Applied Mathematical Sciences, Vol. 2, 2008, no. 43, 2145 - 2156

Controllability of Non-Newtonian Fluids under Homogeneous Extensional Flow

Lynda Wilson, Hong Zhou [1], Wei Kang

Department of Applied Mathematics
Naval Postgraduate School, Monterey, CA 93943-5216, USA

Hongyun Wang

Department of Applied Mathematics and Statistics
University of California, Santa Cruz, CA 95060, USA

Abstract

In the discipline of non-Newtonian materials, the ability to control viscoelastic stresses is very desirable in ascertaining important properties of the influenced materials. We apply the nonlinear geometric control theory to examine the controllability of various popular constitutive models with imposed homogeneous extensional flow. The subsequent constitutive laws considered here include the Phan-Thien-Tanner model, the Johnson-Segalman model, the Giesekus model and the Doi model. This paper provides the first analysis on the effect of extensional flow on these models.

1 Introduction

For the design of certain materials, controllability of viscoelastic fluids is a significant characteristic. In practice, the shape of an extrudate can be controlled by varying the size or shape of an orifice, whereas the advance of the free surface can be controlled by varying the inflow into a mold.

Previous theoretical studies from Renardy [[Renardy, 2005a], [Renardy, 2005b], [Renardy, 2007]] investigated the controllability of flows of linear viscoelastic fluids for the multi-mode Maxwell models, the controllability of the homogeneous shear flow of viscoelastic fluids with several different constitutive models, and the controllability of nonhomogeneous shear flow of an upper convected Maxwell fluid. Very recent studies have examined the controllability of the upper convected Maxwell model under various homogeneous

[1] Corresponding author, hzhou@nps.edu

flows [[Zhou *et al.*, 2007]]. Furthermore, the reachable set for the upper convected Maxwell model under imposed extensional flow was precisely specified in [[Zhou *et al.*, 2007]] even though it is usually very challenging for general cases. In most of these studies the state of the system is characterized by the viscoelastic stresses while the control input is in the form of the flow rate or the body force.

The primary purpose of this paper is to further these earlier studies to consider the controllability of three different model systems under the imposed homogeneous extensional flow. Like shear flow, the extensional flow is also very important in physical applications. For example, extensional flow can be used to locally approximate the flow away from boundaries in extrusion manufacturing or in thin film and sheet manufacturing; an extension-dominated flow occurs in industrial wire-coating processes along the wire-coating region beyond the die. Following earlier studies, we restrict our attention to two-dimensional models. This is mainly due to its mathematical simplicity and is also physically motivated by monolayer thin flims. Admittedly two-dimensional models are ideal situations but we can't hope to understand the controllability of more realistic cases unless we understand the controllability of simpler ones.

For reader's convenience, we now present on overall basic definition and description of *weak controllability* adopted from [[Isidori, 1995]].

Let

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i \tag{1}$$

be a general nonlinear control system, where x is the state variable, and $u_i \in \mathbb{R}$, $i = 1, \cdots, m$, are the control variables. Let M denote the manifold of state variables. A point x_1 in M is *reachable* from a point x_0 in M if there exist piecewise continuous input functions, $u_i = \alpha_i(t)$, such that the trajectory, $x(t)$, of (1) with initial state x_0 reaches x_1 in finite time. The global reachability is usually hard to prove for nonlinear control systems. A feasible solution is to seek weak controllability. The system (1) is *weakly controllable* within some open subset $S \subseteq M$ if for each point $x_0 \in S$, there is an open neighborhood U_0 of x_0 so that the set of points reachable from x_0 along trajectories inside U_0 contains at least an open subset of M.

In (1), $f(x)$ and $g_i(x)$ $(i = 1, \cdots, m)$ are vector fields in M. Under the Lie bracket operation, $[f, g]$, the space of smooth vector fields on M forms a Lie algebra. This Lie algebra, which is the smallest subalgebra containing the vector fields f, g_1, $\cdots$, g_m, is called the *Control Lie Algebra*.

