to w_i is

$$\frac{dJ_i}{dw_i} = \frac{\rho}{\mu}\left[2\rho[H^{\mathcal{P}}_{-i}]^{-1}(x_{-i} - y_{-i}) + y_{-i}\right] - 1$$

if the derivative exists. At points that the function is not differentiable, the right derivative and left derivative exist and are found by taking the above expression and substituting $H^{\mathcal{P}+}_{-i}$ or $H^{\mathcal{P}-}_{-i}$ respectively for $H^{\mathcal{P}}_{-i}$ where $\mathcal{P}+ = \lim_{\epsilon\downarrow 0}\{j : y_j(\mathbf{w}_{-i}, w_i + \epsilon)\}$ and $\mathcal{P}- = \lim_{\epsilon\uparrow 0}\{j : y_j(\mathbf{w}_{-i}, w_i + \epsilon)\}$.

Proof: Differentiating expression (3) for ρ with respect to w_i yields $\frac{d\rho}{dw_i} = \frac{\rho}{\mu}[1 - \rho[H^{\mathcal{P}}_{-i}]^{-1}]$. Note that this expression is valid only where the derivative exists, which henceforth we refer to as a regular point. The derivative jumps at points the set $\mathcal{P}$ changes. From the above form, we see that in any interval of w_i that results in the same set $\mathcal{P}$, ρ monotonically approaches $[H^{\mathcal{P}}_{-i}]$.

Since $x_i = \frac{w_i}{||\mathbf{w}||}$, differentiating yields $\frac{dx_i}{dw_i} = \frac{x_{-i}}{\mu}$. For $i \in \mathcal{P}$, $y_{-i} = a_i^{-1}\rho x_{-i}$. Differentiating yields $\frac{dy_i}{dw_i} = \frac{\rho}{\mu}[H^{\mathcal{P}}_{-i}]^{-1} y_{-i}$ on regular points and consequently y_i is monotone non-decreasing with respect to y_i. Similar analysis shows that $\frac{dy_j}{dw_i} = a_j^{-1}\frac{\rho}{\mu}[y_{-j}a_j[H^{\mathcal{P}}_{-i}]^{-1} - 1]$. Consequently, if y_j is decreasing and approaching 0 as w_i is increased, it must be that $a_j < H^{\mathcal{P}}_{-i}$. After y_j becomes 0, the set $\mathcal{P}$ looses player j and $H^{\mathcal{P}}_{-i}$ increases since a "below average" player has left the set of which the hyperbolic average is taken. Similarly if y_j increases from zero as w_i is increased, it must be that $a_j > H^{\mathcal{P}}_{-i}$. When y_j becomes positive, the set $\mathcal{P}$ gains player j and $H^{\mathcal{P}}_{-i}$ increases relative to its value before player j became positive since an "above average" player has joined the set of which the hyperbolic average is taken. Thus $H^{\mathcal{P}}_{-i}$ increases monotonically in w_i. From this and the above expression for $\frac{dy_j}{dw_i}$, if y_j decreases to zero as w_i increases, it cannot later increase from zero. Consequently, $H^{\mathcal{P}}_{-i}$ changes only a finite number of times with respect to w_i, and thus it is always possible to find a left and right derivative of J_i with respect to w_i.

The objective function is $J_i = a_i y_i + \rho x_i - \rho y_i - w_i = a_i y_i + \rho(y_{-i} - x_{-i}) - w_i$. Differentiating gives

$$\frac{dJ_i}{dw_i} = a_i\frac{\rho}{\mu}[H^{\mathcal{P}}_{-i}]^{-1}y_{-i} + \frac{\rho}{\mu}[1 - \rho[H^{\mathcal{P}}_{-i}]^{-1}](y_{-i} - x_{-i}) + \rho(-\frac{\rho}{\mu}[H^{\mathcal{P}}_{-i}]^{-1}y_{-i} + \frac{x_{-i}}{\mu}) - 1.$$

at regular points, which simplifies to the expression in the lemma statement. Since the left and right derivatives exist at non-regular points, the above calculations can be done with the left and right derivatives respectively to get the result in the lemma statement. ■

B. Analysis of CCFS game

Before starting the proof, we first prove a needed lemmas. The following is a proof of Lemma 4.2 stated in the main text.

