Recap

- Non-dimensionalization, identifying time scale, length scale identifying small parameter,
- Asymptotic series of \( f(x, \epsilon) \)
  IVP of ODE oscillation of pendulum straightforward expansion is invalid for large \( t \).
  period of oscillation

A brief look at Laplace transform

Consider function \( f(t) \).

Laplace transform of \( f(t) \) is defined as

\[
L \left[ f(t) \right](s) \equiv \int_0^\infty e^{-st} f(t) \, dt
\]

Full notation

Other notations: \( F(s), \ L \left[ f(t) \right] \)

We start with the L-transform of \( e^{at} \).

Example:

\[
L \left[ e^{at} \right] = \int_0^\infty e^{-(s-a)t} \, dt = \frac{1}{s-a}
\]

L-transform of derivatives with respect to a parameter

\[
L \left[ \frac{d}{da} f(t,a) \right] = \frac{d}{da} L \left[ f(t,a) \right]
\]

Example:

\[
1 = \left. e^{at} \right|_{a=0}
\]
$$L[1] = \left. \frac{1}{s-a} \right|_{a=0} = \frac{1}{s}$$

Similarly, we can derive

$$L[t^n] = \frac{n!}{s^{n+1}}$$

**Example:**

$$L[\sinh(at)] = L\left[\frac{e^{iat} - e^{-iat}}{2i}\right] = \frac{1}{2i} \left( \frac{1}{s-a} - \frac{1}{s+a} \right) = \frac{a}{s^2 - a^2}$$

$$L[e^{iat}] = \frac{1}{s-ai}$$

$$L[\sin(at)] = L\left[\frac{e^{iat} - e^{-iat}}{2i}\right] = \frac{1}{2i} \left( \frac{1}{s-ai} - \frac{1}{s+ai} \right) = \frac{a}{s^2 + a^2}$$

**L-transform of derivatives with respect to independent variable**

$$L[f'(t)] = sL[f(t)] - f(0)$$

$$L[f''(t)] = s^2L[f(t)] - sf(0) - f'(0)$$

**Finding inverse L-transform using derivative with respect to a parameter**

$$L^{-1}\left[ \frac{d}{da} g(s,a) \right] = \frac{d}{da} L^{-1}[g(s,a)]$$

**Example:**

$$L^{-1}\left[ \frac{s}{(s^2 + a^2)^2} \right] = \frac{t}{2a} \sin(at)$$

$$L^{-1}\left[ \frac{1}{(s^2 + a^2)^2} \right] = \text{?} \quad \text{We use differentiation to find this one.}$$
\[
\frac{1}{(s^2+a^2)^2} = \frac{1}{-2a} \frac{d}{da} \left( \frac{1}{s^2+a^2} \right)
\]

\[
L^{-1} \left[ \frac{1}{(s^2+a^2)^2} \right] = \frac{1}{-2a} L^{-1} \left[ \frac{d}{da} \left( \frac{1}{s^2+a^2} \right) \right] = \frac{1}{-2a} \frac{d}{da} L^{-1} \left[ \frac{1}{s^2+a^2} \right]
\]

\[
= \frac{1}{-2a} \frac{d}{da} \left( \frac{1}{a} \sin(at) \right) = \frac{\sin(at)}{2a^3} - \frac{t \cos(at)}{2a^2}
\]

\[
L^{-1} \left[ \frac{1}{(s^2+a^2)^2} \right] = \frac{\sin(at)}{2a^3} - \frac{t \cos(at)}{2a^2}
\]

**Boundary Value Problem (BVP) of ODE**

**Regular perturbation**

Example:

\[
\begin{cases}
  y'' - \varepsilon y' - 9y = 0 \\
  y(0) = 0, \quad y(1) = 1
\end{cases}
\]

We seek an expansion of the form

\[
y(x, \varepsilon) = a_0(x) + \varepsilon a_1(x) + \cdots
\]

