Congestion Pricing Using a Raffle-Based Scheme

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***Abstract*—We propose a raffle-based scheme for the decongestion of a shared resource. Our scheme builds on ideas from the economic literature on incentivizing contributions to a public good. We formulate a game-theoretic model for the decongestion problem in a setup with a finite number of users, as well as in a setup with an infinite number of non-atomic users. We analyze both setups, and show that the former converges toward the latter when the number of users becomes large. We compare our results to existing results for the public good provision problem. Overall, our results establish that raffle-based schemes are useful in addressing congestion problems.**

***Index Terms*—congestion pricing; raffle-based incentive schemes; public good; probabilistic pricing; demand management**

I. Introduction

Congestion is becoming increasingly frequent in large infrastructures such as the Internet, transportation and power networks. Congestion occurs when a large number of users share a common resource, and user demand varies with time. Environments with frequent congestion include access points of the Internet (*e.g.,* a base-station of a wireless 3G network or a DSLAM for wired residential networks) or transportation network (*e.g.,* the Bay Bridge in California). In these cases, congestion management becomes necessary when the inefficiencies increase and result in a substantial decrease of user utility. In the case of power networks, high peak demand could cause instability and an increase in the costs of provision.

In all these examples, congestion is driven by incentives misalignment of individual users relative to social optimum. Thus, decongestion can be cast as a public good. Indeed, when a user reduces or moves part of his demand to another time, the benefits of the reduced congestion are shared by all the users. With that perspective, ideas from the economic literature on incentivizing contributions to a public good can be brought to bear on the decongestion problem.

In this paper, we propose a raffle-based scheme for the decongestion of a shared resource. We start by formulating the decongestion problem as a public good provision problem using a game-theoretic model in a setup with a finite number of atomic users. Next, we introduce a setup with an infinite number of non-atomic users, and show that the former converges toward the latter when the number of users goes to infinity (in the following, we will simply call these setup the atomic setup and the non-atomic setup respectively).

Our raffle-based scheme builds on the ideas of Morgan [1] who pioneered the economic analysis of public good provision via lotteries. However, the specifics of the congestion management problem require substantial changes from Morgan's model [1]. The idea of lotteries have also been used in other contexts, *e.g.,* for the raffle scheduling technique applied in computer operating systems [2]. Recent interest in application of lotteries to congestion management was facilitated by Mergu, Prabhakar, and Rama who demonstrated with a field study that lottery-based mechanisms can be used to decongest transportation systems [3]. In contrast, our focus is methodological. We approach lotteries as a technical tool of congestion management. We compare our setup(s) to Morgan's scheme of using lotteries for funding public goods.

II. Overview of Morgan's model

Let $N = \{1, \dots, n\}$ be a set of users (players). Each user $i \in N$ has a wealth w_i, and chooses an amount $x_i \in [0, w_i]$ to contribute to a public good. With voluntary contributions, the level of public good $G = \sum_{i=1}^n x_i$ is the sum of contributions, and the objective of each user $i \in N$ is to maximize his utility $U_i = w_i + h_i(G) - x_i$, where $h_i(\cdot)$ is an increasing strictly concave function which reflects user i's valuation of the public good.

The raffle-based scheme (*fixed-price raffle* in [1]) gives a reward $R > 0$ to one or more users and user's expected reward is proportional to his fraction of the total contribution. The scheme is financed by deducting the reward from the total contribution. Then each user $i \in N$ has expected utility

$$U_i(x_i, x_{-i}) = w_i + h_i(G) - x_i + R \cdot \frac{x_i}{\sum_{i=1}^n x_i}, \qquad (1)$$

where the level of public good is now $G = \sum_{i=1}^n x_i - R$, and x_{-i} denotes the vector of contributions of all users but i. Finally, Morgan [1] assumes that if the total contribution is insufficient to cover the prize up to an arbitrarily small amount $\delta > 0$ (*i.e.,* if $\sum_{i=1}^n x_i < R - \delta$), then the raffle is canceled and each user's contribution is returned.

At a Nash equilibrium, each user maximizes his utility (1). We denote with a superscript $^{(\mathrm{eq})}$ all quantities at Nash equilibrium. The aggregate welfare W is

$$W = \sum_{i=1}^n U_i. \qquad (2)$$

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We define social optimum as the point where the aggregate welfare is maximized. All quantities at social optimum are denoted with a superscript *.

Theorem 1 (Summary of Morgan's results [1])**.**

(i) For any $R > 0$, there exists a unique Nash equilibrium, whereas for $R = 0$ (voluntary contributions), there can exist multiple equilibria, all with the same amount of public good.
(ii) There exists a unique level of public good G^*, which maximizes the aggregate welfare, and for any $R > 0$, we have $G^{(eq)}(0) < G^{(eq)}(R) < G^*$.
(iii) The equilibrium amount of public good with the raffle-scheme $G^{(eq)}(R)$ can be made arbitrarily close to the socially optimal level G^* by choosing a sufficiently large reward R.
(iv) For any $R > 0$, $G^{(eq)}(R) > 0$ if and only if $G^* > 0$.

III. Decongestion of a shared resource

To address the problem of decongestion, we first formulate it as a public good provision problem using a game-theoretic model in an atomic setup (with a finite number of users) as in Morgan [1] (Sec. III-A). Then, we propose a non-atomic setup (Sec. III-A), as it is often done in economics to model games with a large number of players [4].

A. Atomic setup

Let $N = \{1, \ldots, n\}$ be the set of users. For simplicity, we assume that each user has identical demand $d = D/n$ for the shared resource, where D is the total demand. There are two times of the day: peak and off-peak, and each user decides how to divide their demand between them. Each user $i \in N$ is endowed with a type $\theta_i \in \Theta$ that characterizes the user's preferences between the two times. Without any incentive scheme, we assume that the utility of a user $i \in N$ who chooses to put a fraction x_i of his demand in the off-peak time is:

$$U_i(x_i, x_{-i}) = d\Big[P_{\theta_i}(1-x_i) + O_{\theta_i}(x_i) - p - (1-x_i)\cdot L_P\big(D - d\sum_j x_j\big) - x_i \cdot L_O\big(d\sum_j x_j\big)\Big],$$

where $P_\theta(\cdot)$ and $O_\theta(\cdot)$ are the utilities that the user gets for his demand in the peak and off-peak periods respectively, while $L_P(\cdot)$ and $L_O(\cdot)$ are the costs of delay in the peak and off-peak periods respectively. These delay costs are per unit of demand, hence they are multiplied by the fraction of demand in each time. The whole expression is scaled by the user's total demand d since this form will later be useful for considering the limit in which each user's fraction of the total demand diminishes. The quantity p is a fixed monthly subscription price and x_{-i} is the vector of choices of all users but i. Finally, we assume that since there are many more off-peak hours than peak hours, congestion at off-peak times never becomes significant. Therefore, $L_O(\cdot) \approx 0$.

