N-Player Cournot and Price-Quantity Function Mixed Competition

Shuang Wu and John Musacchio

Abstract— **We study the value of network providers committing to offering a quantity of bandwidth to a market versus having the amount of bandwidth offered be conditional on the prices that the market settles upon. For instance a cable television ISP has the option to shift capacity from Internet service to television channels if the market price for Internet service is low, and thus such a provider can avoid committing to a fixed capacity being devoted to Internet service. To study the issue, we consider a two-stage game. In the first stage, competing network providers either commit to set a quantity of bandwidth to offer to the market, or choose to offer bandwidth to the market according to a function relating price to quantity. If they choose the later option, the network provider initially chooses a slope for their function. In the second stage, the quantity players choose the quantity to offer, where as the function players choose the offset term of their function. We show that a unique Nash equilibrium exists for the second stage play, and that it is the only subgame-perfect equilibrium for each provider to choose a quantity commitment in the first stage. We also show that a quantity commitment is not always a subgame-perfect equilibrium when demand uncertainty is introduced to the model.**

I. INTRODUCTION

When an Internet service provider enters a market, that provider has to make irreversible investments in capacity. These investments could include laying new fibre or coaxial cable, deploying routers, or investing in DSL acccess multiplexers (DSLAM)s. Some of these investments in capacity have the potential to be used to provide services other than Internet access. For instance a cable television provider that lays fiber optic cable to head end stations in neighborhoods can decide after making that investment how much of the capacity to use for providing Internet services, while the remainder can be used for television service. Other investments are more Internet service specific, such as routers and DSLAMs.

The greater the degree to which an investment is specific to providing Internet service, the greater an ISP making that investment is committing to offer a certain quantity to the market. Conversely, an ISP that makes investments whose capacity can be shifted to other services is making less of a commitment to quantity. Such an ISP has the option to move capacity into the market for Internet service if prices are high in that market, and it has the option to withdraw that capacity when prices are low. The motivation of this paper is to understand the strategic effect of committing to offer a quantity of bandwidth to the market versus signaling to the market that the quantity offered will depend on the prevailing prices in the market.

We develop a simple model to capture this effect, and show that when the demand functions are known, there is an advantage to committing to a quantity as much as possible, as making this commitment affects the actions of the other players. Intuitively, a provider signaling to the market that one will sell a given quantity no matter what makes it harder for a competitor to take its customers by undercutting that provider's price. However, when demand functions are uncertain, the value of commitment is outweighed by the value of the flexibility afforded by being able to reduce quantity offered when prices are low.

We start with a discussion of the literature, including background on the basic classical models of oligopoly competition as well as discussion of related work on supply function competition in energy markets.

II. BACKGROUND

The most important classical models of oligopoly are those of Cournot and Bertrand. In the Cournot model, firms simultaneously choose the quantities they will produce, which they then sell at the market-clearing price, and a Nash equilibrium is determined by the condition that no firm can improve profit by unilaterally changing its quantity. In the Bertrand model, firms simultaneously choose prices and must produce enough output to meet demand after the choices of price become known, and a Nash equilibrium is determined by the condition that no firm can improve profit by unilaterally changing its price. Bertrand and Cournot equilibria yield different outcomes even though there exists a one-to-one correspondence between prices and quantities. Under certain assumptions, Cournot competition can achieve higher prices, lower quantities and higher profits than Bertrand competition. The Cournot and Bertrand model provide equivalent one-stage reductions of complex models. Kreps and Scheinkman [1] considers a two-stage duopoly game where players simultaneously choose capacity, and, later compete on price. They concludes that the unique equilibrium outcome is the Cournot outcome under mild assumptions.

There is a larger class of work comparing Cournot competition with Bertrand competition. Singh and Vives [2] show that in a differentiated duopoly, for each firm, Cournot prices are larger than Bertrand prices if the goods are substitutes; Cournot quantities are smaller than Bertrand quantities if the goods are complements; Vives [3] shows that if the demand structure is symmetric and both Bertrand and Cournot equilibria are unique, then prices and profits are larger and quantities are smaller in Cournot competition than

Research supported in part by NSF grant CNS-0626397.

J. Musacchio and S. Wu are with the Technology and Information Management Program, University of California, Santa Cruz, Santa Cruz, CA 95064, USA {johnm,swu}@soe.ucsc.com

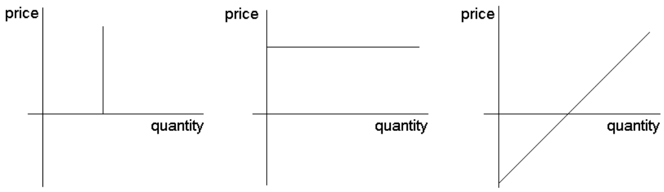

Fig. 1. Cournot, Bertrand and Price-Quantity Function strategy.

in Bertrand competition under certain assumptions. Amir and Jin [4] consider an oligopoly model with linear demand, and a mixture of substitute and complementary products. They provide counter-examples to show that no clear-cut comparison of prices and quantities between Cournot and Bertrand is possible without strategic complementarity in either of Cournot and Bertrand games. (Strategic complementarity of the Bertrand (Cournot) game holds if the cross partial derivative of any firm's profit function with respect to its own price (output) and to the price (output) of any other firm is nonnegative.)

In reality, it is also possible that Cournot and Bertrand firms coexist in the same industry. Bylka and Komar [5] compute Cournot-Bertrand mixed duopoly equilibrium prices in the case of linear demand and cost functions. Singh and Vives [2] show that in a differentiated duopoly, for each firm to choose the quantity (price) strategy is a subgame-perfect equilibrium if the goods are substitutes (complements). Cheng [6] reaches the same result by a geometric approach. Okuguchi [7] compares the equilibrium prices for the Bertrand and Cournot oligopolies with product differentiation. He shows that if all firms have linear demand and cost functions, and if, in addition, the Jacobian matrix of the demand functions has a dominant negative diagonal, the Bertrand equilibrium prices are not higher than the Cournot ones. Sato [8] computes equilibrium prices on Cournot-Bertrand mixed duopolies, and compare them with those of pure Cournot and Bertrand duopolies. Qin and Stuart [9] identify Nash equilibria in a generalized model in which firms choose among Cournot and Bertrand strategies. They show that best responses always exist. For the duopoly case, they show that iterated best responses converge under mild assumptions on initial states either to Cournot equilibrium or to an equilibrium where only one firm plays the Bertrand strategy with price equal to marginal cost and zero sales. Tanaka [10] examines a subgame-perfect equilibrium of a two-stage game where multiple firms produce substitutable goods. In the first stage, the firms choose their strategic variable, price or quantity and in the second stage, they choose the magnitudes of their strategic variables. It shows that a quantity strategy is the best response for each firm when all other firms choose a price strategy and thus the Bertrand equilibrium does not constitute a subgame-perfect equilibrium of the two stage game; a quantity strategy is also the best response for each firm when all other firms choose a quantity strategy and thus the Cournot equilibrium constitutes a subgame-perfect equilibrium. Tanaka reached the result by examining the necessary condition of a Nash equilibrium (the first order condition). Tasnádi [11] studies a similar two-stage game and shows that Cournot equilibrium constitutes a subgame-perfect equilibrium. He assumes that there are n oligopolists with zero fixed costs and constant marginal costs up to an identical capacity constraint $k > 0$.

In contrast to the previous literature, we suppose that players' strategy can be an affine price-quantity function. We formulate the game in the following way. Each player first chooses a slope $t_j \in [0, \infty)$ and later picks an offset z_j where $p_j = t_j q_j + z_j$. Note that a Bertrand strategy is equivalent to $(t_j = 0, z_j > 0)$ and a Cournot strategy is approximately equivalent to $(t_j \to +\infty, z_j \to -t_j q_j)$ where q_j is quantity. The strategies are illustrated in Figure 1.

In oligopoly games, players could make strategic precommitments to alter the conditions of future competition. Examples of strategic precommitments can be investments in capacity (e.g., [12] & [1]), cost reduction (e.g., [13]), price-quantity contract (e.g., [14]), and new product (technology) development (e.g., [15]). A strategic precommitment can have different effects in different models. For example, a reduction in marginal cost might be beneficial in quantity competition and detrimental in price competition. The effect of precommitment can be studied by means of stage games. We are interested in simple-structure models, which allow us to identify those mechanisms that are important in determining the gain from precommitment.