Proof of Lemma 4.2: Set $w_i = 0$ for any player not in $\mathcal{P}$. First consider the case in which the there are 2 players whose slopes are larger than those of any others and w.l.o.g. suppose these have indices 1,2 and $a_1 \geq a_2$. Set $\mathcal{P} = \{1, 2\}$. Now we show there exists a feasible $\mathbf{w}$. Set $w_i = 0$ for any $i \neq 1, 2$. Since for $i = 1, 2$, $a_i^{-1} - [H_{-i}^{\mathcal{P}}]^{-1} = a_i^{-1} - a_i^{-1} = 0$, thus (4) holds for any nonnegative w_1 and w_2. For $j > 2$, $a_j^{-1} - [H_{-i}^{\mathcal{P}}]^{-1} \geq 0$ so (5) holds. By construction (6) holds and one can always choose w_1 and w_2 small enough for (7) to hold.

Next, suppose there are $m > 2$ players with the highest slope and w.l.o.g. suppose they are indexed $1, ..., m$. Let $\mathcal{P} = \{1, ..., m\}$. For $i \leq m$, $a_i^{-1} - [H_{-i}^{\mathcal{P}}]^{-1} = a_i^{-1} - a_i^{-1} = 0$, thus (4) holds for any nonnegative w_i. For $j > 2$, $a_j^{-1} - [H_{-i}^{\mathcal{P}}]^{-1} \geq 0$ so (5) holds. By construction (6) holds and one can always choose $w_1, ..., w_m$ small enough for (7) to hold.

Finally, suppose there is 1 player (indexed 1) with the largest slope and $m > 1$ players with the second highest slope (indexed $2, ..., m + 1$). Let $\mathcal{P} = \{1, ..., m + 1\}$. Note $a_1^{-1} - [H_{-j}^{\mathcal{P}}]^{-1} \leq 0$, thus (4) holds for any nonnegative w_1. Now suppose that $w_i = w$ for $i = 2, ..., m + 1$. Note that $H_{-i}^{\mathcal{P}}$ is the same for all such players, so call that value $H = m/(a_1^{-1} + (m - 1)a_2^{-1})$. (4) requires $H^{-1}w \geq (a_2^{-1} - a_2^{-1})w + (m-1)(a_2^{-1} - H^{-1})w$, which just simplifies to $w \geq 0$. By construction (6) holds and one can always choose w_1, w small enough for (7) to hold. (5) can also be shown to hold. ■

We need the following to later prove quasiconcavity.

Lemma A.4: Suppose that first stage bid vector $\mathbf{w} \in \mathcal{S}(\mathcal{P})$ and ρ is determined by the unique subgame equilibrium $\mathbf{v}^*$ following bids $\mathbf{w}$ in the first stage. If $i \in \mathcal{P}$ and $\rho > [H_{-i}^{\mathcal{P}}]$ then $a_i > \rho$.

Proof: Summing the second stage FOCs yields $\rho = (n-1)[\sum_{\mathcal{P}} a_j^{-1} x_{-j} + \sum_{\mathcal{P}^c} \tilde{a}_k^{-1} x_{-k}]^{-1}$ where $\tilde{a}_k \triangleq \rho x_{-k}$. Let $\tilde{H}_{-l}$ be the hyperbolic mean of the set $(\{a_j\}_{j\in\mathcal{P}} \cup \{\tilde{a}_k\}_{j\in\mathcal{P}^c})\backslash a_l$ (or ... $\backslash\tilde{a}_l$ instead if $l \in \mathcal{P}^c$). By rearranging sums we observe that $\rho^{-1} = A(\{\tilde{H}_{-j}^{-1}\}, \{x_j\})$ where the notation on the right side of the equality denotes the weighted arithmetic mean of the numbers $\tilde{H}_{-1}, ..., \tilde{H}_{-n}$ with respective weights $x_1, ..., x_n$. Then we see that