We assume that, in the absence of latency, users always prefer to use the service at peak-time, *i.e.,* $P'_\theta(1-x)$ is larger than $O'_\theta(x)$ for all user types $\theta \in \Theta$ and $x \in [0,1]$. We may therefore define the notion of the *cost of shifting* as the loss of utility that a user of type $\theta \in \Theta$ incurs when shifting a fraction x of his demand from peak to off-peak time:

$$c_\theta(x) = \bar{u}_\theta - \left(P_\theta(1-x) + O_\theta(x)\right), \quad \forall \theta \in \Theta,$$

where

$$\bar{u}_\theta = P_\theta(1) + O_\theta(0)$$

is the maximal utility that a user could get without shifting any of his demand if there was no congestion. We assume that $c_\theta(\cdot)$ is increasing and strictly convex (this assumption may also be derived from the assumption that $P_\theta(\cdot)$ and $O_\theta(\cdot)$ are strictly concave). This assumption reflects the fact that for a given x_i, users will shift the most easily shiftable parts of their demand before shifting demand that is more costly to shift. We also assume that $c_\theta(\cdot)$ is twice differentiable, and that it has a slope that is bounded away from zero by a constant independent of θ. To simplify the proofs, we also assume that $c'_\theta(\cdot)$ is bounded on $[0,1]$ by a constant independent of θ, but this assumption can likely be relaxed.

We view each players choice of x_i as a contribution to the public good since it represents demand withheld from the peak time. Thus we define

$$G = \sum_{i=1}^{n} x_i d. \tag{3}$$

We may now recast the utility of each user as

$$U_i(x_i, x_{-i}) = d\left[\bar{u}_{\theta_i} + (1-x_i)h(G) - c_{\theta_i}(x_i) - p\right]$$

where $h(G) = -L_P(D-G)$. We assume that $L_P(\cdot)$ is increasing and strictly convex and twice differentiable. This assumption results in $h(\cdot)$ being increasing strictly concave.

The raffle-based scheme introduces for each user a reward proportional to his fraction of the total shifted demand dx_i/G. However, it is financed here by charging an extra price Δp (per unit of d) to each user. With the raffle-based scheme of parameter $R \geq 0$, the utility of user $i \in N$ then becomes

$$U_i(x_i, x_{-i}) = d\Big[\bar{u}_{\theta_i} + (1-x_i)h(G) - c_{\theta_i}(x_i) - p + R \cdot \frac{x_i}{G} - \Delta p\Big], \tag{4}$$

where $\Delta p = \frac{R}{D}$, so that $\sum_{i=1}^n d \cdot \left(R \cdot \frac{x_i}{G} - \Delta p\right) = 0$. Finally, we assume that if no user shifted any of his demand, *i.e.,* $x_i = 0$ for all $i \in N$ (hence $G = 0$), then the reward is not given. The raffle-based scheme could be implemented as a conventional raffle in which the whole prize R is awarded randomly to one player with a probability proportional to that player's contribution to the public good as in [1]. However the scheme could also be implemented deterministically – *i.e.,* each player i could be given a payment or rebate of the amount $R \cdot \frac{dx_i}{G}$.

The marginal utility of user $i \in N$ is

$$\frac{\partial U_i}{\partial x_i} = d \cdot \Big\{ -h(G) + (1-x_i)dh'(G) - c'_{\theta_i}(x_i) + R \cdot \frac{G - x_i d}{G^2}\Big\}. \tag{5}$$

At a Nash equilibrium, each user chooses his contribution x_i to maximize his utility (4). Since utility (4) is strictly concave, for a vector of contributions x to be a pure Nash equilibrium, it is necessary and sufficient that is satisfies the first-order conditions (FOCs)

$$\frac{\partial U_i}{\partial x_i}(x_i, G) \quad \begin{cases} \leq 0, & \forall i \in N : x_i = 0, \\ = 0, & \forall i \in N : x_i \in (0,1), \\ \geq 0, & \forall i \in N : x_i = 1, \end{cases} \tag{6}$$

and equation (3). In general, a Nash equilibrium does not coincide with the social optimum in which (2) is maximized.

B. Non-atomic setup

When the number of users is large, the demand d of each user becomes negligible w.r.t. to the total demand D and a setup with an infinite number of non-atomic users becomes more appropriate [4], [5]. We assume here that the total demand D remains constant and the individual demand tends to zero. We assume that user types are i.i.d. distributed according to a measure μ on the set of types. Since the utilities are strictly concave, two users of the same type have the same contribution at equilibrium. Therefore, we work with the distribution of types directly instead of working with the distribution of users as in [4], [5].

Let $(\Theta, \mathcal{F}, \mu)$ be a measured space; where Θ is the set of user types, $\mathcal{F}$ is a $\sigma-$algebra and μ is a finite measure accounting for the distribution of the user types. We assume that users have identical demand and that each user of type $\theta \in \Theta$ chooses a fraction x_θ of his demand to contribute to the public good. This defines a measurable function $x : \Theta \to [0,1]$ on $(\Theta, \mathcal{F}, \mu)$. In this framework, the total demand is $D = \int_\Theta \mathrm{d}\mu(\theta)$ and the level of public good is

$$G = \int_\Theta x_\theta \mathrm{d}\mu(\theta). \tag{7}$$

With the raffle-based scheme, a user of type $\theta \in \Theta$ has utility

$$\begin{aligned} u_\theta(x_\theta, x_{-\theta}) = \bar{u}_\theta + (1 - x_\theta)h(G) - c_\theta(x_\theta) - p \\ + R \cdot \frac{x_\theta}{G} - \Delta p, \end{aligned} \tag{8}$$

which corresponds to utility (4) normalized by the demand d. Since users are non-atomic, u_θ may also be interpreted as a "density of utility" at type $\theta \in \Theta$. We still assume that if no user contributes, then the reward is not given. However, if a set of users of measure zero contributes, then each contributing user gets an infinite reward distributed so that the integral with respect to the measure of users is R.