Boundary condition:

\[
y(0) = 0 \quad \Rightarrow \quad a_0(0) + \varepsilon a_1(0) + \cdots = 0
\]

\[
\Rightarrow \quad a_0(0) = 0, \quad a_1(0) = 0
\]

\[
y(1) = 1 \quad \Rightarrow \quad a_0(1) + \varepsilon a_1(1) + \cdots = 1
\]

\[
\Rightarrow \quad a_0(1) = 1, \quad a_1(1) = 0
\]

Substituting the expansion into the equation, we have

\[
\left[ a_0'' + \varepsilon a_1'' \right] - \varepsilon \left[ a_0' \right] - 9 \left[ a_0 + \varepsilon a_1 \right] = 0
\]

\[
\Rightarrow \quad \left[ a_0'' - 9a_0 \right] + \varepsilon \left[ a_1'' - 9a_1 - a_0' \right] + \cdots = 0
\]
\(\varepsilon^0:\ \left\{ \begin{array}{l} a_0'' - 9a_0 = 0 \\ a_0(0) = 0, \ a_0(1) = 1 \end{array} \right. \)

We use Laplace transform to solve it.

Let \(A(s) = L[a_0(x)]\)

Recall

\[
L[f'(x)] = sL[f(x)] - f(0)
\]

\[
L[f''(x)] = s^2L[f(x)] - sf(0) - f'(0)
\]

Taking Laplace transform of both sides, we have

\[
s^2A(s) - sa_0(0) - a_0'(0) - 9A(s) = 0
\]

Let \(\alpha = a_0'(0)\), unknown for the time being. We have

\[
(s^2 - 3^2)A(s) = \alpha
\]

\[
\Rightarrow A(s) = \alpha \frac{1}{(s-3)(s+3)}
\]

\[
\Rightarrow A(s) = \frac{\alpha}{6} \left( \frac{1}{s-3} - \frac{1}{s+3} \right)
\]

\[
\Rightarrow a_0(x) = L^{-1} \left[ A(s) \right] = \frac{\alpha}{6} \left( e^{3x} - e^{-3x} \right)
\]

Enforcing the boundary condition \(a_0(1) = 1\) to determine \(\alpha\)

\[
\Rightarrow a_0(x) = L^{-1} \left[ A(s) \right] = \frac{\alpha}{6} \left( e^{3x} - e^{-3x} \right)
\]

\(\varepsilon^1:\ \left\{ \begin{array}{l} a_1'' - 9a_1 = \frac{3}{e^3 - e^{-3}} \left( e^{3x} + e^{-3x} \right) \\ a_1(0) = 0, \ a_1(1) = 0 \end{array} \right. \)

(Skip the derivation in lecture)

We use Laplace transform to solve it.

Let \(A(s) = L[a_1(x)]\)

Taking Laplace transform of both sides, we have
\[ s^2A(s) - sa_1(0) - a_1'(0) - 9A(s) = q \left( \frac{1}{s-3} + \frac{1}{s+3} \right), \quad q = \frac{3}{e^3 - e^{-3}} \]

Let \( \alpha = a_1'(0) \). We have

\[ \Rightarrow \quad (s^2 - 3^2)A(s) = \alpha + q \left( \frac{1}{s-3} + \frac{1}{s+3} \right) \]

\[ \Rightarrow \quad A(s) = \frac{1}{6} \left( \frac{1}{s-3} - \frac{1}{s+3} \right) + \frac{q}{6} \left[ \alpha + \frac{1}{s-3} + \frac{1}{s+3} \right] \]

\[ \Rightarrow \quad A(s) = \frac{\alpha}{6} \left( \frac{1}{s-3} - \frac{1}{s+3} \right) + \frac{q}{6} \left[ \frac{1}{(s-3)^2} - \frac{1}{(s+3)^2} \right] \]

\[ \Rightarrow \quad a_1(x) = L^{-1} \left[ A(s) \right] = \frac{\alpha}{6} (e^{3x} - e^{-3x}) + \frac{q}{6} L^{-1} \left[ \frac{1}{(s-3)^2} - \frac{1}{(s+3)^2} \right] \]

Recall

\[ L \left[ e^{at} \right] = \frac{1}{s-a} \]