A powerful sufficient condition on the weak controllability of a nonlinear control system is the *contollability rank condition (CRC)* [[Isidori, 1995], [Hermann and Krener, 1977]]. Namely, a control system of the form (1) is weakly controllable on an open set S if it satisfies the controllability rank condition on S, i.e.,

$$dim(\Delta_{\mathcal{C}}(x)) \equiv n \tag{2}$$

for all $x \in S$, where n is the dimension of the manifold M, and

$$\Delta_{\mathcal{C}}(x) = \text{span}\{X(x)|X \text{is a vector field in the control Lie algebra}\}.$$

Note that the big advantage of the CRC is that it is purely algebraic and it does not require integrations of the differential equations.

2 The Phan-Thien-Tanner Model

The Phan-Thien-Tanner (PTT) model is one of the most applied differential type of the nonlinear viscoelastic constitutive equations. It contains two parameters that control nonlinearity and has the ability to fit, to some extent, the shear and elongational properties of the viscoelastic materials independently. Thus, the Phan-Thien-Tanner model is physically more realistic than the constant shear viscosity Oldroyd-B model or upper-convected Maxwell model.

The Phan-Thien-Tanner model has the constitutive equation [[Phan-Thien and Tanner, 1977]]

$$\dot{\mathbf{T}} - (\nabla \mathbf{v})\mathbf{T} - \mathbf{T}(\nabla \mathbf{v})^T + \lambda \mathbf{T} + \nu(\text{tr}\mathbf{T})\mathbf{T} = 2\mu \mathbf{D}, \tag{3}$$

where $\mathbf{T}$ is the stress tensor, $\mathbf{v}$ is the velocity vector, $\nabla \mathbf{v}$ is the velocity gradient tensor, λ is the relaxation rate, ν is a constant, the notataion "tr" stands for the trace of the tensor, μ is the elastic modulus and $\mathbf{D}$ is the rate-of-deformation tensor.

We consider 2-D homogeneous viscoelastic fluids and denote the stress tensor and the imposed extensional flow with rate $\dot{\gamma}(t)$ by

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{bmatrix}, \quad \mathbf{v} = (\dot{\gamma}(t)\frac{x}{2}, -\dot{\gamma}(t)\frac{y}{2}), \tag{4}$$

respectively, where T_{11} is the first normal stress difference, T_{22} the second normal stress difference and T_{12} the shear stress. Suppose the control input is denoted by $\dot{\gamma}$ which is closely related to the velocity. Then the general dynamic problem of the PTT model (3) takes the form

$$\dot{\mathbf{T}} = \mathbf{F}(\dot{\gamma}(t), \mathbf{T}), \ \mathbf{T}(0) = \mathbf{T}_0, \ \mathbf{T}(t_{final}) = \mathbf{T}_1, \tag{5}$$

where $\mathbf{T}_0$ and $\mathbf{T}_1$ are the given initial and final states. The state of the system (5) is characterized by viscoelastic stress $\mathbf{T}$ with three components T_{11}, T_{22} and T_{12}.

For the extensional flow in (4), the velocity gradient is

$$\nabla \mathbf{v} = \begin{bmatrix} \dfrac{\dot{\gamma}(t)}{2} & 0 \\ 0 & -\dfrac{\dot{\gamma}(t)}{2} \end{bmatrix}, \tag{6}$$

which implies that the rate-of-strain tensor is

$$\mathbf{D} = \frac{1}{2}[\nabla \mathbf{v} + (\nabla \mathbf{v})^T] = \frac{\dot{\gamma}(t)}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{7}$$