Marginal utility (5) becomes:

$$\frac{\partial u_\theta}{\partial x_\theta} = -h(G) - c'_\theta(x_\theta) + \frac{R}{G}. \tag{9}$$

A Nash equilibrium is defined as a function $x^{(\text{eq})}$ such that for all $\theta \in \Theta$, $u_\theta(x_\theta^{(\text{eq})}, x_{-\theta}^{(\text{eq})}) \geq u_\theta(a, x_{-\theta}^{(\text{eq})}), \forall a \in [0,1]$. It is equivalent to $x^{(\text{eq})}$ satisfying the FOCs

$$\frac{\partial u_\theta}{\partial x_\theta}(x_\theta, G) \quad \begin{cases} \leq 0, & \forall \theta \in \Theta : x_\theta = 0, \\ = 0, & \forall \theta \in \Theta : x_\theta \in (0,1), \\ \geq 0, & \forall \theta \in \Theta : x_\theta = 1, \end{cases} \tag{10}$$

and equation (7). At social optimum, the aggregate welfare

$$W = \int_\Theta u_\theta \mathrm{d}\mu(\theta) \tag{11}$$

is maximized.

IV. Analysis and convergence results

We first analyze the equilibrium and social optimum in the non-atomic setup (Sec. IV-A). Then, in Sec. IV-B, we analyze the equilibrium in the atomic setup and we show that it converges toward equilibrium of the non-atomic setup when the number of users tends to infinity.

A. Analysis of the non-atomic setup

Our first result establishes existence and uniqueness of the Nash equilibrium in the non-atomic setup.

Theorem 2. *For any $R \geq 0$, there exists a unique Nash equilibrium $x^{(eq)}(R)$ in the non-atomic setup.*

Proof: See Appendix A. ∎

The discontinuity in utilities at points where $G = 0$ necessitates some special treatment to prove the existence of a Nash equilibrium. Moreover, the uniqueness property is non-trivial and is derived from the specifics of our model.

The intuition behind the result of Theorem 2 is as follows. For a given value of G, each user of type $\theta \in \Theta$ chooses his best-response contribution $x_\theta^{(\text{resp})}(G) \in [0,1]$ to maximize his utility, *i.e.,* to solve the FOCs (10). Integrating the contribution of each type gives the aggregate best-response $G^{(\text{resp})}(G) = \int_\Theta x_\theta^{(\text{resp})}(G)\mathrm{d}\mu(\theta)$, *i.e.,* the amount of public good that users want to provide in response to a given G. At an equilibrium, both quantities are equal, which corresponds to solving the fixed-point equation

$$G^{(\text{resp})}(G) = G. \tag{12}$$

A key feature of our model is that the marginal utilities (9) are decreasing when G increases. Intuitively, if G is higher, users are less willing to contribute, both because they want to enjoy more the resource if it is less congested (the term $-h(G)$ in (9)) and because the reward per unit of contribution is lower (the term $\frac{R}{G}$ in (9)). Therefore, the aggregate best response $G^{(\text{resp})}(G)$ decreases when G increases. It is also a continuous function of G, due to the strict convexity of the cost functions. This leads to a unique fixed-point $G^{(\text{eq})}(R)$ for equation (12), and then to a unique Nash equilibrium $x^{(\text{eq})}(R)$.

Our next result concerns the social optimum.

Theorem 3. *There exists a function x^*, uniquely determined almost-everywhere, maximizing* (11)*.*

Moreover, $x^{(eq)}(R) = x^*$ almost-everywhere, and hence $G^{(eq)}(R) = G^*$, for $R = R^*$ where

$$R^* = G^* h'(G^*)(D - G^*). \tag{13}$$

Proof: See Appendix B. ∎

Intuitively, the result holds because the externality faced by a user $(-h(G) + \frac{R}{G})$ is independent of his type θ. Therefore, by fixing a reward that is also independent of the type θ, it is possible to achieve social optimum by making users effectively pay a Pigovian tax [6]. Then, Theorem 3 shows that by tuning the reward parameter R, the raffle-based scheme is able to enforce socially optimal contributions from every users at the Nash equilibrium.

The next two propositions further characterize the variations of the Nash equilibrium with the reward parameter R.

Proposition 1. *For any $R > 0$, we have $G^{(eq)}(R) > 0$.*

Proof: If $G = 0$, each user wants to contribute positively (the unit reward is infinite), hence it is not an equilibrium. ■

Note that this result is consistent with Theorem 3: if $G^* = 0$, then the only parameter value that permits to achieve social optimum at Nash equilibrium is $R^* = 0$.

Proposition 2. *For any $R' > R$, $G^{(eq)}(R') \geq G^{(eq)}(R)$; and the inequality is strict if $G^{(eq)}(R) \in (0, D)$.*

Proof: See Appendix C. ■

Proposition 2 finally formalizes the intuition that contributions increase with the reward R, due to the increase of the marginal utility.

B. Analysis of the atomic setup and convergence

In the previous section, we established existence and uniqueness of Nash equilibrium in the non-atomic setup of the decongestion problem model. We now show that there is also a unique Nash equilibrium in the atomic setup, which converges toward the equilibrium of the non-atomic setup.

We start with the existence and uniqueness of the Nash equilibrium for the atomic setup.

Theorem 4. *For any number of users $n \geq 2$, any sequence of types $(\theta_i)_{i \in N}$ and any $R \geq 0$, there exists a unique Nash equilibrium $x^{(eq,n)}(R)$ in the atomic setup.*

Proof: See Appendix D. ■

As for the non-atomic setup, the existence of a Nash equilibrium cannot be derived from continuity and concavity conditions (see *e.g.,* [7]) due to the discontinuity at the point $x = (0, \cdots, 0)$ where no user contributes. However, with our assumptions, as long as $n \geq 2$ (at least two users), this point cannot be a Nash equilibrium. Indeed if no user contributes, each user has a positive incentive to contribute a small fraction of his demand so as to get the entire reward R. Then, our proof goes through the formal steps to show the existence of a unique Nash equilibrium. The discontinuity at the origin created by the presence of the sum of contribution at the denominator also appears in different contexts, *e.g.,* [8], [9], where it is handled through different means such as the introduction of a reservation price in [9].