$$\rho^{-1} = A(\{\tilde{H}_{-j}^{-1}\}, \{x_j\}) = A\left(\left\{\frac{n\tilde{H}^{-1} - a_j^{-1}}{n-1}\right\}, \{x_j\}\right)$$
$$= \frac{n}{n-1}\tilde{H}^{-1} - \frac{1}{n-1}A(\{a_j^{-1}\}, \{x_j\})$$

where with some abuse of notation the set $\{\cdot\}$ above has n elements and uses the $\tilde{a_k}$'s for those indices in $\mathcal{P}^c$. The notation $\tilde{H}$ is the hyperbolic mean of $\{a_j\}_{j\in\mathcal{P}} \cup \{\tilde{a}_k\}_{j\in\mathcal{P}^c}$. This reduces to

$$\rho^{-1} = \tilde{H}_{-i}^{-1} + \frac{1}{n-1}\left[a_i^{-1} - A(\{a_j^{-1}\}, \{x_j\})\right].$$

$\tilde{H}_{-i} < H_{-i}^{\mathcal{P}}$ since the former adds numbers to the average that are all not bigger than any number included in the average $H_{-i}^{\mathcal{P}}$. Thus $\rho\tilde{H}_{-i}^{-1} > 1$. From the above equation, it is necessary that

$$a_i^{-1} < A(\{a_j^{-1}\}, \{x_j\}) \leq A(\{a_j^{-1}\}, \{x_{-j}\}) = \rho^{-1}$$

where the 2nd inequality is because the arithmetic average on the right weights bigger a_j^{-1} values more than arithmetic average on the left as a consequence of (6). ■

The following result is needed in the proof of Theorem 4.1.

Lemma A.5: The function $J(w_i, \mathbf{w}_{-i})$ is quasiconcave with respect to w_i on $\mathcal{S}(\mathcal{P}) \cap \{\mathbf{w} : \epsilon \leq \sum w_j \leq 2\max_i a_i\}$ for any $\epsilon > 0$.

Proof: First we consider $i \in \mathcal{P}$. By the lemma assumptions the derivative $\frac{dJ_i}{dw_i}(w_i; \mathbf{w}_{-i})$ exists and is equal to

$$\frac{\rho}{\mu}\left[2\rho[H_{-i}^{\mathcal{P}}]^{-1}(x_{-i} - y_{-i}) + y_{-i}\right] - 1 \tag{11}$$
$$= 2\frac{\rho^2}{\mu^2}[H_{-i}^{\mathcal{P}}]^{-1}(1 - \rho a_i^{-1})w_{-i} + \frac{\rho^2}{\mu^2}a_i^{-1}w_{-i} - 1.$$

Multiplying both sides by $\frac{\mu 2}{\rho^2}$ gives

$$\frac{\mu^2}{\rho^2}\frac{dJ_i}{dw_i}(w_i; \mathbf{w}_{-i}) = 2[H_{-i}^{\mathcal{P}}]^{-1}(1 - \rho a_i^{-1})w_{-i} + a_i^{-1}w_{-i} - 1.$$

Defining $\psi(w_i; \mathbf{w}_{-i}) \triangleq \frac{\mu^2}{\rho^2}\frac{dJ_i}{dw_i}(w_i; \mathbf{w}_{-i})$ and differentiating with respect to w_i gives

$$\frac{d}{dw_i}\psi(w_i; \mathbf{w}_{-i}) = -2a_i\mathbf{w}_{-i}[H_{-i}^{\mathcal{P}}]^{-1}\frac{d\rho}{dw_i} - 2\frac{\mu}{\rho}[H_{-i}^{\mathcal{P}}]^{-1},$$
$$= -2[H_{-i}^{\mathcal{P}}]^{-1}\left[y_{-i}\left(1 - \rho[H_{-i}^{\mathcal{P}}]^{-1}\right) + \frac{\mu}{\rho}\right].$$