Differentiating with respect to \( a \)

\[ \Rightarrow \quad L \left[ t e^{at} \right] = \frac{1}{(s-a)^2} \quad L^{-1} \left[ \frac{1}{(s-a)^2} \right] = t e^{at} \]

Using this result to calculate \( a_1(x) \), we obtain

\[ a_1(x) = \frac{\alpha}{6} (e^{3x} - e^{-3x}) + \frac{q}{6} x (e^{3x} - e^{-3x}) \]

Enforcing the boundary condition \( a_1(1) = 0 \) to determine \( \alpha \)

\[ \Rightarrow \quad \alpha = -q \]

\[ \Rightarrow \quad a_1(x) = \frac{q}{6} (x-1) (e^{3x} - e^{-3x}) \quad \text{Recall} \quad q = \frac{3}{e^3 - e^{-3}} \]

\[ \Rightarrow \quad a_1(x) = \frac{1}{2(e^3 - e^{-3})} (x-1) (e^{3x} - e^{-3x}) \]

Thus, the solution of BVP has the expansion
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\[ y(x, \epsilon) \sim \frac{e^{3x} - e^{-3x}}{e^x - e^{-x}} \left[ 1 + \frac{1}{2} \epsilon (x - 1) \right] \]

The figure above compares the exact solution and asymptotic expansions for \( \epsilon = 1 \).

In comparison, the exact solution of BVP is

\[ y_{\text{ext}}(x) = \frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{e^{\lambda_1} - e^{\lambda_2}} \]

where

\[ \lambda_1 = \frac{\epsilon + \sqrt{\epsilon^2 + 36}}{2} \approx 3, \quad \lambda_1 = \frac{-\sqrt{\epsilon^2 + 36}}{2} = -3 \]

Singular perturbation

(Boundary layers and matched asymptotic expansions)

We first study the exact solution of the BVP below. The goal of examining the exact solution is to see the behavior of boundary layer and to see what we need to do to capture the singular boundary layer.

\[
\begin{cases}
\epsilon y'' + y' + y = 0 \\
y(0) = 0, \quad y(1) = 1
\end{cases}, \quad \epsilon \to 0^+ 
\]

The characteristic equation is

\[ \epsilon \lambda^2 + \lambda + 1 = 0 \]
It has two distinct roots:
\[ \lambda_1 = -1 + \frac{\sqrt{1 - 4\varepsilon}}{2\varepsilon}, \quad \lambda_2 = -1 - \frac{\sqrt{1 - 4\varepsilon}}{2\varepsilon} \]

A general solution is
\[ y_{\text{ext}}(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \]

Enforcing the boundary condition yields
\[ y_{\text{ext}}(x) = \frac{1}{e^{\lambda_1} - e^{\lambda_2}} (e^{\lambda_1 x} - e^{\lambda_2 x}) \]

Now we look at the behaviors of various terms in the exact solution as \( \varepsilon \to 0^+ \).

As \( \varepsilon \to 0^+ \), we have
\[
\begin{align*}
\lambda_1 &= -1 + \frac{\sqrt{1 - 4\varepsilon}}{2\varepsilon} = -1 + \left( \frac{1}{2} - \frac{1}{2} \varepsilon + O(\varepsilon^2) \right) \\
&= -1 + O(\varepsilon) \to -1 \\

\lambda_2 &= -1 - \frac{\sqrt{1 - 4\varepsilon}}{2\varepsilon} = -1 - \left( \frac{1}{2} - \frac{1}{2} \varepsilon + O(\varepsilon^2) \right) \\
&= -\frac{1}{\varepsilon} + O(1) \to -\infty \\

e^{\lambda_1} &= e^{-1 + O(\varepsilon)} \to e^{-1} \\
e^{\lambda_2} &= e^{-\frac{1}{\varepsilon} + O(1)} \approx 0, \quad x >> O(\varepsilon) \\
e^{\lambda_1 x} &= e^{\frac{1}{\varepsilon} - O(1)} = e^{0}, \quad x = O(\varepsilon) \\
e^{\lambda_2 x} &= e^{\frac{1}{\varepsilon} x + O(1)} = e^{\frac{1}{\varepsilon} x}, \quad 0 \leq x \leq O(\varepsilon) \\