When the number of users is large, it becomes more appropriate to represent the population as a continuum of non-atomic users whose types follow distribution μ. To establish a link between the atomic and non-atomic setups, we assume that in the atomic setup, users have types that are random variables distributed according to μ and let the number of users n go to infinity. Formally, let $(\Omega, \mathcal{F}', \mathbb{P})$ be a probability space. Let $(\theta_i)_{i \geq 1}$ be a sequence of i.i.d. random variables defined on this probability space, taking values in Θ and distributed according to the probability measure $\mu(\cdot)/D$. For any integer n, the atomic setup corresponds to n users whose types are the first n values of the random sequence $(\theta_i)_{i \geq 1}$. Then we have the following result.

Theorem 5. *For any R, as n goes to infinity, we have $\mathbb{P}$-almost surely:*

(i) The equilibrium level of public good in the atomic setup converges toward the equilibrium level of public good in the non-atomic setup:

$$G^{(eq,n)} = \sum_{i \in N} x_i^{(eq,n)} \cdot d \xrightarrow[n \to \infty]{\mathbb{P}-a.s.} G^{(eq)};$$

(ii) The equilibrium contribution of each user in the atomic setup converges toward the equilibrium contribution of the corresponding user type in the non-atomic setup:

$$x_i^{(eq,n)} \xrightarrow[n \to \infty]{\mathbb{P}-a.s.} x_{\theta_i}^{(eq)}.$$

Proof: See Appendix E. ■

Theorem 5 establishes a link between the atomic and non-atomic setups: as n goes to infinity and the demand of each user becomes negligible compared to the total demand, the fraction that user i contributes to the public good at equilibrium in the atomic setup approaches the equilibrium contribution of the type $\theta = \theta_i$ in the non-atomic setup. This justifies the use in practical cases of the non-atomic setup which permits easier analyzes and simulations. Our convergence result is closely related to the results of Bodoh-Creed [10] for Bayesian-Nash equilibrium in games with semi-anonymous utilities (our model is a particular case of such games). Although it is different in spirit, our result is also related to [11], [12] in the context of routing games, where the number of user classes remains fixed and each user is replaced by a growing number of identical copies.

V. Comparison with Morgan's model

Finally, let us compare our results with Morgan's results [1]. The heterogeneity in our model is due to cost heterogeneity ($c_{\theta_i}(\cdot)$) rather than heterogeneity of user valuation of the public good ($h(\cdot)$), as in [1]. Also, our cost of contribution is strictly convex rather than linear as in [1]. However, the most important difference is that in our model, user valuation of the public good ($h(\cdot)$) is reduced by $(1 - x_i)$, reflective of reduced demand for the resource. Another modification of [1] is the financing of the incentive scheme. In [1], the reward is subtracted from the total user contribution, that is the amount available for public good financing is equal to total user contribution net of the reward R. In contrast, our raffle-based scheme is financed by an increase Δp of the subscription price.

The result of Theorem 2 partially contrasts with the Morgan's results (*(i)* of Theorem 1). In our model, the case $R = 0$ leads to a unique equilibrium whereas the case of voluntary contributions can lead to multiple equilibria in [1].

Indeed, in Morgan's model, the equilibrium level of public good could be achievable at multiple combinations of user contributions (when the sum of user contributions remains equal to equilibrium level). This multiplicity of equilibria is driven by the linear cost in [1] and does not occur in our model due the convexity of the cost of shifting.

The result of Theorem 3 differs from Morgan's result (*(ii)* and *(iii)* of Theorem 1) in two respects. First, in [1], only the level of public good is uniquely determined at social optimum, but it may be induced by multiple combinations of contributions. This is again due to the linear costs in [1]. Second, the socially optimal level of public good in our model is exactly achievable, whereas in [1], it can only be achieved in the limit of infinitely large reward. This difference is driven by the difference in the scheme's financing. The difference between the result of Proposition 1 and Morgan's result (*(iv)* of Theorem 1) is also driven by the difference in the scheme's financing.

Proposition 2 implies that our raffle-based scheme can lead to overprovision of the public good. Indeed, the equilibrium level of public good exceeds G^* (hence welfare is suboptimal) if the reward exceeds R^*. However, in reality, this is unlikely to occur because the same welfare could always be achieved with a reward lower than R^* (and public good level lower than G^*).

VI. Concluding remarks

In this paper, we developed a raffle-based scheme for congestion management, building on the economic literature on public good provision by means of lotteries. Our scheme can be viewed as probabilistic pricing: a user's reward depends not only on his contribution but also on the contribution of the other users. It is implementable via lottery-like mechanism.

The design of our scheme by a provider requires estimations of the user utilities to set the parameter to its optimal value. Imperfect information on user utilities may lead to overshooting of the parameter. However, the effect of this overshooting will be limited due to the close-loop mechanism inherent in our raffle-based scheme: the more users contribute, the smaller the incentive to contribute.

Our analysis is valid only if the price increase required by our scheme does not affect user participation. This clearly holds if each user has a sufficiently high utility before the price increase, so that even with an increased monthly price all users continue to buy the service. This also holds if we assume that users are heterogeneous on the timescale of a day but homogeneous on the timescale of a month; *e.g.,* each day, users have their types drawn independently according to a common stationary distribution. Then, for any R for which the raffle-based scheme is welfare improving (around R^*), each user utility over the timescale of a month is larger than it was before the imposition of the scheme.

Our scheme can be applied in various settings for demand management. An application to decongestion of peak-time in broadband access networks is proposed in [13]. A similar scheme could be applied to manage residential demand for electricity. In that case, privacy and security considerations make our scheme advantageous relative to real-time pricing. Indeed, our scheme requires no real-time user-dispatcher communication. In addition, unlike currently suggested real-time pricing schemes (*e.g.,* [14]), our scheme requires only aggregate data. Overall, this makes our raffle-based scheme an attractive tool to improve the allocation of a shared resource.

