Short Wave Instability on Vortex Filaments

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The short wave instability on corotating vortex filaments is investigated. It is shown that the short wave instability always occurs on corotating vortex filaments of fixed core structure. Moreover, when the interfilament distance is smaller than or comparable to the core size, vortex filaments produce short wave unstable modes which lead to wild stretching and folding; when vortex filaments are far apart from each other, unstable modes are bounded by a fraction of the core size and the vortex stretching remains bounded. These findings may be used to explain the smooth behavior of superfluid vortices. They also explain the stretching seen in numerical calculations. [S0031-9007(98)06172-9]

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The instability of vortex pairs or vortex rings has been investigated by many authors. Crow [1] studied the long wave and short wave instability on antiparallel vortex pairs arising in jet wakes. With a slightly different model and a more accurate calculation of the self-induced rotation of the short wave modes, Widnall, Bliss, and Tsai [2] investigated the instability of antiparallel vortex pairs and vortex rings. Recently Klein, Majda, and Damodaran [3] studied the instability of long wave modes on both antiparallel and corotating vortex pairs. It was found that certain long wave modes are unstable on antiparallel vortex pairs but long wave modes are always stable on corotating vortex pairs. However, the study in [3] is based on a simplified model equation derived from an asymptotic analysis, in which one of the assumptions is that the wavelength is large in comparison with the core size. Because of this assumption, short wave modes are excluded from the model equation. In this Letter we present the first study of short wave instability on corotating vortex filaments. We show that (a) there always exist short wave unstable modes on corotating vortex pairs; (b) if the distance between filaments is large compared with the core size, the amplitude of unstable modes is bounded by the core size; and (c) if the filaments are close together, the growth of the unstable modes leads to wild stretching and folding. These results suggest a plausible explanation for the different behavior of classical and superfluid vortices, which has puzzled physicists for quite a long time. Classical vortex filaments stretch and fold wildly, and form small scale structures while superfluid vortex filaments remain smooth [4–7]. It has been found that superfluid vortex filaments have a very small core size [$\sim O(\text{Å})$] and a fixed core structure [5–7]. Result (b) implies that the short wave instability is insignificant for superfluid vortex filaments since the interfilament distance far exceeds the core size. This suggests that it is the tiny core size rather than the quantization of circulation that is responsible for the smooth behavior of superfluid vortices. In the vortex methods [8–12], the vorticity field is represented by many overlapping vortex filaments of fixed core structure. Result (c) explains the wild stretching and the exponential growth of numerical vortex elements observed in numerical calculations.

We first construct a linear stability analysis of short waves on corotating vortex pairs with fixed core structure. We then extend our analysis to the case of a single vortex filament immersed in a corotating vorticity field. If the growth rate of a mode is positive, the amplitude of the mode will grow exponentially, not bounded by any constant multiple of its initial amplitude. We call such modes unstable. In this Letter, "long wave" means the wavelength is large compared to the core size; "short wave" means the wavelength is comparable to or smaller than the core size. The linear analysis will show that short wave unstable modes always exist on corotating vortex filaments. When unstable modes grow to an amplitude comparable to the wavelength, their motion is no longer governed by the linear analysis. Numerical simulations and a refined analysis will reveal the long time evolution of unstable modes: Neighboring vortices induce short wave unstable modes that lead to stretching and folding, but an isolated vortex filament does not create hairpins or wild stretching.

A corotating vortex pair consists of two parallel vortex filaments of the same circulation. The unperturbed pair rotates around its axis of symmetry with an angular velocity Ω_0. To illustrate the evolution of the perturbation wave relative to the unperturbed filaments, we put the z axis parallel to the unperturbed pair and let the x-y plane rotate with the pair.

Now we perturb the vortex pair by a sinusoidal wave of small amplitude. The stability calculation can be performed by considering the motion of the vortex filaments that results from the perturbation. Let us focus on one filament of the pair. The vortex filament moves with a velocity that is a combination of the self-induced rotation of the sinusoidally perturbed filament and the velocity induced by the other filament. For short wave perturbations, the velocity induced at the vortex by the second filament