\Rightarrow y_{\text{ext}}(x) &= \begin{cases} 
0, & x >> O(\varepsilon) \\
\frac{e^{1-x}}{e^{1/\varepsilon} \varepsilon}, & 0 \leq x \leq O(\varepsilon) 
\end{cases}
\]

The figure below shows the boundary layer in the exact solution as \( \varepsilon \to 0^+ \)
The boundary layer suggests that we need two asymptotic expansions:

- **Outer expansion**: outside the boundary layer: \( x \gg O(\varepsilon) \)

- **Inner expansion**: inside the boundary layer: \( x = O(\varepsilon) \)

Below we first use the exact solution to illustrate the outer expansion and inner expansion. Later, we will derive the outer expansion and inner expansion directly from differential equation without using the exact solution.

**Outer expansion:**

At a fixed \( x > 0 \) (outside the boundary layer),

\[
\lim_{\varepsilon \to 0^+} y_{\text{ext}}(x) = e^{1-x} \quad \text{(Here } y_{\text{ext}}(x) \text{ is the exact solution)}
\]

\[
\implies y^{(\text{out})}(x) = e^{1-x} + \ldots
\]

**Observation:**

\( y^{(\text{out})}(x) \) satisfies **only** the boundary condition at \( x = 1 \)

(at the end away from the boundary layer)

The boundary conditions are \( y(0) = 0 \) and \( y(1) = 1 \).

\( y^{(\text{out})}(x) \) satisfies **only** \( y^{(\text{out})}(1) = e^{1-x} \bigg|_{x=1} = 1 \).

At \( x = 0 \), \( y^{(\text{out})}(x) \bigg|_{x=0} = e^{1-x} \bigg|_{x=0} = e \neq 0 \).

**Inner expansion:**
From the exact solution, we know the width of boundary layer = $O(\varepsilon)$ as $\varepsilon \to 0^+$.  
In other words, as $\varepsilon \to 0^+$, the width of boundary layer $\to 0$. 

To get rid of this singularity and to capture the boundary layer as $\varepsilon \to 0^+$, we use scaling to introduce an inner variable

$$u = \frac{x}{\varepsilon}$$

In terms of $u$, the width of the boundary layer is $O(1)$.

**Notation** (Be careful! This notation may be a bit confusing):

$$y_{\text{ext}}(u) \equiv y_{\text{ext}}(x)|_{x=\varepsilon u}$$

At a fixed $u$ (inside the boundary layer)

$$\lim_{\varepsilon \to 0^+} y_{\text{ext}}(u) = \lim_{\varepsilon \to 0^+} e^{(-\varepsilon u - e^{-u})} = e(1-e^{-u})$$

$$\implies y_{\text{inn}}(u) = e(1-e^{-u}) + \ldots$$

**Observation:**

$y_{\text{inn}}(u)$ satisfies only the boundary condition at $x = 0$  
(at the boundary layer)

The boundary conditions are $y(0) = 0$ and $y(1) = 1$. 

$y_{\text{inn}}(u)$ satisfies only

$$y_{\text{inn}}\left(\frac{x}{\varepsilon}\right)|_{x=0} = e\left(1 - e^{-\frac{x}{\varepsilon}}\right)|_{x=0} = 0.$$

At $x = 1$,  

$$y_{\text{inn}}\left(\frac{x}{\varepsilon}\right)|_{x=1} = e\left(1 - e^{-\frac{x}{\varepsilon}}\right)|_{x=1} = e \neq 1.$$

Let us summarize what we learned from examining the exact solution.

**Summary:**

We need two expansions:

$y_{\text{out}}(x)$: outside the boundary layer and

$y_{\text{inn}}(u)$: inside the boundary layer with inner variable $u = \frac{x}{\varepsilon}$.

Only one boundary condition is imposed on each of $y_{\text{out}}(x)$ and $y_{\text{inn}}(u)$.

$y_{\text{out}}(x)$ satisfies only the boundary condition away from the boundary layer. 

$y_{\text{inn}}(u)$ satisfies only the boundary condition at the boundary layer.
Question: In \( \varepsilon y'' + y' + y = 0 \) with \( y(0) = 0 \), \( y(1) = 1 \), what happens if \( \varepsilon \to 0 \)?