The last expression comes from substituting an expression for $\frac{d\rho}{dw_i}$ and the 2nd stage FOCs. The quantity $\psi(w_i; \mathbf{w}_{-i})$ must always have the same sign as $\frac{dJ_i}{dw_i}(w_i; \mathbf{w}_{-i})$. Since $\psi(w_i; \mathbf{w}_{-i})$ is differentiable and has continuous derivatives on $\mathcal{S}(\mathcal{P}) \cap \{\mathbf{w} : \epsilon \leq \sum w_j \leq 2\max_i a_i\}$, any zero crossings must occur after: (i) $\psi(w_i; \mathbf{w}_{-i})$ is positive and decreasing or (ii) $\psi(w_i; \mathbf{w}_{-i})$ is negative and increasing. We now show that (ii) can not happen. Suppose the opposite and $\frac{d}{dw_i}\psi(w_i; \mathbf{w}_{-i}) > 0$ then from the equation above $y_{-i}\left(1 - \rho[H_{-i}^{\mathcal{P}}]^{-1}\right) + \frac{\mu}{\rho} < 0$ so

$$\rho[H_{-i}^{\mathcal{P}}]^{-1} > 1 + y_{-i}^{-1}\frac{\mu}{\rho} > 0 \tag{12}$$

leading to $[H_{-i}^{\mathcal{P}}]^{-1} > 1$. Thus by Lemma A.4, $\rho a_i^{-1} < 1$ and hence $y_i > x_i$ by the 2nd stage FOC, and hence $(x_{-i} - y_{-i}) > 0$. Thus $\rho[H_{-i}^{\mathcal{P}}]^{-1}$ has a positive coefficient in equation (11), so substituting a lower bound for $\rho[H_{-i}^{\mathcal{P}}]^{-1}$ gives a lower bound for $\frac{dJ_i}{dw_i}(w_i; \mathbf{w}_{-i})$. Using bound (12) we have

$$\frac{dJ_i}{dw_i}(w_i; \mathbf{w}_{-i}) > 2\frac{\rho}{\mu}(1 + y_{-i}^{-1}\frac{\mu}{\rho})(x_{-i} - y_{-i}) + \frac{\rho}{\mu}y_{-i} - 1$$
$$= (\frac{\rho}{\mu} - 1) + (y_i - x_i) + 2(y_{-i}^{-1}x_{-i} - 1)$$

The first term of three in the above expression is nonnegative (as a consequence of (7) and the other are positive. Thus $\frac{dJ_i}{dw_i}(w_i; \mathbf{w}_{-i})$ is positive and hence $\psi(w_i; \mathbf{w}_{-i})$ is positive. This rules out possibility (ii). Thus all zero crossings of

$\psi(w_i; \mathbf{w}_{-i})$ occur when it is positive and decreasing. Thus ψ may: (i) cross zero once at one point, (ii) hit zero, stay there, and then go below, (iii) always be above zero, (iv) always be below zero. Case (i) corresponds to a strictly quasiconcave function, Case (ii) corresponds to a non-strict quasi concave function. Cases (iii) and (iv) imply that there is a unique local maximum at an edge of the allowed region at $2\max_i a_i$ or 0 respectively.

The proof of the $i \in \mathcal{P}^c$ case is similar but less complex so it is omitted to save space. ■

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1: Consider the feasible action set $S(\mathcal{P})$. The set is convex, nonempty and contains more than just the vector $\mathbf{0}$. We augment the constraints to create the set $\mathcal{D}_\epsilon = S(\mathcal{P}) \cap \{\mathbf{w} : \epsilon \leq \sum w_j \leq 2\max_i a_i\}$. Set $\mathcal{D}_\epsilon$ is compact, convex, and nonempty for sufficiently small ϵ. By Lemma A.5, $J(w_i, \mathbf{w}_{-i})$ is quasiconcave w.r.t w_i and continuous w.r.t. $\mathbf{w}$ on $\mathcal{D}_\epsilon$. By Berge's theorem, the best response set $w_i^{BR}(\mathbf{w}_{-i}) = \arg\max_{w_i \in \mathcal{D}_\epsilon} J(w_i, \mathbf{w}_{-i})$ is a nonempty, closed, convex-valued, upper-hemicontinuous correspondence [9]. By Kakutani's fixed point theorem there is a fixed point $\mathbf{w}^{eq,\epsilon}$ of the set-valued function $\mathbf{w}^{BR}(\mathbf{w}) = \{\mathbf{z} : z_i \in w_i^{BR}(\mathbf{w}_{-i})\}$. This fixed point is a Nash equilibrium in the coupled constraint game with action space $\mathcal{D}_\epsilon$. It remains to show that this same point is a Nash equilibrium of the game with action space $\mathcal{S}(\mathcal{P})$.