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Appendix A
Proof of Theorem 2

Let R be fixed. For any given demand $G \in [0, D]$ shifted by the population, the best response (solving the FOCs (10)) defines a measurable function $x^{(\text{resp})}(G) : \Theta \to [0, 1]$ given for all $\theta \in \Theta$ by

$$x_\theta^{(\text{resp})}(G)=\begin{cases} 0 \text{ if } \frac{R}{G} - h(G) - c'_\theta(0) \leq 0, \\ 1 \text{ if } \frac{R}{G} - h(G) - c'_\theta(1) \geq 0, \\ (c'_\theta)^{-1}\left(\frac{R}{G} - h(G)\right) \text{ otherwise.} \end{cases} \tag{14}$$

Due to the convexity assumption of $c_\theta(\cdot)$, $c'_\theta(\cdot)$ is strictly increasing, hence invertible and with an increasing inverse function. Therefore, (14) uniquely defines $x^{(\text{resp})}(G)$. Let

$$G^{(\text{resp})}(G) \quad = \quad \int_\Theta x_\theta^{(\text{resp})}(G) \mathrm{d}\mu(\theta) \tag{15}$$

be the aggregate best response, *i.e.,* the total demand that the population wants to shift in response to G. By definition and by strict concavity of the utility functions, a measurable function $x : \Theta \to [0,1]$ is a Nash equilibrium if and only if there exists $G \in [0, D]$ satisfying the fixed-point equation (12) such that $x = x^{(\text{resp})}(G)$.

To conclude the proof of Theorem 2, we show that (12) admits a unique fixed-point.

Lemma 1. *There exists a unique solution of* (12)*.*

Proof: The r.h.s. of (12) (G) is clearly a strictly increasing continuous function of G. It goes from 0 to D as G increases from 0 to D.

For the l.h.s. ($G^{(\text{resp})}(G)$), first note that it is a continuous function of G. Indeed, due to the convexity assumption of $c_\theta(\cdot)$, $(c'_\theta(\cdot))^{-1}(\cdot)$ is strictly increasing continuous, hence $x_\theta^{(\text{resp})}(G)$ is continuous in G for all $\theta \in \Theta$. Moreover, the function $x^{(\text{resp})}(G) : \Theta \to [0,1]$ is dominated by the constant function equal to 1 (*i.e.,* $|x^{(\text{resp})}(G)| \le 1$) which is integrable w.r.t. μ. Therefore, for any $G \in [0, D]$ and for any sequence $(G_n)_{n\ge 0}$ which converges to G, we have $x^{(\text{resp})}(G_n) \xrightarrow[n\to\infty]{} x^{(\text{resp})}(G)$ pointwise (by continuity of $x_\theta^{(\text{resp})}(G)$ w.r.t to G for all $\theta \in \Theta$) and by Lebesgue dominated convergence theorem,

$$\begin{aligned}\lim_{n\to\infty} G^{(\text{resp})}(G_n) &= \lim_{n\to\infty} \int_\Theta x_\theta^{(\text{resp})}(G_n)\mu(\mathrm{d}\theta) \\ &= \int_\Theta \left[\lim_{n\to\infty} x_\theta^{(\text{resp})}(G_n)\right]\mu(\mathrm{d}\theta) \\ &= G^{(\text{resp})}(G).\end{aligned}$$

Clearly, the l.h.s. ($G^{(\text{resp})}(G)$) is also a non-increasing function of G taking values in $[0, D]$. Therefore, there is a unique fixed-point of (12). ■

Appendix B
Proof of Theorem 3

A. Proof of existence and uniqueness of x^*

Let $\mathcal{X}$ be the set of functions $x : \Theta \to \mathbb{R}$ such that $\int_\Theta |x_\theta|\,\mu(\mathrm{d}\theta) < \infty$ and let $\mathcal{X}_0 \subset \mathcal{X}$ be the set of functions $x : \Theta \to [0,1]$. Consider the aggregate welfare (11) as a functional on $\mathcal{X}_0$ taking values in $\mathbb{R}$:

$$\begin{aligned}W(x) &= \left[h\left(\int_\Theta x_\theta\mu(\mathrm{d}\theta)\right)\right]\cdot\left(D - \int_\Theta x_\theta\mu(\mathrm{d}\theta)\right) \\ &\quad - \int_\Theta c_\theta(x_\theta)\mu(\mathrm{d}\theta) + \int_\Theta \bar{u}_\theta\mu(\mathrm{d}\theta) - pD.\end{aligned}$$

Since $\mathcal{X}_0$ is compact and the functional W is continuous, it has a maximum (see Corollary 38.10 of [15, p. 152]). Let $x^* \in \mathcal{X}_0$ be such that W is maximal and let

$$G^* = \int_\Theta x^*_\theta\mu(\mathrm{d}\theta).$$

Define the three subsets of Θ: Θ_1, Θ_2 and Θ_3 where $x^* = 0$, $x^* \in (0,1)$ and $x^* = 1$ respectively. We now derive necessary conditions for x^* to maximize W in each subset.

We start with the subset Θ_2 corresponding to interior points. Let $y \in \mathcal{X}$ be such that $y_\theta = 0$ for all $\theta \in \Theta\backslash\Theta_2$. We define the directional derivative (also called Gâteaux derivative) of W around x^* in the direction y as

$$\mathrm{d}W(x^*, y) = \lim_{t\to 0}\frac{W(x^* + ty) - W(x^*)}{t}.$$

Then, we have

$$\mathrm{d}W(x^*, y) = \int_{\Theta_2} y_\theta\cdot[h'(G^*)(D - G^*) - h(G^*) - c'_\theta(x^*_\theta)]\,\mathrm{d}\mu(\theta),$$

where the exchange between limit and integration in the last term (giving $-y_\theta c'_\theta(x_\theta)$) is justified by Lebesgue's dominated convergence theorem whenever $\int_\Theta |y_\theta \cdot c'_\theta(x_\theta)|\,\mu(\mathrm{d}\theta) < \infty$. This holds here due to the assumption that $c_\theta(\cdot)$ has a bounded slope.

For x^* to be optimal, it is necessary that $\mathrm{d}W(x^*, y) = 0$, *i.e.,*

$$\int_{\Theta_2} y_\theta\cdot[h'(G^*)(D - G^*) - h(G^*) - c'_\theta(x^*_\theta)]\,\mathrm{d}\mu(\theta) = 0.$$

For this to hold for any function y such that $y_\theta = 0$ for all $\theta \in \Theta\backslash\Theta_2$, it is necessary that we have

$$h'(G^*)(D - G^*) - h(G^*) - c'_\theta(x^*_\theta) = 0, \tag{16}$$

for almost-all $\theta \in \Theta_2$, *i.e.,* for almost-all θ such that $x^*_\theta \in (0,1)$.