Applying the velocity gradient (6) and the rate-of-strain tensor (7) to the PTT system (3), we have in component form that

$$\begin{aligned} \dot{T}_{11} &= -[\lambda + \nu(T_{11} + T_{22})]T_{11} + (\mu + T_{11})\dot{\gamma}(t) \\ \dot{T}_{22} &= -[\lambda + \nu(T_{11} + T_{22})]T_{22} - (\mu + T_{22})\dot{\gamma}(t) \\ \dot{T}_{12} &= -[\lambda + \nu(T_{11} + T_{22})]T_{12} \end{aligned} \tag{8}$$

For mathematical convenience, the following notation is introduced:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} T_{11} \\ T_{22} \\ T_{12} \end{bmatrix}. \tag{9}$$

Then the system (8) can be rewritten as

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}) + \vec{g}(\vec{x})u, \tag{10}$$

where

$$\vec{f}(\vec{x}) = -[\lambda + \nu(x_1 + x_2)]\vec{x}, \ \ \vec{g}(\vec{x}) = \begin{bmatrix} \mu + x_1 \\ -(\mu + x_2) \\ 0 \end{bmatrix}, \ \ u = \dot{\gamma}(t). \tag{11}$$

The first Lie bracket is defined by

$$[\vec{f}, \vec{g}] = \nabla\vec{g} \cdot \vec{f} - \nabla\vec{f} \cdot \vec{g} = \begin{bmatrix} \nu x_1(x_1 - x_2) + \mu(\lambda + \nu x_1 + \nu x_2) \\ \nu x_2(x_1 - x_2) - \mu(\lambda + \nu x_1 + \nu x_2) \\ \nu(x_1 - x_2)x_3 \end{bmatrix}. \tag{12}$$

Similarly, the second Lie bracket is given by

$$[\vec{f}, [\vec{f}, \vec{g}]] = \nabla[\vec{f}, \vec{g}]\cdot\vec{f} - \nabla\vec{f}\cdot[\vec{f}, \vec{g}] = \begin{bmatrix} -\nu\lambda x_1(x_1 - x_2) + \mu\lambda[\lambda + \nu(x_1 + x_2)] \\ -\nu\lambda x_2(x_1 - x_2) - \mu\lambda[\lambda + \nu(x_1 + x_2)] \\ -\nu\lambda(x_1 - x_2)x_3 \end{bmatrix}. \tag{13}$$

Utilizing the vectors from $\vec{g}(\vec{x})$, the two Lie brackets (12) and (13), one contructs a matrix. By investigating the rank of this matrix, the weak controllability can be characterized. More specifically, if the matrix is full rank or nonsingular, then the system is weakly controllable; if the matrix is not full rank or singular, then this approach offers no insight to the weak controllability of the system and one has to seek more advanced tools.

We now compute the determinant and obtain

$$\det\left[\vec{g}, [\vec{f}, \vec{g}], [\vec{f}, [\vec{f}, \vec{g}]]\right] = 2\mu\lambda\nu x_3[\lambda + \nu(x_1 + x_2)](x_1 - x_2)^2. \tag{14}$$

The determinant does not equal zero when $x_3 \neq 0$, $x_1+x_2 \neq -\lambda/\nu$ and $x_1 \neq x_2$. It can be concluded that the system of the PTT model under extensional flow satisfies the CRC and thus is weakly controllable when $T_{12} \neq 0$, $T_{11} + T_{22} \neq -\lambda/\nu$ and $T_{11} \neq T_{22}$. This result can be summarized geometrically as follows.

Given $R^3 = \{(x_1, x_2, x_3)|x_1, x_2, x_3 \in R\}$, there exist three surfaces:

$$S_1 = \{(x_1, x_2, x_3)|x_3 = 0\}$$

$$S_2 = \{(x_1, x_2, x_3)|\lambda + \nu(x_1 + x_2) = 0\}$$

$$S_3 = \{(x_1, x_2, x_3)|x_1 = x_2\}$$

such that the PTT model (3) under extensional flow is weakly controllable at all points in $R^3 \setminus \{S_1 \cup S_2 \cup S_3\}$.