There are many types of strategic variables by which precommitments can be made. In a mature market where each firm has excess capacity, price competition is more descriptive and in a new industry where firms invest in capacity, quantity competition is more likely to occur. We are interested in games where strategic variable is price-quantity function. Klemperer and Meyer [16] mentions that commitment to a price-quantity function may be accomplished in practice by the choice of the firm's size and structure, its culture and values, and the incentive systems and decision rules for its employees. It can also accomplished in practice by rules or contracts.

An example of price-quantity function competition is supply function competition in the electricity market. In a number of electricity markets, such as England and Wales, New Zealand, Australia, the Power Exchange in California, and the Pennsylvania-New Jersey-Maryland (PJM) interconnection, the grid operator requires suppliers to offer a price schedule that applies throughout a period, rather than simply put forth a series of quantity bids over the period. This requirement explicitly represents an obligation to bid consistently over an extended time horizon. After submitting their supply schedule of prices, suppliers receive the market-clearing price p, which varies with demand. Green [17] presents the following supply function equilibrium:

Suppose that each electricity supplier i is required to submit a supply function with the linear form $q_i = \beta_i p$ and the production cost function is $C_i = \frac{1}{2} c_i q_i^2$. Each supplier's *profit is thus defined by $pq_i - C_i$. In Nash equilibrium,*

$$\frac{1}{\beta_i} = c_i + (-\frac{dD}{dp} + \sum_{j \neq i} \beta_j)^{-1}, \ D(\cdot)\text{: demand function.} \quad (1)$$

Baldick [18] assumes that $D = N - \gamma p$, $q_i = \beta_i(p - \alpha_i)$ and $C_i = \frac{1}{2}c_i q_i^2 + a_i q_i$. He finds one Nash equilibrium where $\alpha_i = a_i$ and β_i satisfies equation (1). Liu [19] claims that if $\alpha_i = a_i$ is fixed and each player i's strategy is the function $p = \frac{1}{\beta_i}q_i + \alpha_i$, then a unique Nash equilibrium characterized by (1) exists and stable. Note that if we define $p_i = p - \frac{1}{2}c_i q_i - a_i$, Green, Baldick and Liu's supply function is then equivalent to $p_i = (\frac{1}{\beta_i} - \frac{1}{2}c_i)q_i = t_i q_i + 0$. Baldick [20] shows that when the strategic variables α_i and β_i are chosen simultaneously, there are indeed multiple equilibria. The existence of multiple Nash equilibria is very problematic from the perspective of providing a convincing prediction of market outcome. If a focal equilibrium (a equilibrium preferred by every player) happens to exist, then the equilibrium may actually prevail. Baldick [20] describes four alternative specifications of the strategic variables: α-parametrization, β-parametrization with $\alpha = 0$, $(\alpha \propto \beta)$-parametrization and (α, β)-parametrization. By studying numerical examples with two players, he concludes that the profits in the Cournot equilibrium are the same as the limiting cases of the equilibrium for small β_i and β_j and such a limiting equilibrium is a focal equilibrium under (α, β)-parametrization. Baldick [20] also provide numerical examples with quantity constraint. In his example, there is no pure strategy equilibrium under $(\alpha \propto \beta)$-parametrization, there are pure strategy equilibria under α-parametrization (and thus (α, β)-parametrization) and there is a pure strategy Cournot equilibrium such that the transmission constraint is not binding. We focus on the problem of parametrization of price-quantity function, which is critical to the results. Our two stage game is similar to first choosing β and later choosing α. We provide an explicit mathematical proof on the existence and uniqueness of Nash equilibrium of the second stage subgame. We also identify a subgame-perfect equilibrium and a monotone changing relationship between player i's profit (& quantity) and the pre-chosen value of β_i.

We show that in the absence of uncertainty each player has an incentive to choose setting quantity or setting a price-quantity function with a infinite slope. We also show by example that when demand is uncertain, an infinite slope is not the best strategy.

The rest of the paper is organized as follows. Section 3 defines the two-stage game and presents main results. We first show the existence and uniqueness of pure-strategy Nash equilibrium of the competition in the second stage. Later, we consider subgame-perfect equilibrium. Section 4 discuss the effect of demand uncertainty. Section 5 contains conclusions.

III. The Two-Stage Game

Suppose that there are $N \geq 2$ players that compete. In the first stage of the two-stage game, all players simultaneously choose actions from their choice sets. We suppose that the choice set of each player includes two actions: committing to set a quantity or committing to set a price-quantity function. If a player i commits to set a quantity, the player has to supply a quantity q_i, which will be announced at the beginning of the second stage. If a player j commits to set a price-quantity function, the player must also announce a slope $t_j \geq 0$ in the first stage and then has to maintain the following relationship between her price p_j and quantity q_j

$$p_j = t_j q_j + z_j \quad (2)$$

where z_j is a real number that will be announced at the beginning of the second stage. Thus by announcing a t_j in the first stage, a player commits to raise his price as his quantity sold increases with a "rate" of t_j. As we explain in more detail later in the section, our model has two interpretations. In the first interpretation, p_n is the price users see, in a market where users experience congestion effects. In the second interpretation, p_n is the difference of the market price p and player n's average cost of per unit of product (or bit rate supplies). We suppose that there is a prohibitively high cost associated with breaking commitment, so breaking a commitment is not considered as a strategic option. At the end of the first stage, all players observe the stage's action profile. In the second stage, available actions in each player's choice set depend on what has happened previously. All players simultaneously choose actions from their choice sets. For a player i who has committed to set a quantity, the available action is picking a quantity q_i. For a player j who has committed to set a price-quantity function, the available action is picking an "offset" z_j. Each player n seeks to maximize individual profit $\pi_n = p_n q_n$. Since the profits depend on other players' actions, we assume that each player acts according to the Nash equilibrium notion in the second stage. In Nash equilibrium, a quantity-setting player i has no incentive to deviate from her current strategy q_i; a function-setting player j has no incentive to deviate from her current strategy z_j. We denote the strategy of player n in the second stage by s_n. Thus for a quantity setting player i, $s_i = q_i$ and for a function setting player j, $s_j = z_j$. We denote the vector of strategies in the second stage by $\mathbf{s} = [s_1, s_2, ..., s_N]$. More formally, we say a vector of strategies $\mathbf{s}$ is a Nash equilibrium profile if for each n,

$$\pi_i(s_n, s_{-n}) \geq \pi_n(\bar{s}_n, s_{-n}), \quad \forall \ \bar{s}_n \geq 0.$$

We denote by $\mathcal{C}$ the set of players that commit to set a quantity in the first stage. ($\mathcal{C}$ is a mnemonic for Cournot.) Similarly, we denote by $\mathcal{F}$ the set of players that commit to set a price-quantity function.

Our model for how consumers choose between providers has two interpretations, even though the mathematics of each interpretation are identical.