We claim that for small enough ϵ, $||\mathbf{w}^{eq,\epsilon}||_1 > \epsilon$ by the argument that follows, which was adapted from an argument in [10]. Suppose this were not true. By (3), $\rho \geq \frac{1}{2}\min_i a_i$. For players with positive final allocation, consider expression (11) for $\frac{dJ_i}{dw_i}$. Take the weighted average of these expressions across $i \in \mathcal{P}$ with weights $H^{\mathcal{P}}_{-i}$. This procedure yields

$$\frac{\rho}{\mu \sum_{i\in\mathcal{P}} H^{\mathcal{P}}_{-i}}[2\rho(1-x_p) + \sum_{i\in\mathcal{P}} H^{\mathcal{P}}_{-i} y_{-i}] - 1$$
$$\geq \frac{\underline{a}}{2n_p\bar{a}\epsilon}\sum_{i\in\mathcal{P}} \underline{a} y_{-i} \geq \frac{\underline{a}^2(n_p-1)}{2\bar{a}n_p\epsilon} - 1$$

where $\underline{a}$ and $\bar{a}$ are the minimum and maximum of the a_i values respectively. Also note that it can be shown that $n_p \geq 2$ because 2 players must always bid positively in an equilibrium of the spot game. Thus, at least one player must have a value of $\frac{dJ_i}{dw_i}$ as large as the last expression. Since for ϵ small enough, the expression is positive, we conclude that when all the players first stage bids add to ϵ, there will always be one player who could improve their payoff by increasing their bid. If such a player can do so without violating the feasible region $\mathcal{S}(\mathcal{P})$, then the point is not an equilibrium. If he is unable to raise his bid because a "larger" player is bidding the same amount (constraint (6)) , one can show that this "larger" player would also want to increase his bid.

We now argue that $\mathbf{w}^{eq,\epsilon}$ is an equilibrium of the game with constraints $S(\mathcal{P}) \cap \{\mathbf{w} : \sum w_j \leq 2\max_i a_i\}$. Suppose this were not true. Then at least one player could improve their payoff by reducing their bid to $\tilde{w}_i$ so that $\sum_{j\neq i} w_j^{eq,\epsilon} + \tilde{w}_i < \epsilon$ and $(\tilde{w}_i, \mathbf{w}_{-i}^{eq,\epsilon}) \in S(\mathcal{P})$. Constraints (5) and (6) could not be tight for player i in $\mathbf{w}^{eq,\epsilon}$ or otherwise the player would not be able to lower his bid. Also the constraint $\sum w_i \geq \epsilon$ is not tight by the argument in the preceding paragraph. Thus, for this player it must be that $\frac{dJ_i}{dw_i}(\mathbf{w}^{eq,\epsilon}) \geq 0$. Having $J_i(\tilde{w}_i, \mathbf{w}_{-i}^{eq,\epsilon}) > J_i(w_i^{eq,\epsilon}, \mathbf{w}_{-i}^{eq,\epsilon})$ would thus violate the quasiconcavity property that we have already established.