We again use the exact solution to answer this question.

\[
y_{\text{ext}}(x) = \frac{e^{\lambda_2 x} - e^{\lambda_1 x}}{e^{\lambda_2} - e^{\lambda_1}}
\]

As \( \varepsilon \to 0 \), we have

\[
\lambda_1 = \frac{-1 + \sqrt{1 - 4\varepsilon}}{2\varepsilon} = -1 + \frac{1 - \frac{1}{2} \cdot 4\varepsilon + O(\varepsilon^2)}{2\varepsilon} \\
= -1 + O(\varepsilon) \to -1
\]

\[
\lambda_2 = \frac{-1 - \sqrt{1 - 4\varepsilon}}{2\varepsilon} = -1 - \frac{1 - \frac{1}{2} \cdot 4\varepsilon + O(\varepsilon^2)}{2\varepsilon} \\
= -\frac{1}{\varepsilon} + O(1) \to +\infty
\]

We re-write the exact solution as

\[
y_{\text{ext}}(x) = e^{\lambda_2 x} - e^{\lambda_1 x} = e^{\lambda_2(x-1)} - e^{\lambda_1 x} - \lambda_2 \\
e^{\lambda_1 - \lambda_2} \to 0 \\
e^{\lambda_1 x - \lambda_2} \to 0
\]

\[
e^{\lambda_2(x-1)} = e^{\left(-\frac{1}{\varepsilon} + O(1)\right)(x-1)} \approx \begin{cases} 
0, & (1-x) >> O(\varepsilon) \\
e^{-\frac{1}{\varepsilon}(x-1)}, & (1-x) = O(\varepsilon)
\end{cases}
\]

\[
\implies y_{\text{ext}}(x) \approx \begin{cases} 
0, & (1-x) >> O(\varepsilon) \\
e^{-\frac{1}{\varepsilon}(x-1)}, & (1-x) = O(\varepsilon)
\end{cases}
boundary layer
\]

The figure below shows the boundary layer in the exact solution as \( \varepsilon \to 0 \).

Remark:
\( \varepsilon \to 0^+ \) and \( \varepsilon \to 0^- \) lead to very different solutions.

In particular, the boundary layer is at different locations.

Now we derive asymptotic expansions directly from the differential equation without using the exact solution.

We proceed with two assumptions:

1) we know that the boundary layer is at \( x = 0 \) and
2) we know that the width of the boundary layer is \( O(\varepsilon) \).

Later on, we will discuss how to find the location and the width of a boundary layer.

Example

(Deriving the outer expansion, the inner expansion, and the composite expansion):

\[
\begin{cases}
\varepsilon y'' + y' + y = 0 \\
y(0) = 0, \quad y(1) = 1
\end{cases} \quad \varepsilon \to 0^+
\]

Outer expansion (outside the boundary layer)

We seek an expansion of the form

\[
y^{(\text{out})}(x) = a_0(x) + \varepsilon a_1(x) + \cdots
\]

On \( y^{(\text{out})}(x) \), we enforce the boundary condition away from the boundary layer.

Boundary condition at \( x = 1 \): \( y(1) = 1 \)
$$\Rightarrow a_0(1) + \varepsilon a_1(1) + \cdots = 1$$

$$\Rightarrow a_0(1) = 1, \quad a_1(1) = 0$$

**Note:** When the boundary layer is at \(x = 0\), only the boundary condition at \(x = 1\) is imposed on the outer expansion.

Substituting into equation

$$\varepsilon(\alpha'' + \cdots) + (\alpha' + \varepsilon \alpha' + \cdots) + (\alpha_0 + \varepsilon a_1 + \cdots) = 0$$

$$\Rightarrow \begin{bmatrix} \alpha' + a_0 \\ \alpha_0 \end{bmatrix} + \varepsilon \begin{bmatrix} \alpha' + a_1 + a_0'' \\ \alpha_0 \end{bmatrix} + \cdots = 0$$

\(\varepsilon^0:\)

- \(\alpha' + a_0 = 0\)
- \(a_0(1) = 1\)

$$\Rightarrow a_0(x) = e^{1-x}$$

\(\varepsilon^1:\)

- \(\alpha' + a_1 = -a_0'' = -e^{1-x}\)
- \(a_1(1) = 0\)

(Skip the derivation in lecture.)