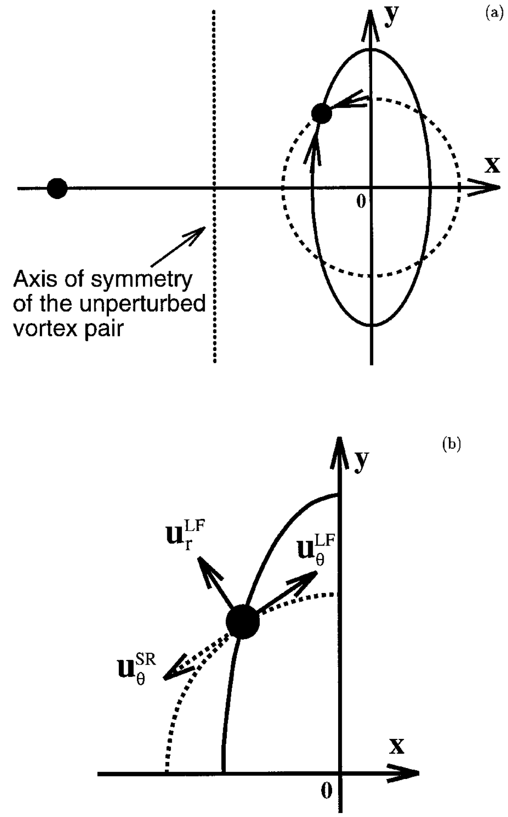

FIG. 1. (a) The coordinate system. (b) Motion of the perturbed vortex.

can be well approximated by treating the second filament as unperturbed. For simplicity, we put the origin of the x-y plane at the unperturbed position of the vortex we focus on. Figure 1(a) shows a cross section of the perturbed vortex pair. The filled circle on the right represents the perturbed vortex. The second vortex is shown as unperturbed by the filled circle on the left. As far as the motion of the right filament is concerned, the contribution from the perturbation on the left filament is negligible.

The induced velocity at the right vortex (x, y) by the left vortex observed in the nonrotating reference system is given by the Biot-Savart law [13]

$$\mathbf{u}_0(x, y) = \frac{\Gamma}{2\pi} \frac{f(d/\delta)}{d^2} \begin{bmatrix} -y \\ b + x \end{bmatrix}, \tag{1}$$

where $d = \sqrt{(b + x)^2 + y^2}$, b is the separation of the unperturbed pair, δ is the core size, Γ is the circulation, and $f(r)$ is a two-dimensional cutoff function determined by the core vorticity distribution [10,11].

The angular velocity of the unperturbed pair in the nonrotating reference system is

$$\Omega_0 = \frac{\Gamma}{\pi b^2} f\left(\frac{b}{\delta}\right). \tag{2}$$

In the rotating reference system of Fig. 1(a), the induced velocity is

$$\mathbf{u}(x, y) = \mathbf{u}_0(x, y) - \Omega_0 \begin{bmatrix} -y \\ b/2 + x \end{bmatrix}. \tag{3}$$

Expanding $\mathbf{u}(x, y)$ around $(0, 0)$, the position of the unperturbed vortex, one has

$$\mathbf{u}(x, y) = \frac{\Gamma}{2\pi b^2} \begin{bmatrix} c_1 y \\ -c_2 x \end{bmatrix} + O(x^2 + y^2)$$
$$\stackrel{\text{def}}{=} \mathbf{u}^{\mathrm{LF}} + O(x^2 + y^2), \tag{4}$$

where $c_1 = f(b/\delta)$ and $c_2 = 3f(b/\delta) - (b/\delta)f'(b/\delta)$. In the neighborhood of $(0, 0)$, the velocity field $\mathbf{u}(x, y)$ can be approximated by the linear flow (denoted as LF) $\mathbf{u}^{\mathrm{LF}}$. This linear flow is a special case of the stagnation-point flows studied by Criminale *et al.* [14,15]. The streamlines of $\mathbf{u}^{\mathrm{LF}}$ are a family of ellipses [solid lines in Figs. 1(a) and 1(b)] described by $x^2/c_1 + y^2/c_2 = \alpha$.

In the cylindrical (r, θ) coordinate system, the radial and the tangential components of $\mathbf{u}^{\mathrm{LF}}$ are

$$u_r^{\mathrm{LF}} = -r \frac{\Gamma}{2\pi b^2} \frac{c_2 - c_1}{2} \sin 2\theta, \tag{5}$$

$$u_\theta^{\mathrm{LF}} = -r \frac{\Gamma}{2\pi b^2} \left(\frac{c_1 + c_2}{2} + \frac{c_2 - c_1}{2} \cos 2\theta\right). \tag{6}$$

In addition to $\mathbf{u}^{\mathrm{LF}}$, the perturbed vortex is also subject to a self-induced rotation (denoted as SR)

$$u_\theta^{\mathrm{SR}} = -r \frac{\Gamma}{2\pi \delta^2} \Omega(k\delta), \tag{7}$$

where k is the wave number and Ω is the nondimensional rotation frequency of the perturbation wave mode. The streamlines of u_θ^{SR} are a family of circles [dashed lines in Figs. 1(a) and 1(b)].