Finally, we argue that $\mathbf{w}^{eq,\epsilon}$ is an equilibrium of the game with constraints $S(\mathcal{P})$. Suppose this were not true. Then at least one player could improve their payoff so that the new first stage price $\mu \geq 2\max_i a_i$. Since $\rho \geq \mu$ this player pays an amount per unit capacity larger than the steepest utility, so he must finish with a nonpositive payoff, which cannot be an improvement over his payoff when the bid vector is $\mathbf{w}^{eq,\epsilon}$ since one can show that each player gets a payoff of at least 0 in that situation. ■

C. Lemmas Used in the Proof of Theorem 5.1

Lemma A.6: In any SPE of the forward-spot game (unconstrained version), $\frac{\mu}{\rho} \leq 1$.

Proof: Note that there exist at least one user j with $y_j \leq x_j$, because if not the the resource constraint $\sum_i x_i = \sum_i y_i$ is violated. If this user has a positive final allocation, the FOC requires that the left derivative of the payoff with respect w_i be greater than or equal to zero. From Lemma A.3, we have

$$0 \leq \frac{d_- J_i}{dw_i} = \frac{\rho}{\mu}\left[2\rho[H^{\mathcal{P}-}_{-i}]^{-1}(x_{-i} - y_{-i}) + y_{-i}\right] - 1,$$
$$\frac{\mu}{\rho} \leq 2\rho[H^{\mathcal{P}-}_{-i}]^{-1}(x_{-i} - y_{-i}) + y_{-i} \leq 1.$$

The last inequality follows because $y_{-i} \leq 1$ and the other term in the summation is non-positive. The proof in the case that $y_i = 0$ is similar and omitted for space. ■

Lemma A.7: In the unconstrained forward-spot game, if user i finishes with positive spot allocation $y_i^* > 0$, then he will have positive forward allocation $x_i^* > 0$ at forward-spot equilibrium.

Proof: Suppose user i finishes with $y_i > 0$, and his forward position is $x_i = 0$, which implies $w_i = 0$. Then, after substituting $\rho = a_i x_{-i}^{-1} y_{-i}$ from the 2nd stage FOC and with some rearrangements we have

$$\frac{d_+ J_i(w_i, \mathbf{w}_{-i})}{dw_i} = 2[H^{\mathcal{P}+}_{-i}]^{-1}\frac{\rho^2}{\mu}[1 - y_{-i}] + y_{-i}\frac{\rho}{\mu} - 1. \quad (13)$$

From the 2nd stage FOCs for player i, $a_i = \rho y_{-i}^{-1} > \rho$. Note,

$$\rho[H^{\mathcal{P}+}_{-i}]^{-1} = \frac{\sum_{k\in\mathcal{P}+} a_k^{-1} - a_i^{-1}}{(n_p-1)\rho^{-1}} = \frac{\sum_k a_k^{-1}}{(n_p-1)\rho^{-1}} - \frac{\rho}{a_i(n_p-1)}$$
$$= \frac{\sum_k a_k^{-1}}{\sum_k a_k^{-1} x_{-k}} - \frac{\rho}{a_i(n_p-1)} > 1 - \frac{1}{n_p-1} = \frac{n_p-2}{n_p-1}$$

where in the above $n_p = |\mathcal{P}+|$. From the last inequality we have $\rho[H^{\mathcal{P}}_{-i}]^{-1} \geq \frac{1}{2}$ for $n_p \geq 3$. Substituting this into (13) yields us $\frac{d_+ J_i}{dw_i} \geq 0$ at $w_i = 0$. Hence, $w_i = 0$ is not an equilibrium for player i.

Now consider the case of $n_p = 2$. Again $a_i = \rho y_i^{-1} > \rho$. Since ρ is equal to a weighted hyperbolic average of the two a's times a constant less than 1, it must be that $a_j \leq a_i$ for $j \neq i$. Also note that $[H^{\mathcal{P}+}_{-i}]^{-1} = a_j^{-1}$. Hence $[H^{\mathcal{P}+}{}_{-i}]^{-1} > \rho^{-1}$. Substitute this with the fact that $\rho/\mu \geq 1$ from Lemma A.6 into (13), we again note $\frac{d_+ J_i}{dw_i} \geq 0$. Hence, $w_i = 0$ is not an equilibrium for $n_p = 2$ also. Hence proved. ■