3 The Johnson-Segalman model

The Johnson-Segalman model characterizes the behavior of non-Newtonian fluids, including special cases of Newtonian and Maxwell fluids. Additionally, it is a viscoelastic fluid model which was developed to allow non-affine deformations [[Johnson and Segalman, 1977]].

The Johnson-Segalman (JS) model gives a viscoelastic constitutive equation in the form

$$\dot{\mathbf{T}} - \frac{a+1}{2}\left[(\nabla\mathbf{v})\mathbf{T} + \mathbf{T}(\nabla\mathbf{v})^T\right] - \frac{a-1}{2}\left[(\nabla\mathbf{v})^T\mathbf{T} + \mathbf{T}(\nabla\mathbf{v})\right] + \lambda\mathbf{T} = 2\mu\mathbf{D}. \tag{15}$$

Here a is a parameter describing polymer slip where $-1 < a < 1$; When $a = 1$, the model (15) reduces to the Oldrody-B model.

Analogous to the previous model, we now apply the gradient of the homogeneous extensional flow (6) to the JS system (15) which leads to

$$\begin{aligned} \dot{T}_{11} &= -\lambda T_{11} + (\mu + aT_{11})\dot{\gamma}(t) \\ \dot{T}_{22} &= -\lambda T_{22} - (\mu + aT_{22})\dot{\gamma}(t) \\ \dot{T}_{12} &= -\lambda T_{12} \end{aligned} \tag{16}$$

Note that the third equation in (16) can be solved exactly such that $T_{12} = T_{12}(0)e^{-\lambda t}$. Consequently, the Johnson-Segalman model (16) is not weakly controllable under extensional flow. Nonetheless, the state space has a stable invariant subspace $T_{12} = 0$. It is in this subspace that all trajectories of the system, under any control input, asymptotically move toward the subspace $T_{12} = 0$. As a result, the decisive behavior of the control system (16) can be characterized by a reduced subsystem on this stable subspace:

$$\begin{aligned} \dot{T}_{11} &= -\lambda T_{11} + (\mu + aT_{11})\dot{\gamma}(t), \\ \dot{T}_{22} &= -\lambda T_{22} - (\mu + aT_{22})\dot{\gamma}(t), \end{aligned} \tag{17}$$

which can be put in a compact form

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}) + \vec{g}(\vec{x})u,$$

where

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} T_{11} \\ T_{22} \end{bmatrix}, \ \vec{f}(\vec{x}) = -\lambda\,\vec{x}, \ \vec{g}(\vec{x}) = \begin{bmatrix} \mu + ax_1 \\ -(\mu + ax_2) \end{bmatrix}, \ u = \dot{\gamma}(t). \tag{18}$$

We compute the Lie bracket and find

$$[\vec{f}, \vec{g}] = \begin{bmatrix} \lambda\,\mu \\ -\lambda\,\mu \end{bmatrix}.$$

It follows immediately that

$$\det\left[\vec{g}, [\vec{f}, \vec{g}]\right] = a\lambda\mu(x_2 - x_1). \tag{19}$$

The determinant does not vanish when $x_1 \neq x_2$. Therefore, the subsystem (17) of the Johnson-Segalman model under extensional flow satisfies CRC and is weakly controllable when $T_{11} \neq T_{22}$. In geometrically terms, the system (17) is weakly controllable at all points in $R^2 \setminus S_1$ where

$$R^2 = \{(x_1, x_2)|x_1, x_2 \in R\}, \ S_1 = \{(x_1, x_2)|x_1 = x_2\}.$$

In [[Zhou *et al.*, 2007]] we have characterized the set of reachable states for a subsystem of the upper-convected Maxwell model under extensional flow. We have found that for the system

$$\begin{aligned} \dot{T}_{11} &= -(\lambda - \dot{\gamma}(t))T_{11} + \mu\,\dot{\gamma}(t) \\ \dot{T}_{22} &= -(\lambda + \dot{\gamma}(t))T_{22} - \mu\,\dot{\gamma}(t) \end{aligned} \tag{20}$$