We now treat the case of subset Θ_1, which corresponds to the points of the lower boundary. Let $y \in \mathcal{X}$ be such that $y_\theta \ge 0$ for all $\theta \in \Theta_1$ and $y_\theta = 0$ for all $\theta \in \Theta\backslash\Theta_1$; that is y is a direction that "pushes up" the values of x^* that are at zero. The directional derivative of W around x^* in the direction y is defined similarly to the previous case but with a limit $t > 0$:

$$\mathrm{d}W(x^*, y) = \lim_{t\to 0^+}\frac{W(x^* + ty) - W(x^*)}{t},$$

which gives

$$\mathrm{d}W(x^*, y) = \int_{\Theta_1} y_\theta\cdot[h'(G^*)(D - G^*) - h(G^*) - c'_\theta(x^*_\theta)]\,\mathrm{d}\mu(\theta).$$

Here, for x^* to be optimal, it is necessary that $\mathrm{d}W(x^*, y) \le 0$, that is

$$\int_{\Theta_1} y_\theta\cdot[h'(G^*)(D - G^*) - h(G^*) - c'_\theta(x^*_\theta)]\,\mathrm{d}\mu(\theta) \le 0.$$

For this to hold for any function y such that $y_\theta \ge 0$ for all $\theta \in \Theta_1$ and $y_\theta = 0$ for all $\theta \in \Theta\backslash\Theta_1$, it is necessary that

$$h'(G^*)(D - G^*) - h(G^*) - c'_\theta(x^*_\theta) \le 0, \tag{17}$$

for almost-all $\theta \in \Theta_1$, that is for almost all θ such that $x^*_\theta = 0$.

The case of subset Θ_3 is handled similarly and yields the necessary condition:

$$h'(G^*)(D - G^*) - h(G^*) - c'_\theta(x^*_\theta) \ge 0, \tag{18}$$

for almost all θ such that $x^*_\theta = 1$.

In summary, (16)-(18) show that for function x^* to maximize W, it is necessary that x^* is solution for almost all $\theta \in \Theta$ of the FOCs (10) where the unit reward $\frac{R}{G}$ is replaced by $h'(G)(D - G)$. By the assumption on $h(\cdot)$, this is a decreasing function of G. Therefore the same proof as for Theorem 2 applies to show that x^* is uniquely determined almost everywhere.

B. Proof of coincidence with Nash equilibrium

From the previous proof (App. B-A), it is clear that if $R = R^*$, then the FOCs (10) at a Nash equilibrium coincide with the optimality condition, which immediately brings the desired conclusion.

Appendix C
Proof of Proposition 2

Case 1: If $G^{(\mathrm{eq})}(R) = 0$, then the result is obvious.

Case 2: If $G^{(\mathrm{eq})}(R) = D$, then we have $\frac{R}{D} - c'_\theta(1) \geq 0$ for almost all $\theta \in \Theta$, which implies $\frac{R'}{D} - c'_\theta(1) \geq 0$. Hence $G^{(\mathrm{eq})}(R') = D$.

Case 3: If $G^{(\mathrm{eq})}(R) \in (0, D)$, for a given G, $x^{(\mathrm{resp})}(G)$ of (14) is non-decreasing when R increases to R', and strictly increasing for θ's s.t. $x^{(\mathrm{resp})}(G^{(\mathrm{eq})}(R)) \in (0, D)$. Since the set of such θ's is of positive measure, the new fixed-point has $G^{(\mathrm{eq})}(R') > G^{(\mathrm{eq})}(R)$.

Appendix D
Proof of Theorem 4

A. Existence

For $n = 1$, a Nash equilibrium may not exist. For example, if the user's utility without the raffle-based scheme is decreasing in his contribution, there is no equilibrium. Indeed, if the user has a positive contribution, he wants to reduce it (and still get the entire reward); whereas if he has a null contribution he wants to increase it to get the reward. Note however that the one-user case is not of interest to us since G is simply the user's contribution which is not a *public* good.

For $n \geq 2$, the utility is discontinuous at $x = (0, \cdots, 0)$, so that existence of an equilibrium cannot be derived from a standard existence theorem based on continuity assumptions (see *e.g.,* [7]). Clearly, $x = (0, \cdots, 0)$ is not a Nash equilibrium because any user can improve his utility by removing a small fraction of his demand. We now show that there exists a Nash equilibrium where at least one user shifts a positive fraction of his demand.

Let $\epsilon > 0$ and consider the convex closed bounded domain $\mathcal{D}_\epsilon = \left\{x \in [0,1]^n : \sum_{i \in N} x_i \geq \epsilon\right\}$. In $\mathcal{D}_\epsilon$, the utility (4) is continuous in x and concave in x_i for each fixed value of x_{-i}. Then, by Theorem 1 of Rosen [16], there exists an equilibrium point $x^{(\mathrm{eq},n)} \in \mathcal{D}_\epsilon$ such that for all $i \in N$:

$$U_i(x_i^{(\mathrm{eq},n)}, x_{-i}^{(\mathrm{eq},n)}) = \max_{x_i : (x_i, x_{-i}^{(\mathrm{eq},n)}) \in \mathcal{D}_\epsilon} U_i(x_i, x_{-i}^{(\mathrm{eq},n)}). \quad (19)$$

If ϵ is small enough and $n \geq 2$, this equilibrium point is not on the corner of $\mathcal{D}_\epsilon$, *i.e.,* $G^{(\mathrm{eq},n)} = \sum_{i \in N} x_i^{(\mathrm{eq},n)} \cdot d > \epsilon$. Indeed, suppose that we had an equilibrium with $G^{(\mathrm{eq},n)} = \epsilon$. Then, for at least one user $i \in N$ (*e.g.,* a user such that $x_i^{(\mathrm{eq},n)} \leq \epsilon/2$), the marginal utility of that user

$$\frac{\partial U_i}{\partial x_i} = d \cdot \Big\{ - h(\epsilon) + (1 - x_i^{(\mathrm{eq},n)}) d h'(\epsilon) - c'_{\theta_i}(x_i^{(\mathrm{eq},n)}) + R \frac{\epsilon - x_i^{(\mathrm{eq},n)} \cdot d}{\epsilon^2} \Big\}$$

is positive (for ϵ small enough); which contradicts the equilibrium property.