A. Model Interpretation 1

In the first interpretation, each competing provider has a congestion effect, such as latency on his network, that grows linearly with quantity (bit rate) consumed. The latency on provider n's network is $a_n q_n$, where a_n is a positive constant. Consumers then choose which network to connect

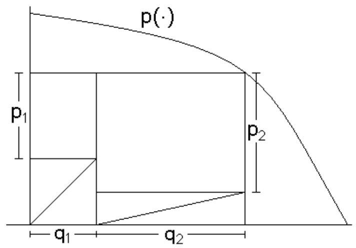

Fig. 2. Two-player competition in the second stage. The lines illustrate the latency functions of each player. The concave down function is the inverse demand (disutility) function. Note that for all players that carry flow, the sum of latency and price have the same value.

to on the basis of which network has the lowest sum of price and latency, which we call disutility. We suppose that users are nonatomic, meaning that each individual user's contribution to latency is negligible. Therefore, the users reach an equilibrium across providers such that each provider that carries positive flow has the same disutility. This is known as a Wardrop equilibrium. Demand is elastic in that as the best available disutility increases, the fraction of users that are willing to tolerate that disutility decreases. The function $p(q)$ is the disutility enough users will tolerate (be willing to connect) in order to have a total quantity consumed (traffic flow) of q, where $q \triangleq \sum_n q_n$. The function p: $[0,\infty) \mapsto [0,\infty)$ is decreasing, continuously differentiable and concave on the interval where it has positive value, satisfies $p(0) > 0$, and there exists some $\bar{q} > 0$ such that $p(q) = 0$ for all $q \geq \bar{q}$. Thus for a given second stage strategy profile $\mathbf{s}$, a quantity setting player i will be able to extract a price p_i from the market given by the following relation

$$p_i = p(q) - a_i q_i, \quad i \in \mathcal{C}. \tag{3}$$

For players that chose the function strategy, their price increases as more quantity is sold. In equilibrium, the quantity such a player sells will depend on the slope t_j and offset z_j they chose, the slope of their latency function a_j, and $p(q)$ the disutility that all the used paths share. The dependence has the relation

$$q_j = \frac{(p(q) - z_j)_+}{t_j + a_j}, \quad j \in \mathcal{F}. \tag{4}$$

An instance of how these relations can be thought of geometrically is illustrated in Figure 2.

B. Interpretation 2

In the second interpretation of our model, there is not any latency that the users see directly. Instead, providers have a quadratic cost function, $C(q_n) = a_n q_n^2$. Consumers choose which provider to buy from solely based on the end consumer price. To clear a total quantity of q the price must be $p(q)$ where $p(q)$ is a concave decreasing functions with the same properties as what we assumed in interpretation 1. Therefore $p(q)$ is an inverse demand function (it relates quantity to price rather than the other way around.) The average cost for each unit of quantity sold is $a_n q_n$, and therefore if we define p_n to be the average margin on each unit of quantity sold, then for a quantity setting player i, p_i is determined by relationship (3).

For function setting players, we again suppose that p_j is an average margin rather than the price the consumer sees. The consumer sees a total price of $p_j + a_j q_j$, where $a_j q_j$ is the providers average cost per unit, and chooses which provider to use based on that quantity. Therefore all providers that succeed in selling a nonzero quantity must have the same value of $p_j + a_j q_j$. These facts make it so that expression (4) described the quantity sold of a function setting player, just as it did in interpretation 1 of the model.

To analyze the two-stage game, we first identify the Nash equilibrium conditions for the second stage subgame, assuming that the first stage play is given. We then use these results to analyze the decisions in the first stage. This is the standard backwards induction approach in game theory [21].

We will apply Kakutani's fixed point theorem to prove the existence of pure Equilibria of the second stage subgame. Before that, we need a sequence of lemmas to show the following: each players profit is continuous in $\mathbf{s}$; each player's best response is in a non-empty, compact and convex region; the set-value function has a closed graph; the function value set is non-empty and convex. A necessary condition of the Nash equilibria is derived to support the uniqueness of Nash equilibrium and subgame-perfect equilibrium.

C. Existence of Pure Equilibria in the Second Stage

Given the group $\mathcal{C}$ of quantity-setting players, the group $\mathcal{F}$ of function-setting players and the vector $[t_j]_{j\in\mathcal{F}}$, a simultaneous move game among the players takes place in the second stage. We are interested in finding a Nash equilibrium for the payoffs prevailing under the action profile of the first stage. We first show that a pure Nash equilibria of the simultaneous move game in the second stage exists.

Lemma 1: Given a strategy vector $\mathbf{s}$, the inverse demand p is determined uniquely. Also, each player's price and quantity are determined uniquely.

Proof: For convenience, let

$$h(q;\mathbf{s}) \triangleq q - [\sum_{i\in\mathcal{C}} q_i + \sum_{j\in\mathcal{F}} \frac{(p(q) - z_j)_+}{a_j + t_j}].$$

From $q = \sum_n q_n$ and (4), we have $h(q;\mathbf{s}) = 0$. We next show that given $\mathbf{s}$, q has a unique value satisfying $h(q;\mathbf{s}) = 0$. Since $p(q)$ is non-increasing with q,

$$\sum_{j\in\mathcal{F}} \frac{(p(q) - z_j)_+}{a_j + t_j}$$

is also non-increasing with q. Thus $h(q;\mathbf{s})$ is increasing with q. Obviously $h(0;\mathbf{s}) \leq 0$, and $h(q;\mathbf{s}) \to +\infty$ as $q \to +\infty$. Since $h(q;\mathbf{s})$ is continuous and increasing with q, there must be a *unique* non-negative value of q such that $h(q;\mathbf{s}) = 0$. Consequently, the inverse demand p and each player's price and quantity are determined uniquely. ■

In Lemma 2 and Lemma 3, we show the consequences of a unilateral change of strategy.

Lemma 2: Let $[p_1, ..., p_N]$ be a price vector with corresponding quantity vector $[q_1, ..., q_N]$. Suppose that $n \in \mathcal{C}$ and player n decreases q_n to $q'_n < q_n$ while the other players keep their strategies unchanged. Denote the new quantity vector by $[q'_1, ..., q'_N]$ and the new price vector by $[p'_1, ..., p'_N]$. Then

$$p'_n > p_n; \quad (5)$$
$$p'_i \geq p_i, i \neq n, i \in \mathcal{C}; \quad (6)$$
$$q'_j \geq q_j, j \in \mathcal{F}; \quad (7)$$
$$q' \leq q, \text{ where } q' = \sum q'_n. \quad (8)$$

If player 1 increases her quantity, the symmetric claim holds.

Proof: Relation (5) follows from the following reasoning. Suppose $p'_n \leq p_n$, then the inverse demand must decrease. For a function-setting player j, z_j does not change and thus the quantity must not increase. For any quantity-setting player other than player 1 the quantity does not change. Therefore decrease of total quantity contradicts decrease of inverse demand.

To arrive at (6) consider the following. Suppose a quantity-setting player $i \neq n$ has a $p'_i < p_i$. Since q_i is unchanged, the inverse demand must decrease. For a function-setting player j, z_j does not change and therefore the quantity must not increase. For any quantity-setting player other than player 1, the quantity does not change. Therefore decrease of total quantity contradicts decrease of inverse demand.

Relation (7) follows from the following reasoning. Suppose a function-setting player j has a $q'_j < q_j$. Since z_j is unchanged, the inverse demand must decrease. For any function-setting player other than j, the quantity must not increase. For any quantity-setting player other than player 1, the quantity does not change. Therefore decrease of total quantity contradicts decrease of inverse demand.

To arrive at (8) consider the following. Suppose that $q' > q$. Since player 1 has decreased her quantity, and other quantity-setting players, if any, do not change their quantities, then there must be a function-setting player j whose quantity is increased. Therefore the inverse demand must increase, which contradicts to $q' > q$. ∎

Lemma 3: Let $[p_1, ..., p_N]$ be a price vector with corresponding quantity vector $[q_1, ..., q_N]$. Suppose that $n \in \mathcal{F}$ and player n increases z_n to $z'_n > z_n$ while the other players keep their strategies unchanged. Denote the new quantity vector by $[q'_1, ..., q'_N]$ and the new price vector by $[p'_1, ..., p'_N]$. Then

$$q'_n \leq q_n; \quad (9)$$
$$q'_j \geq q_j, j \neq n, j \in \mathcal{F}; \quad (10)$$
$$p'_i \geq p_i, i \in \mathcal{C}; \quad (11)$$
$$q' \leq q, \text{ where } q' = \sum q'_n. \quad (12)$$

The symmetric claim holds if player 2 decreases z_n.

Proof: Relation (9) follows from the following reasoning. Suppose $q'_n > q_n$, then the inverse demand must increase. For any function-setting player other than player 2, the quantity must not decrease. For any quantity-setting player, the quantity does not change. Therefore the total quantity increases while the inverse demand increases, which is a contradiction.

To arrive at (10) consider the following. Suppose a function-setting player $j \neq n$ has a $q'_j < q_j$. Since z_j is unchanged, the inverse demand must decrease. For any function-setting player other than player n and j, the quantity must not increase. For any quantity-setting player, the quantity does not change. Therefore decrease of total quantity contradicts decrease of inverse demand.