the reachable set is $R_1 + R_0$ where R_1 and R_0 are given by

$$R_1 = \left\{ (T_{11}(t_f), T_{22}(t_f)) \;\middle|\; \begin{array}{l} \sqrt{(\frac{T_{11}(t_f)}{\mu}+1)(\frac{T_{22}(t_f)}{\mu}+1)} - 1 \\ > e^{-\lambda t_f}[\sqrt{(\frac{T_{11}(0)}{\mu}+1)(\frac{T_{22}(0)}{\mu}+1)} - 1] \end{array} \right\}$$

and

$$R_0 = \left\{ (T_{11}(t_f), T_{22}(t_f)) \;\middle|\; \begin{array}{l} \sqrt{(\frac{T_{11}(t_f)}{\mu}+1)(\frac{T_{22}(t_f)}{\mu}+1)} - 1 \\ = e^{-\lambda t_f}[\sqrt{(\frac{T_{11}(0)}{\mu}+1)(\frac{T_{22}(0)}{\mu}+1)} - 1] \\ \text{and} \quad T_{11}(0) = T_{22}(0),\ T_{11}(t_f) = T_{22}(t_f) \end{array} \right\}$$

Note that if we redefine the two parameters μ and $\dot{\gamma}(t)$ as $\mu_{new} = \frac{\mu_{old}}{a}$, $\dot{\gamma}_{new}(t) = a\dot{\gamma}_{old}(t)$, then the system (17) has the same form as (20). Therefore, the reachable set for the subsystem of the Johnson-Segalman model (17) under extensional flow is $R_1 + R_0$ where μ is replaced by μ/a.

4 The Giesekus Model

Another typical nonlinear viscoelastic fluid model is the Giesekus model [[Giesekus, 1982]]. The constitutive relation for the Giesekus model is given by

$$\dot{\mathbf{T}} - (\nabla\mathbf{v})\mathbf{T} - \mathbf{T}(\nabla\mathbf{v})^T + \lambda\mathbf{T} + \nu\mathbf{T}^2 = 2\mu\mathbf{D}. \tag{21}$$

Applying the gradient of the homogeneous extensional flow (6) to (21), we obtain

$$\begin{aligned} \dot{T}_{11} &= -\lambda T_{11} - \nu(T_{11}^2 + T_{12}^2) + (\mu + T_{11})\dot{\gamma}(t) \\ \dot{T}_{22} &= -\lambda T_{22} - \nu(T_{12}^2 + T_{22}^2) - (\mu + T_{22})\dot{\gamma}(t) \\ \dot{T}_{12} &= -[\lambda + \nu(T_{11} + T_{22})]T_{12} \end{aligned} \tag{22}$$

which in turn can be put in the compact form

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}) + \vec{g}(\vec{x})u. \tag{23}$$

Here

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} T_{11} \\ T_{22} \\ T_{12} \end{bmatrix},\ \vec{f}(\vec{x}) = \begin{bmatrix} -\lambda x_1 - \nu(x_1^2 + x_3^2) \\ -\lambda x_2 - \nu(x_2^2 + x_3^2) \\ -[\lambda + \nu(x_1 + x_2)]x_3 \end{bmatrix}, \tag{24}$$

$$\vec{g}(\vec{x}) = \begin{bmatrix} \mu + x_1 \\ -(\mu + x_2) \\ 0 \end{bmatrix},\ u = \dot{\gamma}(t).$$