To conclude, we show that an equilibrium $x^{(\mathrm{eq},n)}$ of the concave n-person game defined by $\mathcal{D}_\epsilon$ for ϵ small enough so that $G^{(\mathrm{eq},n)} > \epsilon$ is also a Nash equilibrium of the initial game corresponding to the raffle-based scheme (with strategy space $[0,1]$ for each user). First for $i \in N$ such that $\sum_{j \in N \setminus i} x_j^{(\mathrm{eq},n)} \geq \epsilon$, we have $\{x_i : (x_i, x_{-i}^{(\mathrm{eq},n)}) \in \mathcal{D}_\epsilon\} = [0,1]$ so that (19) implies

$$U_i(x_i^{(\mathrm{eq},n)}, x_{-i}^{(\mathrm{eq},n)}) = \max_{x_i \in [0,1]} U_i(x_i, x_{-i}^{(\mathrm{eq},n)}). \quad (20)$$

Second for $i \in N$ such that $\sum_{j \in N \setminus i} x_j^{(\mathrm{eq},n)} < \epsilon$, since $G^{(\mathrm{eq},n)} > \epsilon$, $x_i^{(\mathrm{eq},n)}$ is not on the lower boundary of the domain $\{x_i : (x_i, x_{-i}^{(\mathrm{eq},n)}) \in \mathcal{D}_\epsilon\}$. Therefore, for (19) to be satisfied, we must have $\frac{\partial U_i}{\partial x_i}(x_i^{(\mathrm{eq},n)}, x_{-i}^{(\mathrm{eq},n)}) \leq 0$. Since for $x_{-i}^{(\mathrm{eq},n)}$ fixed, $U_i(x_i, x_{-i}^{(\mathrm{eq},n)})$ is a strictly concave function of x_i on $[0,1]$, this implies that (20) is satisfied. In summary, (20) is satisfied for all $i \in N$ which means that $x^{(\mathrm{eq},n)}$ is a Nash equilibrium.

B. Uniqueness

First recall that, as for the non-atomic setup, for a fixed G, the best responses are uniquely determined due to the strict concavity of the the utility (4) in x_i.

Contrarily to the non-atomic setup, the marginal utilities may not be decreasing in G due to the"raffle term" of (5): $R\frac{G - x_i \cdot d}{G^2}$, which is increasing for $G \in [x_i \cdot d, 2x_i \cdot d]$ and decreasing for $G > 2x_i \cdot d$. Suppose that two equilibria exist with contributions $x^{(1)}$ and $x^{(2)}$ levels of public good G_1 and G_2. Without loss of generality we assume that $G_1 < G_2$ and we distinguish the two possibles cases (which are exhaustive).

Case 1. Suppose that for all users $i \in N$, $x_i^{(1)} \leq \frac{G_1}{2}$.

Then, for all users $i \in N$ we have $\frac{\partial U_i}{\partial x_i}(x_i^{(1)}, G_2) \leq \frac{\partial U_i}{\partial x_i}(x_i^{(1)}, G_1)$, so that at the second equilibrium, we must have $x_i^{(2)} \leq x_i^{(1)}$ for all $i \in N$, hence $G_2 \leq G_1$; which is a contradiction.

Case 2. Suppose that for one user $i \in N$, $x_i^{(1)} > \frac{G_1}{2}$.

Then, for all other users $k \in N \setminus i$, we have $x_k^{(1)} \leq \frac{G_1}{2}$ and therefore $x_k^{(2)} \leq x_k^{(1)}$. If $x_i^{(1)} = 1$, then $G_2 \leq G_1$ which is a contradiction. If $x_i^{(1)} < 1$, then we have

$$\frac{\partial U_i}{\partial x_i}(x_i^{(1)}, G_1) = d \cdot \Big\{ - h(G_1) + (1 - x_i^{(1)}) h'(G_1) - c'_{\theta_i}(x_i^{(1)}) + R \frac{\sum_{k \in N \setminus i} x_k^{(1)} \cdot d}{G_1^2} \Big\} \leq 0,$$

which implies $\frac{\partial U_i}{\partial x_i}(x_i^{(1)}, G_2) < \frac{\partial U_i}{\partial x_i}(x_i^{(1)}, G_1) \leq 0$. Therefore, we must have $x_i^{(2)} \leq x_i^{(1)}$ which leads to $G_2 \leq G_1$, a contradiction.

Appendix E
Proof of Theorem 5

A. Proof of (i)

For a given $G \in [0, D]$, the individual best response for the non-atomic setup is defined by (14). For the atomic setup with

n players, it is defined by

$$x_i^{(\text{resp},n)}(G) = \begin{cases} 0, \text{ if } g_{\theta_i,G}^{(n)}(0) \leq 0, \\ 1, \text{ if } g_{\theta_i,G}^{(n)}(1) \geq 0, \\ \left(g_{\theta_i,G}^{(n)}\right)^{-1}(0), \text{ otherwise;} \end{cases}$$

where for all $\theta \in \Theta$, function $g_{\theta,G}^{(n)}$ is defined for all $x \in [0,1]$ by

$$g_{\theta,G}^{(n)}(x) = -h(G) + (1-x)dh'(G) - c'_\theta(x) + R \cdot \frac{G - x \cdot d}{G^2}.$$

Note that for all $G \in [0, D]$, $x_i^{(\text{resp},n)}(G)$ is well defined since by our assumptions on $h(\cdot)$ and $c_\theta(\cdot)$, $g_{\theta,G}^{(n)}(\cdot)$ is strictly decreasing continuous.

Let $G^{(\text{eq})}$ be the equilibrium level of public good in the non-atomic setup. Recall that, by Proposition 1, we have $G^{(\text{eq})} > 0$. Let $\eta \in (0, G^{(\text{eq})})$. Recall that $d = D/n$ where D is a constant. We now provide the proof of Theorem 5 in three steps. In the first step, we prove the uniform convergence of the atomic setup's response function, $x_i^{(\text{resp},n)}(\cdot)$, to the non-atomic setup's response function, $x_{\theta_i}^{(\text{resp})}(\cdot)$. The second step establishes the almost sure convergence of the aggregate best responses $G^{(\text{resp},n)}(\cdot) = \sum_{i \in N} x_i^{(\text{resp},n)}(\cdot)$ of the population of n users sampled from μ/D to the aggregate best response $G^{(\text{resp})}(\cdot)$. Finally, in the third step, we prove that at equilibrium, the level of public good for the sampled population, $G^{(eq,n)}$, almost surely converges to $G^{(eq)}$ as $n \to \infty$.