Relation (11) follows from the following reasoning. Suppose a quantity-setting player i has a $p'_i < p_i$. Since q_i is unchanged, the inverse demand must decrease. For any function-setting player other than player n, the quantity must not increase. For any quantity-setting player, the quantity does not change. Therefore non-increase of total quantity contradicts decrease of inverse demand.

To arrive at (12) consider the following. Suppose $q' > q$. There must be a $j \in \mathcal{F}, j \neq n$ such that $q'_j > q_j$. Consequently, the inverse demand must increase. Thus increase of inverse demand contradicts $q' > q$. ∎

Lemma 1 shows that given a strategy profile $\mathbf{s}$, each player's price and quantity are uniquely determined, which implies that each player's profit is uniquely determined. We next show that each player's profit is continuous in $\mathbf{s}$.

Lemma 4: Let $\pi : \mathbb{R}^N \to \mathbb{R}^N$ be the function mapping a vector of strategies $\mathbf{s}$ to the vector of profits, i.e. $\pi(\mathbf{s}) = (\pi_1(\mathbf{s}), ..., \pi_N(\mathbf{s}))$. Then $\pi(\cdot)$ is a continuous function of $\mathbf{s}$.

Proof: In Lemma 1 we show that given a strategy vector $\mathbf{s}$, there is a unique $q(\mathbf{s}) \geq 0$ satisfies $h(q; \mathbf{s}) = 0$. We first show that $q(\mathbf{s})$ is continuous in $\mathbf{s}$.

Suppose $q(\mathbf{s})$ is discontinuous at some $\mathbf{s}_0$. Then there must exist a $\Delta > 0$, such that for any $\epsilon > 0$, there is a $\mathbf{s}'$ satisfying

$$||\mathbf{s}' - \mathbf{s}_0|| < \epsilon \text{ and } |q(\mathbf{s}') - q(\mathbf{s}_0)| \geq \Delta.$$

Since $h(q; \mathbf{s})$ is continuous in $\mathbf{s}$ and q, we can pick a small ϵ such that

$$|h(q(\mathbf{s}_0); \mathbf{s}_0) - h(q(\mathbf{s}_0); \mathbf{s}')| = |0 - h(q(\mathbf{s}_0); \mathbf{s}')| < \Delta.$$

We claim that given any $\mathbf{s}$, $|h(y_1; \mathbf{s}) - h(y_2; \mathbf{s})| \geq \Delta$ for any $y_1, y_2 \geq 0$ satisfying $|y_1 - y_2| \geq \Delta$. Proof is as below.

$$h(y_1; \mathbf{s}) - h(y_2; \mathbf{s}) =$$
$$(y_1 - y_2) + (\sum_{j \in \mathcal{F}} \frac{(p(y_2) - z_j)_+}{a_j + t_j} - \sum_{j \in \mathcal{F}} \frac{(p(y_1) - z_j)_+}{a_j + t_j}).$$

If $y_1 - y_2 \geq \Delta$, then $p(y_2) \geq p(y_1)$. Consequently,

$$\sum_{j \in \mathcal{F}} \frac{(p(y_2) - z_j)_+}{a_j + t_j} \geq \sum_{j \in \mathcal{F}} \frac{(p(y_1) - z_j)_+}{a_j + t_j}.$$

Therefore we have

$$h(y_1; \mathbf{s}) - h(y_2; \mathbf{s}) \geq \Delta.$$

If $y_1 - y_2 \leq -\Delta$, in a similar way, we can show that

$$h(y_1; \mathbf{s}) - h(y_2; \mathbf{s}) \leq -\Delta.$$

Since $|q(\mathbf{s}') - q(\mathbf{s}_0)| \geq \Delta$, we have

$$|h(q(\mathbf{s}');\mathbf{s}') - h(q(\mathbf{s}_0);\mathbf{s}')| = |0 - h(q(\mathbf{s}_0);\mathbf{s}')| \geq \Delta,$$

which contradicts $|h(q(\mathbf{s}_0);\mathbf{s}')| < \Delta$. Therefore it must be that $q(\mathbf{s})$ is continuous in $\mathbf{s}$.

Since the inverse demand $p(q)$ is continuous in q, which is continuous in $\mathbf{s}$, the inverse demand must be continuous in $\mathbf{s}$. For a quantity-setting player i, the profit $\pi_i = (p(\mathbf{s}) - a_i q_i) q_i$, which is continuous in $\mathbf{s}$. For a function-setting player j, $q_j = (p(\mathbf{s}) - z_j)_+/(a_j + t_j)$, which is continuous in $\mathbf{s}$. Consequently, $p_j = z_j + t_j q_j$ is also continuous in $\mathbf{s}$. Therefore the profit $\pi_j = p_j \cdot q_j$ is continuous in $\mathbf{s}$. ■

In the following lemma, we identify a region for each player's best response.

Lemma 5: Let $q_{max} = \min\{q : p(q) = 0\}$. For any s_{-i}, a quantity-setting player i's best response $q_i \in \mathcal{S}_i := [0, q_{max}]$; For any s_{-j}, a function-setting player j's best response $z_j \in \mathcal{S}_j := [-q_{max} t_j, p(0)]$.

Proof: If a quantity-setting player i sets $q_i > q_{max}$, then $q \geq q_i > q_{max}$ and therefore $p(q) = 0$. Consequently we have that $p_i = p(q) - a_i q_i < 0$, which implies a negative profit. Since a 0 profit is guaranteed by setting $q_i = 0$, it is not a best response that $q_i > q_{max}$.

If a function-setting player j sets $z_j > p(0)$, then $q_j = 0$ and thus $\pi_j = 0$, which can be achieved by setting $z_j = p(0)$. If $z_j < -q_{max} t_j$, then $\pi_j < 0$. The reason is stated below.

(a) If $q_j \leq q_{max}$, then

$$p_j = z_j + t_j q_j < -q_{max} t_j + q_{max} t_j = 0.$$

Also $q_j > 0$ due to $p(q) \geq 0 > z_j$.

(b) If $q_j > q_{max}$, then $p(q) = 0$ and consequently

$$p_j = p(q) - a_j q_j < 0.$$

Since a 0 profit is guaranteed by setting $z_j = p(0)$, it is not a best response that $z_j < -q_{max} t_j$. ■

Lemma 5 shows that each player's best response must be in a non-empty, compact and convex region. Let the non-empty, compact and convex product space be

$$\mathcal{S} = \times_n \mathcal{S}_n.$$

Let the best response set of player i for s_{-i} be

$$\beta_i(\mathbf{s}_{-i}) = \{s_i' \in \mathcal{S}_i | \pi_i(s_i', \mathbf{s}_{-i}) = \max_{s_i} \pi_i(s_i, \mathbf{s}_{-i})\}.$$

Let the set-valued function on $\mathcal{S}$ be

$$F : \mathcal{S} \to 2^{\mathcal{S}} \quad \text{s.t.} \quad F_i(\mathbf{s}) = \beta_i(\mathbf{s}_{-i}),\ \forall i.$$

A Nash equilibrium is a fixed point $\mathbf{s} \in F(\mathbf{s})$. Kakutani's fixed point theorem implies that if $F(\cdot)$ has a close graph and $F(\mathbf{s})$ is non-empty and convex for any $\mathbf{s} \in \mathcal{S}$ where $\mathcal{S}$ is a non-empty, compact and convex subset of some Euclidean space R^n, then $F(\mathbf{s})$ has a fixed point.

Lemma 6: $F(\cdot)$ has a closed graph.

