1. Use a perturbation method to solve the BVP (boundary value problem)
\[ \begin{cases} \varepsilon y'' + y' + y = 0 \\ y(0) = e, \quad y(l) = 0 \end{cases}, \quad \varepsilon \to 0_+ \]

Find the first two terms (i.e., $O(1)$ term and $O(\varepsilon)$ term) in the composite expansion.

2. Use a perturbation method to solve the BVP (boundary value problem)
\[ \begin{cases} \varepsilon y'' + y' + e^y = 0 \\ y(0) = 1, \quad y(l) = -\ln 2 \end{cases}, \quad \varepsilon \to 0 + \]

Find the leading term in the composite expansion.

Hint: First find a general solution of $y' + e^y = 0$.

3. Recall the example we studied in lecture
\[ \begin{cases} \varepsilon y'' + y' + y = 0 \\ y(0) = 0, \quad y(l) = 1' \end{cases}, \quad \varepsilon \to 0 + \]

The leading term composite expansion is
\[ y_a(x) = e^{1-x} - e^{\frac{1-x}{\varepsilon}} \]

The exact solution is
\[ y_e(x) = \frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{e^{\lambda_1} - e^{\lambda_2}} \]

where
\[ \lambda_1 = -1 + \frac{\sqrt{1 - 4\varepsilon}}{2\varepsilon}, \quad \lambda_2 = -1 - \frac{\sqrt{1 - 4\varepsilon}}{2\varepsilon} \]

Consider the difference between the exact solution and the asymptotic solution
\[ |y_e(x) - y_a(x)| \]

For $\varepsilon = 2^{-3}, 2^{-4}, 2^{-5}, \cdots, 2^{-25}$, calculate numerically
\[ x(\varepsilon) = \arg \max_x |y_e(x) - y_a(x)| \]
\[ err(\varepsilon) = \max_x |y_e(x) - y_a(x)| \]
Use loglog to plot $x(\epsilon)$ as a function of $\epsilon$.

Use loglog to plot $err(\epsilon)$ as a function of $\epsilon$.

Use semilogx to plot $(err(\epsilon)/\epsilon)$ as a function of $\epsilon$. Is $err(\epsilon)$ proportional to $\epsilon$?

4. (Optional) In problem 2 above, find the first two terms (i.e., $O(1)$ term and $O(\epsilon)$ term) in the composite expansion.

Note: The $O(\epsilon)$ term in the expansion will involve the integral $\int_0^\epsilon \frac{e^{-u}}{v} \exp(v) - 1 \, dv$. 