Step 1. First we show that for any integer i, we have $x_i^{(\text{resp},n)}(\cdot) \xrightarrow[n\to\infty]{} x_{\theta_i}^{(\text{resp})}(\cdot)$, uniformly on $[\eta, D]$ and uniformly in θ. From the definition of $x_i^{(\text{resp},n)}$ and $x_{\theta_i}^{(\text{resp})}$, we have, for all $G \in [\eta, D]$,

$$\begin{aligned} &\left| c'_{\theta_i}\left(x_i^{(\text{resp},n)}(G)\right) - c'_{\theta_i}\left(x_{\theta_i}^{(\text{resp})}(G)\right)\right| \\ &\quad \leq \left| \left(1 - x_i^{(\text{resp},n)}(G)\right) \cdot h'(G) - R \cdot \frac{x_i^{(\text{resp},n)}(G)}{G^2}\right| \cdot \frac{D}{n}, \\ &\quad \leq \left| h'(\eta) + \frac{R}{\eta^2}\right| \cdot \frac{D}{n}. \end{aligned}$$

Due to strict convexity, $c'_\theta(\cdot)$ is invertible and due to the assumption that $c_\theta(\cdot)$ has a slope bounded away from 0, $(c'_\theta)^{-1}(\cdot)$ is Lipschitz continuous with a factor independent of θ. We conclude that for all $G \in [\eta, D]$ and all type $\theta_i \in \Theta$,

$$\left| x_i^{(\text{resp},n)}(G) - x_{\theta_i}^{(\text{resp})}(G)\right| \leq K \cdot \frac{1}{n}, \tag{21}$$

where K is a constant independent of G and θ_i.

Step 2. Next, we show that $G^{(\text{resp},n)}(\cdot) \xrightarrow[n\to\infty]{a.s.} G^{(\text{resp})}(\cdot)$, uniformly on $[\eta, D]$.

For all $G \in [\eta, D]$, we have

$$\begin{aligned} &\left| G^{(\text{resp},n)}(G) - G^{(\text{resp})}(G)\right| \\ &\quad \leq \frac{D}{n} \cdot \left| \sum_{i\in N} \left(x_i^{(\text{resp},n)}(G) - x_{\theta_i}^{(\text{resp})}(G)\right)\right| \\ &\qquad + \frac{D}{n} \cdot \left| \sum_{i \in N}\left(x_{\theta_i}^{(\text{resp})}(G) - \mathbb{E}_{\frac{\mu}{D}} x_\theta^{(\text{resp})}(G)\right)\right|. \end{aligned} \tag{22}$$

From (21), it is clear that the first part converges towards zero, uniformly (in G) on $[\eta, D]$. For the second part, consider the set of functions $\mathcal{FU} = \{f : \Theta \to [0,1] \text{ s. t. } f(\cdot) = x_\cdot^{(\text{resp})}(G) \text{ for some } G \in [\eta, D]\}$. By application of a Glivenko-Cantelli theorem (*e.g.,* Theorem 1 in [17, p. 837] whose conditions can be verified with the result (21) of step 1), we obtain

$$\sup_{f \in \mathcal{FU}} \left| \frac{1}{n} \sum_{i\in N} f(\theta_i) - \mathbb{E}_{\frac{\mu}{D}} f(\theta)\right| \xrightarrow[n\to\infty]{a.s.} 0.$$

This exactly means that the second part of (22) converges almost surely towards zero, uniformly (in G) on $[\eta, D]$.

Step 3. We finally show that $G^{(\text{eq},n)} \xrightarrow[n\to\infty]{a.s.} G^{(\text{eq})}$.

We work on a set of $\mathbb{P}$-measure 1 where $G^{(\text{resp},n)}(\cdot)$ converges uniformly towards $G^{(\text{resp})}(\cdot)$ on $[\eta, D]$ (see step 2). Recall that by Theorems 2 and 4, $G^{(\text{eq},n)}$ and $G^{(\text{eq})}$ are uniquely determined if $n \geq 2$. The uniform convergence property implies that

$$\begin{aligned} &\left| G^{(\text{resp},n)}(G^{(\text{eq},n)}) - G^{(\text{resp})}(G^{(\text{eq},n)})\right| \\ &\qquad + \left| G^{(\text{resp},n)}(G^{(\text{eq})}) - G^{(\text{resp})}(G^{(\text{eq})})\right| \xrightarrow[n\to\infty]{} 0, \end{aligned}$$

that is, by definition of $G^{(\text{eq},n)}$ and $G^{(\text{eq})}$,

$$\begin{aligned} &\left| G^{(\text{resp})}(G^{(\text{eq},n)}) - G^{(\text{eq},n)}\right| \\ &\qquad + \left| G^{(\text{resp},n)}(G^{(\text{eq})}) - G^{(\text{eq})}\right| \xrightarrow[n\to\infty]{} 0. \end{aligned} \tag{23}$$

Let $\phi : [\eta, D] \to \mathbb{R}$ be the strictly decreasing function such that for all $G \in [\eta, D]$, $\phi(G) = G^{(\text{resp})}(G) - G$. Suppose that $G^{(\text{eq},n)}$ does not converge toward $G^{(\text{eq})}$. Then there exists $\epsilon > 0$ and subsequence $\{n_j\}$ such that $\left|G^{(\text{eq},n_j)} - G^{(\text{eq})}\right| \geq \epsilon \;\forall j$. Therefore, $\left|\phi(G^{(\text{eq},n_j)})\right| \geq \delta > 0$ where $\delta = \min\left\{\phi\left(G^{(\text{eq})} + \epsilon\right), \phi\left(G^{(\text{eq})} - \epsilon\right)\right\}$ if $G^{(\text{eq})} \leq D - \epsilon$ and $\delta = \phi(D - \epsilon)$ if $G^{(\text{eq})} > D - \epsilon$. Hence, there exists $\epsilon > 0$ and subsequence $\{n_j\}$ such that $\left|G^{(\text{resp})}(G^{(\text{eq},n_j)}) - G^{(\text{eq},n_j)}\right| \geq \delta > 0 \quad \forall j$ which contradicts (23).

B. Proof of (ii)

We have

$$\begin{aligned} \left| x_i^{(\text{eq},n)} - x_{\theta_i}^{(\text{eq})}\right| &\leq \left| x_i^{(\text{resp},n)}(G^{(\text{eq},n)}) - x_{\theta_i}^{(\text{resp})}(G^{(\text{eq},n)})\right| \\ &\quad + \left| x_{\theta_i}^{(\text{resp})}(G^{(\text{eq},n)}) - x_{\theta_i}^{(\text{resp})}(G^{(\text{eq})})\right|. \end{aligned}$$

From (21), the first part goes to zero and from step 3 and the continuity of $x_{\theta_i}^{(\text{resp})}$, the second part goes to zero.