We use the Laplace transform to solve it.

Let \(A(s) = L[a_1(x)]\)

Taking Laplace transform of both sides yields

$$sA(s) - a_1(0) + A(s) = \frac{-e}{s+1}$$

Let \(\alpha = a_1(0)\). We have

$$\left(s+1\right)A(s) = \alpha - \frac{e}{s+1}$$

$$\Rightarrow A(s) = \frac{\alpha}{s+1} - \frac{e}{(s+1)^2}$$

$$\Rightarrow a_1(x) = L^{-1}\left[A(s)\right] = \alpha e^{-x} - xe^{1-x}$$

Imposing condition \(a_1(1) = 0\)

$$\Rightarrow a_1(x) = (1-x) e^{1-x}$$
The outer expansion is
\[ y^{(\text{out})}(x) \sim e^{1-x} \left[ 1 + \varepsilon (1-x) \right] \]

**Inner expansion** (inside the boundary layer)

Rescaling: consider the inner variable
\[ u = \frac{x}{\varepsilon} \]

Note: The relation between the inner variable \( u \) and original variable \( x \) depends on 1) the location and 2) the width of the boundary layer.

**Notation** (may be a bit confusing):
\[ y(u) \equiv y(x) \big|_{x=\varepsilon u} \]

Let us derive the differential equation for \( y(u) \).
\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{1}{\varepsilon} \\
\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} \cdot \frac{1}{\varepsilon^2}
\]

Substituting into equation, we have
\[ \varepsilon \cdot \frac{1}{\varepsilon^2} \frac{d^2y}{du^2} + \frac{1}{\varepsilon} \frac{dy}{du} + y = 0 \]
\[ \Rightarrow y''(u) + y'(u) + \varepsilon y(u) = 0 \]

Remark: In terms of inner variable \( u \), it is a regular perturbation problem.

We seek an expansion of the form
\[ y^{(\text{inn})}(u) = a_0(u) + \varepsilon a_1(u) + \cdots \]

On \( y^{(\text{inn})}(u) \), we enforce the boundary condition at the boundary layer.

Boundary condition at \( u = 0 \): \( y(0) = 0 \)
\[ \Rightarrow a_0(0) + \varepsilon a_1(0) + \cdots = 0 \]
\[ \Rightarrow a_0(0) = 0, \quad a_1(0) = 0 \]

Note: Only the boundary condition at the boundary layer is imposed on the inner expansion.

Substituting the expansion into equation
\[
\left( a_0'' + \varepsilon a_1'' + \cdots \right) + \left( a_0' + \varepsilon a_1' + \cdots \right) + \varepsilon \left( a_0 + \cdots \right) = 0
\]

\[
\begin{align*}
\Rightarrow & \quad \left[ a_0'' + a_0' \right] + \varepsilon \left[ a_1'' + a_1' + a_0 \right] + \cdots = 0
\end{align*}
\]

\[\varepsilon^0: \begin{cases} 
    a_0'' + a_0' = 0 \\
    a_0(0) = 0
\end{cases}\]

Result #1

The solution of \[\begin{cases} 
    a_0'' + a_0' = 0 \\
    a_0(0) = 0
\end{cases}\] is

\[a_0(u) = c_0 \left( 1 - e^{-u} \right)\]

(See Appendix A for the derivation).

\[\varepsilon^1: \begin{cases} 
    a_1'' + a_1' = -a_0 = -c_0 \left( 1 - e^{-u} \right) \\
    a_1(0) = 0
\end{cases}\]

Result #2

The solution of \[\begin{cases} 
    a_1'' + a_1' = -a_0 = -c_0 \left( 1 - e^{-u} \right) \\
    a_1(0) = 0
\end{cases}\] is

\[a_1(u) = c_1 \left( 1 - e^{-u} \right) - c_0 u \left( 1 + e^{-u} \right)\]

(See Appendix B for the derivation).