The two Lie brackets are computed as follows:

$$[\vec{f},\vec{g}] = \nabla\vec{g}\cdot\vec{f} - \nabla\vec{f}\cdot\vec{g} = \begin{bmatrix} \nu(x_1^2 - x_3^2) + \lambda\mu + 2\nu\mu x_1 \\ \nu x_2(-x_2 - 2\mu) - \lambda\mu + \nu x_3^2 \\ \nu(x_1 - x_2)x_3 \end{bmatrix}, \qquad (25)$$

and

$$[\vec{f},[\vec{f},\vec{g}]] = \nabla[\vec{f},\vec{g}]\cdot\vec{f} - \nabla\vec{f}\cdot[\vec{f},\vec{g}] = \begin{bmatrix} \nu(x_1^2 - x_3^2)(-\lambda + 2\mu\nu) + \mu\lambda(\lambda + 2\nu x_1) \\ \nu(x_2^2 - x_3^2)(\lambda - 2\mu\nu) - \mu\lambda(\lambda + 2\nu x_2) \\ \nu x_3(x_1 - x_2)(2\nu\mu - \lambda) \end{bmatrix}. \qquad (26)$$

As a result, the determinant is found to be

$$\det\left[\vec{g},[\vec{f},\vec{g}],[\vec{f},[\vec{f},\vec{g}]]\right] = 2\mu\nu x_3(x_1 - x_2)(\lambda - \nu\mu)^2(\lambda - 2\nu\mu). \qquad (27)$$

Clearly, the determinant does not vanish when $x_3 \neq 0$, $\lambda \neq \nu\mu$, $\lambda \neq 2\nu\mu$ and $x_1 \neq x_2$. We conclude that the system of the Giesekus model under extensional flow satisfies the CRC and thus is weakly controllable when $T_{12} \neq 0$, $T_{11} \neq T_{22}$ and the parameters satisfy the constraints $\lambda \neq \nu\mu$ or $2\nu\mu$. This result can be summarized geometrically as follows.

Define two surfaces

$$S_1 = \{(x_1, x_2, x_3)|x_3 = 0\},$$
$$S_2 = \{(x_1, x_2, x_3)|x_1 = x_2\}.$$

Then the Giesekus model (21) under extensional flow becomes weakly controllable at all points in $R^3 \setminus \{S_1 \cup S_2\}$ provided that the parameters satisfy the constraints $\lambda \neq \nu\mu$ or $2\nu\mu$.

5 The Doi Model

The Doi model for rodlike liquid crystal polymers in a solvent is well-known for its capability to describe both the isotropic and nematic phases and phase transition between them. A fundamental element of the model is the single molecule orientation distribution function. Interactions between molecules are represented by a mean-field potential. The rodlike molecules are also subject to Brownian force due to the fact that they interact with other rodlike molecules and with the flow. Generally, the model is a microscopic Smoluchowski equation or Fokker-Planck type equation for the dynamics of the orientational distribution function coupled with a macroscopic hydrodynamic equation [[Doi and Edwards, 1986]]. The Smoluchowski equation depicts the convection, rotation and diffusion of the rodlike molecules.

The full Doi orientation tensor theory is developed after the kinetic Smoluchowski equation is projected onto a second-moment description using various closure rules. The major ingredient in this tensor theory is the second-moment tensor which describes the orientational distribution of the ensemble of rodlike macromolecules. The orientation tensor is traceless and symmetric. The physical and practical significance of the orientation tensor is that it is the basis for micro-scale light scattering measurements of primary axes ("directors"), degrees of molecular alignment ("birefringence"), and normal and shear stress measurements. The study of two-dimensional liquid crystal polymers has been physically inspired by monolayer films. Thin films of liquid crystal polymers are used as alignment layers for liquid crystal displays because of their stability and nonlinear optical properties. A lot of theoretical and experimental studies have been devoted to the two-dimensional Doi model (for example, see [[Lee *et al.*, 2006]] and references therein).