Instability occurs when the total tangential velocity $u_\theta^{\mathrm{LF}} + u_\theta^{\mathrm{SR}}$ is zero and the vortex diverges in the radial direction driven by u_r^{LF}, as shown in Fig. 1(b).

Solving $\cos 2\theta$ from $u_\theta^{\mathrm{LF}} + u_\theta^{\mathrm{SR}} = 0$ and using the fact that $\cos 2\theta$ is between -1 and 1, we obtain the instability condition

$$-c_2 \frac{\delta^2}{b^2} < \Omega(k\delta) < -c_1 \frac{\delta^2}{b^2}. \tag{8}$$

When the self-induced rotation does not satisfy (8), the perturbation wave evolves periodically in time and its maximum amplitude is bounded by a constant multiple of the initial amplitude, which means the vortex pair is stable outside the range of (8).

For a vortex filament whose core vorticity distribution is derived from the second order cutoff function $f(r) = 1 - \exp(-r^3)$ [10], the dispersion relation, $\Omega(k\delta)$, is shown in Fig. 2. The dispersion relations for other core vorticity distributions are similar to the one shown here and are discussed in [16]. For each value of b/δ, an instability interval of $k\delta$ is solved from inequality (8) and the dispersion relation. The shaded area in Fig. 3 marks the region of instability in the $(b/\delta, k\delta)$ plane. Figure 3 indicates that for any value of b/δ, there are always unstable wave modes for the corotating vortex pair.

We have carried out numerical simulations of the evolution of small sinusoidal perturbations on a corotating

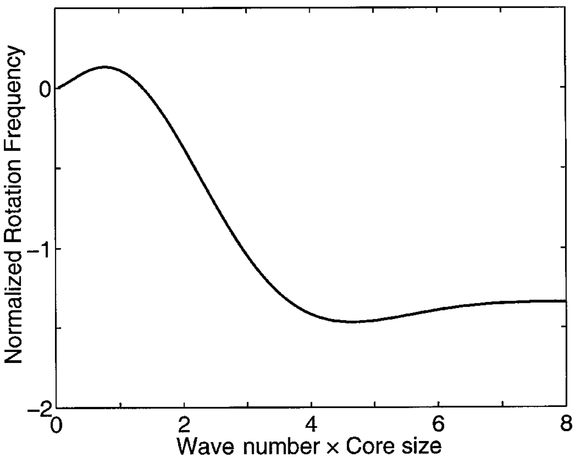

FIG. 2. Dispersion relation $\Omega(k\delta)$.

vortex pair using the thin-tube vortex filament method [17]. The instability condition (8) is verified in numerical simulations [16]. Furthermore, the numerical simulations reveal the long time behavior of the unstable mode for different values of b/δ. When b/δ is above 5, the amplitude of the unstable mode grows to a maximum, falls back, then starts growing and repeats the pattern. The vortex filaments stretch and contract alternately. No wild stretching is observed [Fig. 4(a)]. When b/δ is below 2, the unstable mode grows without bound [Fig. 4(b)]. The vortex filaments stretch catastrophically and develop hairpin-shaped small scale structures [Fig. 4(b) inset]. At the early stage of instability, the unstable mode grows exponentially forming hairpins on corotating filaments. After that, the vortex stretching and folding are more fueled by the pairing of antiparallel parts of each hairpin. Siggia and Pumir [18,19] showed that pairing of two antiparallel pieces leads to the collapse of a vortex filament.

Similarly, one can study the short wave instability on a single vortex filament with fixed core structure immersed in a corotating vorticity field. The motion of the filament is the combination of the self-induced rotation and the velocity induced by the surrounding vorticity field. The velocity field induced by a second order Gaussian vorticity distribution, if linearized around a point and viewed in the reference system attached to that point, is a straining flow [16]. When the surrounding vorticity field is discretized and represented by a set of numerical vortex filaments, our numerical results demonstrate that the streamlines of the linearized flow are ellipses. As the vorticity field is approximated by more numerical vortex filaments, these ellipses become flatter, and the linearized flow is approximately a straining flow [16]. The minimum tangential velocity of a straining flow is zero. From Fig. 2, we see that there are short wave modes whose rotation frequency is negative (opposing the straining flow) and close to zero. Therefore there are always short wave modes whose self-induced rotation can balance the straining flow in the tangential direction. When $u_\theta^{\mathrm{LF}} + u_\theta^{\mathrm{SR}} = 0$, the perturbation mode stops rotating and diverges along the radial direction, driven by the radial component of the straining flow