Proof: If the graph of $F(\cdot)$ is not closed, then there exists a n and two sequences $s_n^k \to s_n$ and $s_{-n}^k \to s_{-n}$ where $s_n = \lim_{k\to\infty} s_n^k$, $s_n^k \in \beta_n(s_{-n}^k)$ and $s_n \notin \beta_n(s_{-n})$. For a $x_n \in \beta_n(s_{-n})$, we have that

$$\pi_n(s_n^k, s_{-n}^k) \geq \pi_n(x_n, s_{-n}^k), \tag{13}$$

$$\pi_n(x_n, s_{-n}) > \pi_n(s_n, s_{-n}). \tag{14}$$

Since $\pi(\cdot)$ is continuous, $\pi_n(x_n, s_{-n}^k)$ and $\pi_n(x_n, s_{-n})$ are close; $\pi_n(s_n^k, s_{-n}^k)$ and $\pi_n(s_n, s_{-n})$ are close. We can choose a large enough k such that

$$|\pi_n(x_n, s_{-n}^k) - \pi_n(x_n, s_{-n})| < \frac{\pi_n(x_n, s_{-n}) - \pi_n(s_n, s_{-n})}{3},$$

$$|\pi_n(s_n^k, s_{-n}^k) - \pi_n(s_n, s_{-n})| < \frac{\pi_n(x_n, s_{-n}) - \pi_n(s_n, s_{-n})}{3}.$$

The above two relations contradict (13) and (14). ■

Lemma 7: $F(\mathbf{s})$ is non-empty and convex for any $\mathbf{s} \in \mathcal{S}$.

Proof: We start by showing that the best response set of player i for any $\mathbf{s}_{-i}$, $\beta_i(\mathbf{s}_{-i})$, is non-empty and convex. Without loss of generality, we consider the best response of player 1. Player 1's quantity is

$$\begin{aligned} q_1(p) &= q(p) - \sum_{i \in \mathcal{C}, i \neq 1} q_i - \sum_{j \in \mathcal{F}, j \neq 1} q_j \\ &= q(p) - \sum_{i \in \mathcal{C}, i \neq 1} q_i - \sum_{j \in \mathcal{F}, j \neq 1} \frac{(p - z_j)_+}{t_j + a_j} \end{aligned}$$

where $q(p) = p^{-1}(p)$ which maps the inverse demand to the total quantity. For convenience, let

$$q_{-1}(p) = \sum_{i \in \mathcal{C}, i \neq 1} q_i + \sum_{j \in \mathcal{F}, j \neq 1} \frac{(p - z_j)_+}{t_j + a_j}.$$

We identify player 1's best response set in two cases.

Case 1: $\sum_{i \in \mathcal{C}, i \neq 1} q_i + \sum_{j \in \mathcal{F}, j \neq 1} \frac{(-z_j)_+}{t_j + a_j} \geq q_{max}$.

In this case, no matter what player 1's strategy is, the inverse demand is always 0. This is because that for any $p \geq 0$, the total quantity of other players $q_{-1}(p) \geq q_{-1}(0) \geq q_{max}$. Consequently, $q = q_1 + q_{-1} \geq q_{max}$ and thus $p(q) = 0$. If $1 \in \mathcal{C}$, then player 1's best response is $q_1^* = 0$. If $q_1 > 0$, then $p_1 = p - a_1 q_1 = -a_1 q_1 < 0$ and $\pi_1 < 0$. If $1 \in \mathcal{F}$, then player 1's best response is $z_1^* \in [0, p(0)]$. If $z_1 < 0$, then $q_1 > 0$. For either quantity-setting or function-setting, the best response set $\beta_1(\mathbf{s}_{-1})$ is convex.

Case 2: $\sum_{i \in \mathcal{C}, i \neq 1} q_i + \sum_{j \in \mathcal{F}, j \neq 1} \frac{(-z_j)_+}{t_j + a_j} < q_{max}$.

Recall that

$$q_1(p(0)) = q(p(0)) - q_{-1}(p(0)) = -q_{-1}(p(0)) \leq 0,$$

$$q_1(0) = q(0) - q_{-1}(0) > 0.$$

Since $q_1(p)$ is a continuous decreasing function for $p \in [0, p(0)]$, there must exist a $\bar{p} \in (0, p(0)]$ such that $q_1(\bar{p}) = 0$ and $q_1(p) \geq 0$ for $p \in [0, \bar{p}]$. We next maximize π_1 subject to $p \in [0, \bar{p}]$. Recall player 1's price is $p_1(p) = p - a_1 q_1(p)$. Consequently, player 1's profit is

$$\pi_1(p) = p_1(p) q_1(p) = (p - a_1 q_1(p)) q_1(p).$$

Obviously, player 1's best response is not a strategy such that $p = 0$. If $p = 0$, then $q \geq q_{max}$. Consequently

$$q_1 = q - (\sum_{i \in \mathcal{C}, i \neq 1} q_i + \sum_{j \in \mathcal{F}, j \neq 1} \frac{(-z_j)_+}{t_j + a_j}) > 0$$

and $p_1 = p - a_1 q_1 = -a_1 q_1 < 0$, which implies $\pi_1 < 0$. Let $-b$ be the slope at $(q, p(q))$. We assume $0 \leq b < \infty$. Note that $q_1(p)$ is not differentiable at $p = z_j$ for any $z_j > 0$. Let the left and right derivative of $\pi_1(p)$ with respect to p be $v^-(p)$ and $v^+(p)$ respectively. For all $j \in \mathcal{F}$, $j \neq 1$,

$$\begin{aligned} v^-(p) &= (1 - a_1 \frac{\partial q_1(p)}{\partial p}) q_1(p) + (p - a_1 q_1(p)) \frac{\partial q_1(p)}{\partial p} \\ &= q_1(p) + (p - 2a_1 q_1(p)) \frac{\partial q_1(p)}{\partial p} \\ &= q_1(p) - (p - 2a_1 q_1(p)) (\frac{1}{b} + \sum_{j: z_j < p} \frac{1}{t_j + a_j}). \end{aligned} \tag{15}$$

$$v^+(p) = q_1(p) - (p - 2a_1 q_1(p)) (\frac{1}{b} + \sum_{j: z_j \leq p} \frac{1}{t_j + a_j}). \tag{16}$$

For convenience, let

$$f(p) = p - 2a_1 q_1(p),$$

which continuously increases with p. Since

$$\begin{aligned} f(0) &= 0 - 2a_1 q_1(0) < 0, \\ f(\bar{p}) &= \bar{p} - 2a_1 q_1(\bar{p}) = \bar{p} > 0, \end{aligned}$$

there must exist a $p' \in (0, \bar{p})$ such that $f(p') = 0$. We investigate $v^-(p)$ and $v^+(p)$ in the following three regions.

Region 1: $p \in [0, p')$.
In this case, $f(p) < 0$ and $q_1(p) > 0$. Thus $v^-(p) > 0$ and $v^+(p) > 0$.

Region 2: $p = p'$.
In this case, $f(p') = 0$. Thus $v^-(p) = v^+(p) = q_1(p') > 0$.

Region 3: $p \in (p', \bar{p}]$.
In this case, $f(p) > 0$. Consequently $v^-(p) < q_1(p')$ and $v^+(p) < q_1(p')$. Since b decreases with p, we have that

$$(\frac{1}{b} + \sum_{j: z_j < p} \frac{1}{t_j + a_j}) \text{ and } (\frac{1}{b} + \sum_{j: z_j \leq p} \frac{1}{t_j + a_j})$$

increase with p (possibly with jumps). Recalling that $q_1(p)$ decreases with p, we have that $v^-(p)$ and $v^+(p)$ decrease with p. Hence, $\pi_1(p)$ is strictly concave in this region.

We maximize π_1 subject to $p \in [0, \bar{p}]$. Since $\pi_1(p)$ increases with p when $p \in [0, p')$ and $\pi_1(p)$ is strictly concave when $p \in [p', \bar{p}]$, there is a unique $p^* \in [p', \bar{p}]$ that maximizes π_1. Therefore the best response set of player 1 is a singleton. If $1 \in \mathcal{C}$, then $q_1^* = q(p^*) - q_{-1}(p^*)$. If $1 \in \mathcal{F}$, then $z_1^* = p^* - q_1^*(a_1 + t_1)$ where $q_1^* = q(p^*) - q_{-1}(p^*)$. ■

With Lemma 4, 5, 6 and 7, we can apply Kakutani's fixed point theorem to prove the following.

Theorem 1: The competition in the second stage of the two-stage game has a Nash equilibrium.

D. Uniqueness of Nash Equilibrium

In this section, we first show that in any Nash equilibrium of the competition in the second stage, every player has a positive profit. Later we conclude that there is a unique Nash equilibrium.

