1. Use a perturbation method to solve the BVP (boundary value problem)
\[
\begin{cases}
\varepsilon y'' + y' + y = 0 \\
y(0) = e, \ y(1) = 0 \quad \varepsilon \to 0
\end{cases}
\]
Find the first two terms (i.e., O(1) term and O(\varepsilon) term) in the composite expansion.

2. Use a perturbation method to solve the BVP (boundary value problem)
\[
\begin{cases}
\varepsilon y'' + y' + e^y = 0 \\
y(0) = 1, \ y(1) = -\ln 2 \quad \varepsilon \to 0
\end{cases}
\]
Find the leading term in the composite expansion.

Hint: First find a general solution of \( y' + e^y = 0 \).

3. Recall the example we studied in lecture
\[
\begin{cases}
\varepsilon y'' + y' + y = 0 \\
y(0) = 0, \ y(1) = 1 \quad \varepsilon \to 0
\end{cases}
\]
The leading term composite expansion is
\[
y_a(x) = e^{1-x} - e^{1-x/\varepsilon}
\]
The exact solution is
\[
y_e(x) = \frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{e^{\lambda_1} - e^{\lambda_2}}
\]
where
\[
\lambda_1 = -1 + \sqrt{1 - 4\varepsilon}, \quad \lambda_2 = -1 - \sqrt{1 - 4\varepsilon}
\]
Consider the difference between the exact solution and the asymptotic solution
\[
\left| y_e(x) - y_a(x) \right|
\]
For \( \varepsilon = 2^{-3}, 2^{-4}, 2^{-5}, ..., 2^{-25} \), calculate numerically
\[ x(\epsilon) = \arg \max_x \left| y_e(x) - y_a(x) \right| \]
\[ \text{err}(\epsilon) = \max_x \left| y_e(x) - y_a(x) \right| \]

Use loglog to plot \( x(\epsilon) \) as a function of \( \epsilon \).
Use loglog to plot \( \text{err}(\epsilon) \) as a function of \( \epsilon \).
Use semilogx to plot \( (\text{err}(\epsilon)/\epsilon) \) as a function of \( \epsilon \). Is \( \text{err}(\epsilon) \) proportional to \( \epsilon \)?

4. (Optional) In problem 2 above, find the first two terms (i.e., \( O(1) \) term and \( O(\epsilon) \) term) in the composite expansion.

Note: The \( O(\epsilon) \) term in the expansion will involve the integral \( \int_0^\epsilon \frac{\exp(v)-1}{v} dv \).