The two-dimensional Doi model is given by [[Lee *et al.*, 2006]]

$$\dot{\mathbf{Q}} = \mathbf{\Omega}\mathbf{Q} - \mathbf{Q}\mathbf{\Omega} + a[\mathbf{D}\mathbf{Q} + \mathbf{Q}\mathbf{D}] + a\mathbf{D} - 2a\mathbf{D} : \mathbf{Q}(\mathbf{Q} + \frac{\mathbf{I}}{2}) - 6D_r F(\mathbf{Q}), \quad (28)$$

where $\mathbf{Q}$ is the orientation tensor such that

$$\mathbf{Q} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \ Q_{11} = -Q_{22}, \ Q_{12} = Q_{21};$$

$\mathbf{\Omega}$ is the vorticity tensor such that $\mathbf{\Omega} = \frac{1}{2}[\nabla\mathbf{v} - \nabla\mathbf{v}^T]$; a is a dimensionless parameter which depends on the molecular aspect ratio; $\mathbf{D}$ is the rate-of-strain tensor; D_r is the rotary diffusivity; $F(\mathbf{Q})$ is defined by

$$F(\mathbf{Q}) = (1 - \frac{N}{2})\mathbf{Q} - N\mathbf{Q}^2 + N\mathbf{Q} : \mathbf{Q}(\mathbf{Q} + \frac{\mathbf{I}}{2})$$

and N is a dimensionless concentration of nematic polymers.

As with the previous models, the gradient of the homogeneous extensional flow (6) is applied to the Doi system (28), yielding

$$\begin{aligned} \dot{Q}_{11} &= -6D_r\left[(1 - \tfrac{N}{2})Q_{11} + 2NQ_{11}(Q_{11}^2 + Q_{12}^2)\right] + a(\tfrac{1}{2} - 2Q_{11}^2)\dot{\gamma}(t), \\ \dot{Q}_{12} &= -6D_r Q_{12}\left[(1 - \tfrac{N}{2}) + 2N(Q_{11}^2 + Q_{12}^2)\right] - 2aQ_{11}Q_{12}\dot{\gamma}(t). \end{aligned} \quad (29)$$

Using the nematic relaxation time scale $\frac{1}{D_r}$, the flow field and orientation dynamics of (28) can be non-dimensionalized. The key dimensionless parameters are then the Peclet number $Pe(t) = \dot{\gamma}(t)/D_r$ (the shear rate normalized with respect to nematic relaxation rate) and the dimensionless concentration parameter N. Rescaling time as $\bar{t} = tD_r$, the nematodynamic model (28) in the dimensionless form becomes

$$\begin{aligned} \dot{Q}_{11} &= -6Q_{11}\left[1 - \tfrac{N}{2} + 2N(Q_{11}^2 + Q_{12}^2)\right] + a(\tfrac{1}{2} - 2Q_{11}^2)Pe(t), \\ \dot{Q}_{12} &= -6Q_{12}\left[1 - \tfrac{N}{2} + 2N(Q_{11}^2 + Q_{12}^2)\right] - 2aQ_{11}Q_{12}Pe(t). \end{aligned} \quad (30)$$

The system (30) can be written as

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}) + \vec{g}(\vec{x})u,$$

where

$$\begin{aligned} \vec{x} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} Q_{11} \\ Q_{12} \end{bmatrix}, \ \vec{f}(\vec{x}) = -6\left[1 - \tfrac{N}{2} + 2N(x_1^2 + x_2^2)\right]\vec{x}, \\ \vec{g}(\vec{x}) &= \begin{bmatrix} a(\frac{1}{2} - 2x_1^2) \\ \\ -2ax_1x_2 \end{bmatrix}, \ u = Pe(t). \end{aligned} \tag{31}$$

The Lie bracket is calculated to be

$$[\vec{f}, \vec{g}] = \begin{bmatrix} 6a\left\{2x_1^2[1 - \frac{N}{2} - 2N(x_1^2 + x_2^2)] + \frac{1}{2} - \frac{N}{4} + N(3x_1^2 + x_2^2)\right\} \\ \\ 12ax_1x_2[1 + \frac{N}{2} - 2N(x_1^2 + x_2^2)] \end{bmatrix}.$$