Lemma 8: In any Nash equilibrium,

1. $\pi_n \geq 0, \quad \forall\, n;$
2. $p(q) > 0.$

Proof: 1) For any s_{-n}, a quantity-setting player can get 0 profit by setting $q_n = 0$; For any s_{-n}, by setting $z_n = 0$, a function-setting player can get a $\pi_n = t_n q_n^2 \geq 0$.
2) Suppose $p(q) = 0$, then $q \geq p^{-1}(0)$, which implies that there exist a $q_n > 0$. Consequently, $p_n = p(q) - a_n q_n < 0$, which implies $\pi_n = p_n q_n < 0$. This contradicts $\pi_n \geq 0$. ■

Lemma 9: In any Nash equilibrium, $q_n > 0$ for all n.

Proof: If a quantity-setting player i's quantity $q_i = 0$, then $p_i = p(q)$. Note that $p(q) > 0$ by Lemma 8. Suppose that player i unilaterally increases her quantity. The total quantity will not decrease and thus the inverse demand will not increase. Player i can always pick a $q_i' > 0$ such that $p(q') - a_i q_i' > 0$ to get a positive profit.

If a function-setting player j's quantity $q_j = 0$, then

$$p_j = z_j \geq p(q) > 0.$$

Suppose that player j unilaterally changes z_j to

$$z_j' = p(q) - \epsilon > 0$$

for a small $\epsilon > 0$. If $q_j' = 0$, then

$$p(q') \leq z_j' = p(q) - \epsilon < p(q).$$

For any other function-setting player, the quantity must not increase. Consequently, $q' \leq q$ contradicts $p(q') < p(q)$. Therefore it must be that $q_j' > 0$ and $p_j' = z_j' + t_j q_j' > 0$, which implies $\pi_j' > 0$. ■

Lemma 10: In any Nash equilibrium, $p_n \neq 0$ for all n.

Proof: If a quantity-setting player i's price $p_i = 0$, then $a_i q_i = p(q) > 0$, which implies $q_i > 0$. Suppose that player i unilaterally decreases her quantity to $q_i' = q_i - \epsilon > 0$. By Lemma 2, we have $p_i' > p_i = 0$. This results in a positive profit for player i.

If a function-setting player j's price $p_j = 0$, then $a_j q_j = p(q) > 0$ and thus $z_j < 0$. Suppose that player j unilaterally increases z_j. By Lemma 3, $q_j' \leq q_j$ and $q' \leq q$. z_j can be increased by a small amount such that $q_j' > 0$. Consequently, $p_j' > 0$. This results in a positive profit for player j. ■

Lemma 8, 9 and 10 imply the following result.

Theorem 2: In any Nash equilibrium, $\pi_n > 0$ for all n.

The positivity of profit in Nash equilibrium allows us to prove the following result.

Lemma 11: In the second stage of the two-stage game, the following condition holds if $\mathbf{s}$ is a Nash equilibrium profile.

$$\frac{p_n}{q_n} = a_n + (\frac{1}{b} + \sum_{j \in \mathcal{F}, j \neq n} \frac{1}{a_j + t_j})^{-1}, \quad \forall\, n.$$

Proof: Theorem 1 shows that a Nash equilibrium exists and Theorem 2 shows that in any Nash equilibrium, $p_n > 0$, $q_n > 0$ for all n. Therefore if $\mathbf{s}$ is a Nash equilibrium profile, then we have $p > z_j \ \forall j \in \mathcal{F}$ from equation (4). Recall equation (15) and equation (16), we have

$$\frac{\partial \pi_1(p)}{\partial p} = q_1(p) - (p - 2a_1 q_1(p))(\frac{1}{b} + \sum_{j\in\mathcal{F}, j\neq 1} \frac{1}{t_j + a_j}).$$

If $\mathbf{s}$ is a Nash equilibrium profile, then it must be that $\partial\pi_1(p)/\partial p = 0$. If $\partial\pi_1(p)/\partial p > 0$, then player 1 can get a higher profit by changing his strategy such that the inverse demand increases. If $\partial\pi_1(p)/\partial p < 0$, then player 1 can get a higher profit by changing his strategy such that the inverse demand decreases. From

$$\begin{aligned} q_1 &= (p - 2a_1 q_1)(\frac{1}{b} + \sum_{j\in\mathcal{F}, j\neq 1} \frac{1}{t_j + a_j}) \\ &= (p_1 - a_1 q_1)(\frac{1}{b} + \sum_{j\in\mathcal{F}, j\neq 1} \frac{1}{t_j + a_j}), \end{aligned}$$

we have that

$$\frac{p_1}{q_1} = a_1 + (\frac{1}{b} + \sum_{j\in\mathcal{F}, j\neq 1} \frac{1}{t_j + a_j})^{-1}.$$

For other players, similar necessary condition applies. Therefore if $\mathbf{s}$ is a Nash equilibrium profile, then

$$\frac{p_n}{q_n} = a_n + (\frac{1}{b} + \sum_{j\in\mathcal{F}, j\neq n} \frac{1}{a_j + t_j})^{-1} \quad \forall\, n.$$

■

For notational convenience, let

$$b_n = (\frac{1}{b} + \sum_{j\in\mathcal{F}, j\neq n} \frac{1}{a_j + t_j})^{-1} \quad \forall\, n.$$

Lemma 11 shows that in any Nash equilibrium of the second stage subgame, the following conditions hold

$$\begin{aligned} p_n &= a_n q_n + b_n q_n, \\ p(q) &= p_n + a_n q_n \quad \forall\, n. \end{aligned} \tag{17}$$

By combining the above equations, we have

$$q_n = \frac{p(q)}{2a_n + b_n} \quad \forall\, n. \tag{18}$$

We will use Lemma 1 and the above equation to show uniqueness of the Nash equilibrium.

Theorem 3: The Nash equilibrium of the competition in the second stage of the two-stage game is unique.

Proof: Suppose $\mathbf{s}$ and $\mathbf{s}'$ are two different Nash equilibrium profiles. If $p > p'$, then $q < q'$ and $b < b'$. Thus $b_n < b'_n$ for all n. From equation (18), we have that $q_n > q'_n$ for all n, which contradicts to $q < q'$. Similarly, if $p < p'$, then $q_n < q'_n$ for all n, which contradicts to $q > q'$.

If $p = p'$, then $b = b'$ and thus $b_n = b'_n$ for all n. From (18), we have $q_n = q'_n$ for all n. Consequently we have that $p_n = p'_n$ for all n from (17). Therefore $q_i = q'_i, i \in \mathcal{C}$ and $z_j = z'_j, j \in \mathcal{F}$, which implies $\mathbf{s} = \mathbf{s}'$. Therefore there must be a unique Nash equilibrium. ■

E. Subgame-perfect Equilibrium

We have shown that any second stage subgame always has a unique Nash equilibrium. In this section, we claim that "committing to quantity strategy" is the only subgame-perfect equilibrium (SPE). We prove it by showing that regardless of what any other players do, the quantity strategy always earns a player a larger payoff than any other.

Theorem 4: In the two-stage game, it is the only subgame-perfect equilibrium for each player to commit to set a quantity.