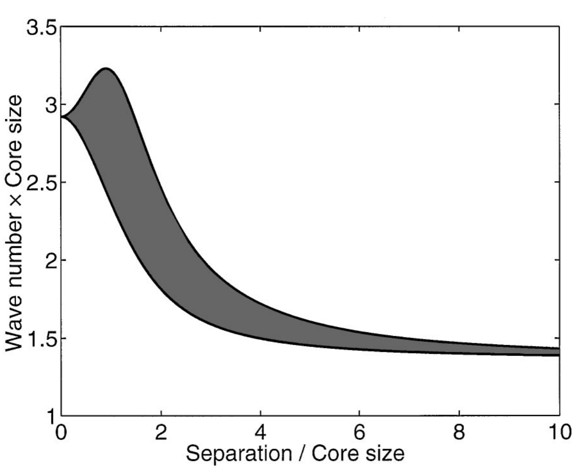

FIG. 3. Region of instability in the $(b/\delta, k\delta)$ plane.

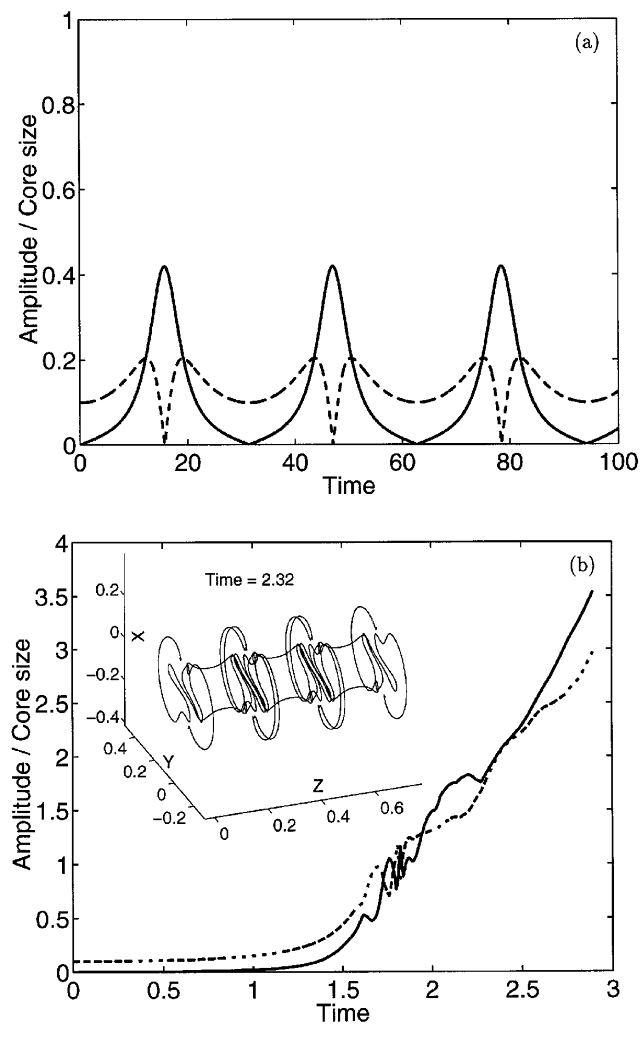

FIG. 4. Time evolution of unstable modes in the coordinate system of Fig. 1. Dashed line: amplitude in the x direction. Solid line: amplitude in the y direction. (a) $b/\delta = 10$. (b) $b/\delta = 2$. The inset shows the configuration of the vortex pair.

[see Fig. 1(b)]. Thus the short wave instability always occurs on a vortex filament immersed in a corotating vorticity field.

As a direct application, our study of short wave instability can be used to explain the smooth behavior of superfluid vortices. For superfluid vortex filaments, the interfilament distance is usually much larger than the core size. Hence the short wave instability does not cause the catastrophic stretching and folding. This implies that the tiny core size of superfluid vortex filaments is more important in accounting for their nonclassical dynamics than the quantization of circulation. An explanation for the different behavior of superfluid and classical vortices has been proposed by Chorin using statistical theories [4]. Our study of the short wave instability provides an explanation from vortex dynamics.

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