Thus, the inner expansion is

\[y^{\text{inn}}(u) = c_0 \left( 1 - e^{-u} \right) + \varepsilon \left[ c_1 \left( 1 - e^{-u} \right) - c_0 u \left( 1 + e^{-u} \right) \right]\]

where coefficients \(c_0\) and \(c_1\) are to be determined in matching.

Matching

\(y^{\text{out}}(x)\) and \(y^{\text{inn}}(u)\) need to be matched in the transition zone between the boundary layer and the outer region.
To facilitate the discussion, we write the outer expansion and inner expansion in the general abstract form.

**Outer expansion:**

\[ y^{(\text{out})}(x) = y^{(\text{out})}_0(x) + \varepsilon y^{(\text{out})}_1(x) + \cdots \]

**Inner expansion:**

\[ y^{(\text{inn})}(\varepsilon x) = y^{(\text{inn})}_0(\varepsilon x) + \varepsilon y^{(\text{inn})}_1(\varepsilon x) + \cdots \]

We are going to look at three matching methods.

- **Prandtl's matching:**
  
  It works only for the leading term.

- **Matching by an intermediate variable:**
  
  It does not produce a composite expansion.

- **Van Dyke's matching**
  
  It works for multiple terms and produces a composite expansion.

1) **Prandtl's matching (It works only for the leading term!)**

We require

\[ \lim_{x \to 0} y^{(\text{out})}_0(x) = \lim_{\varepsilon \to \infty} y^{(\text{inn})}_0(\varepsilon x) \]

We like to have a unified expansion that is valid both inside the boundary layer and outside the boundary layer.

Let

\[ y^{(m)} = \lim_{x \to 0} y^{(\text{out})}_0(x) = \lim_{\varepsilon \to \infty} y^{(\text{inn})}_0(\varepsilon x) \]

**Note:** \( y^{(m)} \) is a scalar, not a function.

The composite expansion:

\[ y^{(c)}(x) = y^{(\text{out})}_0(x) + \left( y^{(\text{inn})}_0(\varepsilon x) - y^{(m)} \right)_{\varepsilon x = \frac{x}{\varepsilon}} \]

**Remarks:**

1) In the composite expansion, it is more convenient if we first calculate \( y^{(\text{inn})}_0(\varepsilon x) - y^{(m)} \) and then evaluate it at \( u = \frac{x}{\varepsilon} \).

2) The composite expansion is one expression valid for the whole interval \([0, 1]\).

Outside the boundary layer, we have \( u = \frac{x}{\varepsilon} >> 1 \) and
\[ y_0^{(\text{inn})}(u) \approx y^{(m)} \]
\[ y^{(c)}(x) = y_0^{(\text{out})}(x) + \left(y_0^{(\text{inn})}(u) - y^{(m)}\right) \bigg|_{u=\frac{x}{\varepsilon}} \approx y_0^{(\text{out})}(x) \]

Inside the boundary layer, we have \( x = O(\varepsilon) \to 0 \) and
\[ y_0^{(\text{out})}(x) = y^{(m)} \]
\[ y^{(c)}(x) = y_0^{(\text{out})}(x) + \left(y_0^{(\text{inn})}(u) - y^{(m)}\right) \bigg|_{u=\frac{x}{\varepsilon}} \approx y_0^{(\text{inn})}\left(\frac{x}{\varepsilon}\right) \]

Now let us apply Prandtl’s matching to the example we are working on.