Proof: We have identified existence and uniqueness of Nash equilibrium of any second stage subgame. We now consider the first stage (stage 0). Without loss of generality, we pick an arbitrary player i.e., player 1. Let s_1^0 denote player 1's strategy. Let s_{-1}^0 denote an arbitrary strategy profile of other players. Let c denote the action "commit to set a quantity". Let f denote the action "commit to set a price-quantity function". To prove committing to a quantity strategy is a SPE, we need to show that a quantity strategy is always better than any other strategy, for any profile of other players' actions:

$$\pi_1(s_1^0 = c, s_{-1}^0) > \pi_1(s_1^0 = f, s_{-1}^0).$$

For notational convenience, we let

$$J = \sum_{j\in\mathcal{F}, j\neq 1} \frac{1}{a_j + t_j}.$$

If $s_1^0 = c$, let q_n^c denote player n's quantity, let q^c denote the total quantity and let $-b^c$ denote the slope at $(q^c, p(q^c))$. Consequently, we introduce the following definitions:

$$\begin{aligned} b_1^c &= (\frac{1}{b^c} + J)^{-1}, \\ b_i^c &= (\frac{1}{b^c} + J)^{-1} \text{ for } i \in \mathcal{C}, \\ b_j^c &= (\frac{1}{b^c} + J - \frac{1}{a_j + t_j})^{-1} \text{ for } j \in \mathcal{F}, \\ D^c &= \frac{1}{2a_1 + b_1^c} + \sum_{i\neq 1, i\in\mathcal{C}} \frac{1}{2a_i + b_i^c} + \sum_{j\in\mathcal{F}} \frac{1}{2a_j + b_j^c}. \end{aligned}$$

If $s_1^0 = f$, let q_n^f denote player n's quantity, let q^f denote the total quantity and let $-b^f$ denote the slope at $(q^f, p(q^f))$. Consequently, we introduce the following definitions:

$$\begin{aligned} b_1^f &= (\frac{1}{b^f} + J)^{-1}, \\ b_i^f &= (\frac{1}{b^f} + J + \frac{1}{a_1 + t_1})^{-1} \text{ for } i \in \mathcal{C} \\ b_j^f &= (\frac{1}{b^f} + J + \frac{1}{a_1 + t_1} - \frac{1}{a_j + t_j})^{-1} \text{ for } j \in \mathcal{F}, \\ D^f &= \frac{1}{2a_1 + b_1^f} + \sum_{i\in\mathcal{C}} \frac{1}{2a_i + b_i^f} + \sum_{j\neq 1, j\in\mathcal{F}} \frac{1}{2a_j + b_j^f}. \end{aligned}$$

From (18), we have

$$q^c = p(q^c)D^c, \quad q^f = p(q^f)D^f. \tag{19}$$

We show that $q^c < q^f$ by the following reasoning. If $q^c \geq q^f$, then $b^c \geq b^f$ by concavity of $p(\cdot)$. Consequently, we have

$b_1^c \geq b_1^f$ and $b_n^c > b_n^f$ for all $n \neq 1$, which implies $D^c < D^f$. Since $p(q)$ is decreasing, $p(q^c) \leq p(q^f)$. From (19), we have that $q^c < q^f$, which contradicts to the assumption $q^c \geq q^f$. Therefore $q^c < q^f$. From (18), we have

$$q_1^c = \frac{p(q^c)}{2a_1 + b_1^c}, \quad q_1^f = \frac{p(q^f)}{2a_1 + b_1^f}.$$

Consequently,

$$\begin{aligned}\frac{\pi_1(c, s_{-1}^0)}{\pi_1(f, s_{-1}^0)} &= (\frac{q_1^c}{q_1^f})^2 \frac{a_1 + b_1^c}{a_1 + b_1^f} \\ &= (\frac{p(q^c)}{p(q^f)})^2 (\frac{2a_1 + b_1^f}{2a_1 + b_1^c})^2 \frac{a_1 + b_1^c}{a_1 + b_1^f} \\ &> (\frac{2a_1 + b_1^f}{2a_1 + b_1^c})^2 \cdot \frac{a_1 + b_1^c}{a_1 + b_1^f}.\end{aligned}$$

Recall

$$b_1^c = (\frac{1}{b^c} + J)^{-1} < (\frac{1}{b^f} + J)^{-1} = b_1^f.$$

Consequently,

$$\begin{aligned}&(2a_1 + b_1^f)^2(a_1 + b_1^c) - (2a_1 + b_1^c)^2(a_1 + b_1^f) \\ &\qquad = (b_1^f - b_1^c)(b_1^f b_1^c + a_1(b_1^f + b_1^c)) > 0\end{aligned}$$

and thus $\frac{\pi_1(c, s_{-1}^0)}{\pi_1(f, s_{-1}^0)} > 1$. This result implies that regardless of what any other players do, committing to set a quantity earns player 1 a larger profit than committing to set a price-quantity function. ∎

The analysis above also yields the following results.

Corollary 1: Given the first stage strategy profile of other players, s_{-j}^0, player j's quantity and profit increase with t_j.

Corollary 2: Suppose $|\mathcal{C}| = 0$. If $\mathbf{t} > \mathbf{t}'$, $q(\mathbf{t}) < q(\mathbf{t}')$.

Proof: If $q \geq q'$, then $b \geq b'$, which implies $b_n > b_n'$ because $t_n > t_n'$. Consequently, we have $D < D'$ and thus $q < q'$, which contradicts to $q \geq q'$. ∎

IV. The Effect of Demand Uncertainty

In early sections, we found that committing to set a quantity (or equivalent to set a price-quantity function with infinite slope) constitute the only SPE when inverse demand is certain. Since the price-quantity function strategy is more general than Cournot strategy, we now just assume that every player chooses setting a price-quantity function. In this section, we investigate how demand uncertainty affects players' decision on the slope of price-quantity function.

We assume that inverse demand function is linear and has the following probability distribution

$$\begin{cases} k - bq & \text{with probability } x \\ \bar{k} - \bar{b}q & \text{with probability } 1 - x \end{cases}$$

Each player i's profit is thus

$$\pi_i = xq_i(t_i q_i + z_i) + (1 - x)\bar{q}_i(t_i \bar{q}_i + z_i)$$

where q_i, $\bar{q}_i$ and z_i satisfy

$$k - b\sum q_i = (a_i + t_i)q_i + z_i \quad \text{if } q_i > 0; \tag{20}$$

$$k - b\sum q_i \leq (a_i + t_i)q_i + z_i \quad \text{if } q_i = 0; \tag{21}$$

$$\bar{k} - \bar{b}\sum \bar{q}_i = (a_i + t_i)\bar{q}_i + z_i \quad \text{if } \bar{q}_i > 0; \tag{22}$$

$$\bar{k} - \bar{b}\sum \bar{q}_i \leq (a_i + t_i)\bar{q}_i + z_i \quad \text{if } \bar{q}_i = 0. \tag{23}$$

A strategy profile $(\mathbf{t}, \mathbf{z})$ is a subgame-perfect equilibrium if for each i, the following conditions are satisfied

$$\pi_i(z_i(\mathbf{t}), z_{-i}(\mathbf{t})|\mathbf{t}) \geq \pi_i(z_i'(\mathbf{t}), z_{-i}(\mathbf{t})|\mathbf{t}) \;\; \forall z_i' \in \mathbb{R};$$
$$\pi_i(t_i, t_{-i}) \geq \pi_i(t_i', t_{-i}) \;\; \forall t_i' \in \mathbb{R}^+.$$

In the second stage, each player i's decision is

$$\max_{z_i} \pi_i = \max_{z_i} xq_i(t_i q_i + z_i) + (1 - x)\bar{q}_i(t_i\bar{q}_i + z_i)$$

In any Nash equilibrium of the second stage subgame, if $q_i > 0$ & $\bar{q}_i > 0 \; \forall i$, the following first order condition holds

$$\begin{aligned}&(1-x)\bar{q}_i - (1-x)(2t_i\bar{q}_i + z_i)\frac{1 + \bar{b}\sum_{j\neq i}\frac{1}{a_j + t_j}}{a_i + t_i + \bar{b} + \bar{b}\sum_{j\neq i}\frac{a_i + t_i}{a_j + t_j}} \\ &+ xq_i - x(2t_i q_i + z_i)\frac{1 + b\sum_{j\neq i}\frac{1}{a_j + t_j}}{a_i + t_i + b + b\sum_{j\neq i}\frac{a_i + t_i}{a_j + t_j}} = 0.\end{aligned}$$

When $b = \bar{b}$, the above equation becomes

$$\frac{xp_i + (1-x)\bar{p}_i}{xq_i + (1-x)\bar{q}_i} = a_i + (\frac{1}{b} + \sum_{j\in\mathcal{F}, j\neq i}\frac{1}{a_j + t_j})^{-1}. \tag{24}$$

Corollary 3: Suppose that each player can only choose setting quantity or setting price in the two-stage game. Also suppose that $b = \bar{b}$. If $q_i > 0$ and $\bar{q}_i > 0$ for all i in any Nash equilibrium of the second stage subgame, then it is a SPE to choose setting quantity.