Finally, we have

$$\det\left[\vec{g}, [\vec{f}, \vec{g}]\right] = 12a^2 x_1 x_2. \tag{32}$$

The determinant does not equal zero when $x_1 \neq 0$ and $x_2 \neq 0$. So the Doi model (30) under extensional flow satisfies CRC and is weakly controllable when $Q_{11} \neq 0$ and $Q_{12} \neq 0$. In terms of geometric words, the system (30) is weakly controllable at all points in $R^2 \setminus \{S_1 \cup S_2\}$ where

$$R^2 = \{(x_1, x_2)|x_1, x_2 \in R\}, \ S_1 = \{(x_1, x_2)|x_1 = 0\}, \ S_2 = \{(x_1, x_2)|x_2 = 0\}.$$

6 Concluding Remarks

We have applied the controllability rank condition to the vector fields in the Phan-Thien-Tanner model, the Johnson-Segalman model, the Giesekus model and the Doi model to study the controllability of non-Newtonian fluids driven by imposed homogeneous extensional flow. In our control system, the state variable is the stress for the Phan-Thien-Tanner model, the Johnson-Segalman model and the Giesekus model and the orientation tensor for the Doi model. The extensional flow rate corresponds to the available control. We have derived sufficient conditions for the weak controllability of each model.

Acknowledgment

This research was supported in part by the Air Force Office of Scientific Research and the National Science Foundation.

References

[Bird *et al.*, 1987] [1] R. B. Bird, R. C. Armstrong and O. Hassager, *Dynamics of Polymeric Liquids*, 2nd edition, Wiley (1987).

[Doi and Edwards, 1986] [2] M. Doi and S. F. Edwards, *The Theory of Polymer Dynamics*, Oxford University Press, New York, 1986.

[Giesekus, 1982] [3] H. Giesekus, "A unified approach to a variety of constitutive models for polymer fluids based on the concept of configuration dependent molecular mobility," Rhelo. Acta 21 (1982), 366-375.

[Hermann and Krener, 1977] [4] R. Hermann and A. Krener, "Nonlinear controllability and observability," IEEE Trans. Automat Contr. AC-22 (1977), 728-740.

[Isidori, 1995] [5] A. Isidori, *Nonlinear Control Systems*, 3rd edition, Springer (1995).

[Johnson and Segalman, 1977] [6] M. W. Johnson and D. Segalman, "A model for viscoelastic fluid behavior which allows non-affine deformation," J. Non-Newtonian Fluid Mech. 2 (1977), 255-270.

[Larson, 1998] [7] R. G. Larson, *The Structure and Rheology of Complex Fluids*, Oxford (1998).

[Lee *et al.*, 2006] [8] J. H. Lee, M. G. Forest and R. Zhou, "Alignment and rheo-oscillator criteria for sheared nematic polymer films in the monolayer limit," Discrete and continuous dynamical systems-series B 6 (2006), 339-356.

[Nijmeijer and van der Shaft, 1990] [9] H. Nijmeijer and A. J. van der Schaft, *Nonlinear Dynamical Control Systems*, Springer (1990).

[Phan-Thien and Tanner, 1977] [10] N. Phan-Thien and R. I. Tanner, "A new constitutive equation derived from network theory," J. Non-Newtonian Fluid Mech. 2 (1977), 353-365.

[Renardy, 2005a] [11] M. Renardy, "Are viscoelastic flows under control or out of control?" Systems & Control Letters 54 (2005a) 1183-1193.

[Renardy, 2005b] [12] M. Renardy, "Shear flow of viscoelastic fluids as a control problem," J. Non-Newtonian Fluid Mech. 131 (2005b) 59-63.

[Renardy, 2007] [13] M. Renardy, "On control of shear flow of an upper convected Maxwell fluid," Z. Angew. Math. Mech. 87 (2007) 213-218.

[Zhou *et al.*, 2007] [14] H. Zhou, W. Kang, A. Krener and H. Wang, "Homogeneous flow field effect on the control of Maxwell materials", J. Non-Newtonian Fluid Mech. 150 (2008) 104-115.

Received: January 26, 2008