Recall that in the example we obtained
\[ y_0^{(\text{out})}(x) = e^{1-x} \]
\[ y_0^{(\text{inn})}(u) = c_0(1 - e^{-u}) \]

Calculating the limits, we have
\[ \lim_{x \to 0} y_0^{(\text{out})}(x) = e \]
\[ \lim_{u \to \infty} y_0^{(\text{inn})}(u) = c_0 \]

Prandtl’s matching condition:
\[ \lim_{x \to 0} y_0^{(\text{out})}(x) = \lim_{u \to \infty} y_0^{(\text{inn})}(u) \]
\[ \implies c_0 = e \]

The composite expansion:
\[ y^{(m)} = e \]
\[ \implies y_0^{(\text{inn})}(u) - y^{(m)} = -e^{1-u} \]
\[ \implies y^{(c)}(x) = y_0^{(\text{out})}(x) + \left(y_0^{(\text{inn})}(u) - y^{(m)}\right) \bigg|_{u=\frac{x}{\varepsilon}} = e^{1-x} - e^{\frac{1-x}{\varepsilon}} \]

Finally, we arrive at the leading term composite expansion:
\[ y^{(c)}(x) = e^{1-x} - e^{\frac{1-x}{\varepsilon}} \]
Appendix A

Derivation of Result #1:

The solution of
\[
\begin{align*}
\frac{d^2 a}{dt^2} + a' &= 0 \\
\frac{d a}{dt}(0) &= 0
\end{align*}
\]

is

\[
a(0) = c_0 (1 - e^{-u})
\]

Solution:

Let \( A(s) = L[a(0)] \)

Taking Laplace transform of both sides, we have

\[
s^2 A(s) - sa(0) - a'(0) + sA(s) - a(0) = 0
\]

Let \( c_0 = a'(0) \). We write \( A(s) \) as

\[
\begin{align*}
\Rightarrow \quad (s^2 + s) A(s) &= c_0 \\
\Rightarrow \quad A(s) &= c_0 \frac{1}{s(s+1)} \\
\Rightarrow \quad A(s) &= c_0 \left[ \frac{1}{s} - \frac{1}{s+1} \right] \\
\Rightarrow \quad a(0) &= L^{-1}[A(s)] = c_0 (1 - e^{-u})
\end{align*}
\]

Appendix B

Derivation of Result #2:

The solution of
\[
\begin{align*}
\frac{d^2 a}{dt^2} + a' &= -c_0 (1 - e^{-u}) \\
\frac{d a}{dt}(0) &= 0
\end{align*}
\]

is

\[
a(0) = c_1 (1 - e^{-u}) - c_0 u(1 + e^{-u})
\]

Solution:

Let \( A(s) = L[a(0)] \)

Taking Laplace transform of both sides, we have
\[ s^2A(s) - sa_1(0) - a_1'(0) + sA(s) - a_1(0) = -c_0 \left[ \frac{1}{s} - \frac{1}{s+1} \right] \]

Let \( c_1 = a_1'(0) \). We write \( A(s) \) as

\[
(\text{for } s^2 + s)A(s) = c_1 - c_0 \left[ \frac{1}{s} - \frac{1}{s+1} \right]
\]

\[
A(s) = c_1 \left[ \frac{1}{s} - \frac{1}{s+1} \right] - c_0 \left[ \frac{1}{s^2} - \frac{2}{s(s+1)} + \frac{1}{(s+1)^2} \right]
\]

\[
= (2c_0 + c_1) \left[ \frac{1}{s} - \frac{1}{s+1} \right] - c_0 \left[ \frac{1}{s^2} + \frac{1}{(s+1)^2} \right]
\]

(Renaming \( 2c_0 + c_1 \) as \( c_1 \))

\[
= c_1 \left[ \frac{1}{s} - \frac{1}{s+1} \right] - c_0 \left[ \frac{1}{s^2} + \frac{1}{(s+1)^2} \right]
\]

\[
\Rightarrow a_1(u) = L^{-1} \left[ A(s) \right] = c_1 (1 - e^{-u}) - c_0 u (1 + e^{-u})
\]

In the above, we have used

\[
L \left[ e^{-au} \right] = \frac{1}{s+a}
\]

\[
\Rightarrow L^{-1} \left[ \frac{1}{s+a} \right] = e^{-au}
\]

Differentiating with respect to \( a \), we obtain

\[
L \left[ u e^{-au} \right] = \frac{1}{(s+a)^2}
\]

\[
\Rightarrow L^{-1} \left[ \frac{1}{(s+a)^2} \right] = u e^{-au}
\]

Note: Differentiating with respect to a parameter is a good way of deriving more formulas for the forward Laplace transform and for the inverse Laplace transform.