Proof: If $b = \bar{b}$ and for all i, $q_i > 0$ and $\bar{q}_i > 0$, recall equation (24), we have that

$$\frac{xp_i + (1-x)\bar{p}_i}{q_i} = a_i + (\frac{1}{b} + \sum_{j\in\mathcal{F}, j\neq i}\frac{1}{a_j})^{-1},$$
$$\pi_i = (xp_i + (1-x)\bar{p}_i)q_i$$

if player i chooses to setting quantity;

$$\frac{p_i}{xq_i + (1-x)\bar{q}_i} = a_i + (\frac{1}{b} + \sum_{j\in\mathcal{F}, j\neq i}\frac{1}{a_j})^{-1},$$
$$\pi_i = p_i(xq_i + (1-x)\bar{q}_i)$$

if player i chooses to setting price. Therefore we can apply the analysis in the proof of Theorem 4 to complete the proof. ∎

We provide numerical examples to show that infinite slope is not always a SPE when the inverse demand is uncertain.

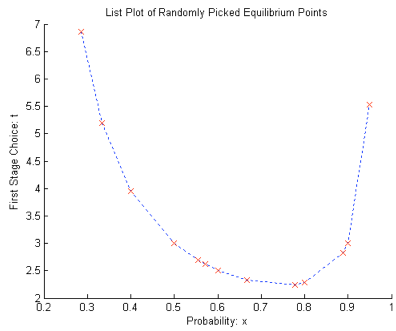

Fig. 3. Example 2: first stage choice t versus probability x.

Example 1: Suppose that $a_1 = a_2 = 1$ and the inverse demand function has the following probability distribution

$$\begin{cases} 5 - q & \text{with probability } \frac{1}{3} \\ 5 - \frac{1}{3}q & \text{with probability } \frac{2}{3} \end{cases}$$

We first apply the first order conditions to find $\mathbf{z}$ as a function of $\mathbf{t}$. Then we find the optimal choice of $\mathbf{t}$ by iteratively evaluating each players best response to the other. Since this process converges in this example, this example must have a Nash equilibrium. The result is $(\mathbf{t} = 0.98764, \mathbf{z} = 0.57334)$. The profits are $\pi_1^* = \pi_2^* = 3.08659$. To test the result, we solve the following optimization problem:

$$\begin{aligned} \max_{z_1} \quad & \pi_1 \\ \text{s.t.} \quad & (20)...(23); t_1 = t_2 = 0.98764; z_2 = 0.57334; \end{aligned}$$

The result is that $z_1^* = 0.57334$ and $\pi_1^* = 3.08659$. Therefore the strategy profile $(\mathbf{t} = 0.98764, \mathbf{z} = 0.57334)$ is a SPE.

Example 2: Suppose that $a_1 = a_2 = 0$ and the inverse demand function has the following probability distribution

$$\begin{cases} 1 - q & \text{with probability } x \\ 3 - q & \text{with probability } 1 - x \end{cases}$$

We generate multiple equilibrium points with different x in the same way described in Example 1. Please see Figure 3 for result. This example demonstrates that there is an advantage of choosing a smaller slope due to uncertainty. The intuition can be explained in the following way. A player that commits to a steep slope is close to committing to a quantity, with the limit being that an infinite slope corresponds to a quantity commitment. When demand is uncertain, a commitment to a quantity is undesirable, because if demand ends up being low, the price received for the quantity the player committed to will be very low. With a price-quantity function, the quantity offered to the market decreases as prices decrease, and so this better responds to the different outcomes of the uncertainty in demand. The need for being adaptable to demand uncertainty competes with the benefits of committing to quantity when demand is certain. Recall that a commitment to quantity signals to other players that they cannot reduce the quantity sold of the other player by undercutting them on price. Thus with uncertain demand, the best slope becomes steeper as the uncertainty in demand lessens.

V. Conclusion

In this paper, we study a two-stage game where $N \geq 2$ players compete for substitutable goods. In the first stage, the players choose their strategic variable, quantity or price-quantity function. In the second stage, they choose the magnitudes of their strategic variables. We first show that a unique Nash equilibrium exists in any second stage subgame. Later we show that it is the only subgame-perfect equilibrium for each player to commit to set a quantity. We also consider the effect of demand uncertainty.

Acknowledgment

We thank Libin Jiang and Nirvikar Singh for valuable comments and suggestions on this work.

References

[1] D. M. Kreps and J. A. Scheinkman, "Quantity Precommitment and Bertrand Competition Yield Cournot Outcomes," *Bell Journal of Econ*, vol. 14, pp. 326-37, 1983.
[2] N. Singh and X. Vives, "Price and Quantity Competition in a Differentiated Duopoly," *RAND J. Econ.*, vol. 15, no. 4, pp. 546-54, 1984.
[3] X. Vives, "On the Efficiency of Bertrand and Cournot Equilibria with Product Differentiation," *Journal of Econ Theory*, vol. 35, no. 1, pp. 166-75, 1985.
[4] R. Amir and J. Y. Jin, "Cournot and Bertrand Equilibria Compared: Substitutability, Complementarity and Concavity," *Int. Journal of Industrial Organization*, vol. 19, no. 3-4, pp. 303-17, 2001.
[5] S. Bylka and J. Komar, "Cournot-Bertrand Mixed Oligopolies," Warsaw Fall Seminars in Mathematical Economics, New York Springer-Verlag, pp. 22-33, 1975.
[6] L. Cheng, "Comparing Bertrand and Cournot equilibria - a Geometric Approach," *RAND J. Econ.*, vol. 16, no. 1, pp. 146-52, 1985.
[7] K. Okuguchi, "Equilibrium Prices in the Bertrand and Cournot Oligopolies," *Journal of Econ Theory*, vol. 42, no. 1, pp. 128-39, 1987.
[8] T. Sato, "On Cournot-Bertrand Mixed Duopolies," *The Japanese Economic Review*, vol. 47, no. 4, 1996.
[9] C. Z. Qin and C. Stuart, "Bertrand versus Cournot Revisited," *Economic Theory*, vol. 10, pp. 497-507, 1997.
[10] Y. Tanaka, "Profitability of Price and Quantity Strategies in an Oligopoly," *Journal of Mathematical Econs*, vol. 35, pp. 409-18, 2000.
[11] A. Tasnádi, "Price vs. Quantity in Oligopoly Games," *International Journal of Industrial Organization*, vol. 24, pp. 541- 54, 2006.
[12] D. Acemoglu, K. Bimpikis, A. Ozdaglar, "Price and Capacity Competition," 44th Annual Allerton Conference on Communication and Control, Monticello, IL, September 2006.
[13] D. Laussel, "Strategic Commercial Policy Revisited: A Supply-Function Equilibrium Model," *American Economic Review*, vol 82, no. 1, pp. 84-99, 1992.
[14] S. Ohlendorf, "Expectation Damages, Divisible Contracts, and Bilateral Investment," SFB/TR 15 Governance and the Efficiency of Economic Systems, Discussion Papers with number 231, 2008.
[15] K. Zhu and J. Weyant, "Strategic Decisions of New Technology Adoption under Asymmetric Information," *Decision Sciences*, vol. 34, no. 4, pp. 643-75, 2003.
[16] P. Klemperer, M. Meyer, "Supply Function Equilibria in Oligopoly under Uncertainty," *Econometrica*, vol. 57, no. 6, pp. 1243-77, 1989.
[17] R. Green, "Increasing competition in the British electricity spot market," *J. Ind. Economics.*, vol. 44, no. 2, pp. 205-16, 1996.
[18] R. Baldick, R. Grant and E. Kahn, "Theory and Application of Linear Supply Function Equilibrium in Electricity Markets," *Journal of Regulatory Economics*, vol. 25, no. 2, pp. 143-67, 2004.
[19] Y. Liu, Y. X. Ni and F. F. Wu, "Existence, Uniqueness, Stability of Linear Supply Function Equilibrium in Electricity Markets," *Proceedings of the Power Engineering Society General Meeting*, IEEE, vol. 1, no. 6-10, pp. 249-54, 2004.
[20] R. Baldick, "Electricity Market Equilibrium Models: The Effect of Parametrization," *IEEE Transactions on Power Systems*, vol. 17, no. 4, 2002.
[21] D. Fudenberg, J. Tirole, *Game Theory*, The MIT Press, Cambridge, Massachusetts, 1991.