Recap of Lecture 1:

- Bisection method: it has guaranteed convergence
- Newton’s method: it may diverge

Questions:

- Question #1: When does Newton’s method converge?
- Question #2: If it converges, does it converge to a solution of $f(x) = 0$?

To answer these questions, we study a class of methods, called fixed point iterative methods.

Fixed point iterative methods (for solving $f(x) = 0$)

The goal is to find one solution of $f(x) = 0$.

Strategy:

Start with $x_0$ (an initial approximation to a solution of $f(x) = 0$)

Use a mapping $g(x)$ to improve the approximation

\[
\begin{align*}
  x_1 &= g(x_0) \\
  x_2 &= g(x_1) \\
  &\vdots \\
  x_{n+1} &= g(x_n) \\
  &\vdots
\end{align*}
\]

This class of methods is called fixed point iterative methods.

$g(x)$ is called the iteration function.

As we will see, Newton’s method is a fixed point iterative method.

We study two issues: consistency and convergence.

Consistency (addressing Question #2)

Suppose the iteration function $g(x)$ is continuous.

Suppose the iteration $x_{n+1} = g(x_n)$ converges.
\[ \lim_{n \to \infty} x_n = x^* \]

Taking limits as \( n \) goes to infinity, at both sides of \( x_{n+1} = g(x_n) \), we have

\[ \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} g(x_n) \]

Since function \( g(x) \) is continuous, we obtain

\[ x^* = g(x^*) \]

\( x^* \) with this property has a special name.

**Definition:**

If \( x^* \) satisfies \( x^* = g(x^*) \), then \( x^* \) is called a fixed point of \( g(x) \).

In the above, we derived that

if \( x_{n+1} = g(x_n) \) converges, it converges to a fixed point of \( g(x) \).

**Question:** How can we make sure that if \( x_{n+1} = g(x_n) \) converges, it converges to a solution of \( f(x) = 0 \)?

**Answer:** Consistency condition

**Consistency condition:**

If all fixed points of \( g(x) \) are solutions of \( f(x) = 0 \), then we say the fixed point iterative method \( x_{n+1} = g(x_n) \) satisfies the consistency condition.

The consistency condition guarantees that if \( x_{n+1} = g(x_n) \) converges, it converges to a solution of \( f(x) = 0 \).

**Example:**

Newton’s method is a fixed point iterative method and satisfies the consistency condition.

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

We write it in the form of fixed point iterative methods:

\[ x_{n+1} = g(x_n), \]

\[ g(x) = x - \frac{f(x)}{f'(x)} \]

Let us check the consistency condition.

Let \( x^* \) be a fixed point of \( g(x) \).

\[ x^* = g(x^*) \]
\[ \Rightarrow \quad x^* = x^* - \frac{f(x^*)}{f'(x^*)} \]

\[ \Rightarrow \quad 0 = -\frac{f(x^*)}{f'(x^*)} \]

\[ \Rightarrow \quad f(x^*) = 0 \]

\[ \Rightarrow \quad \text{Newton’s method satisfies the consistency condition.} \]

Therefore, we answered Question #2 for Newton’s method.

**Answer to Question 2:**

If Newton’s method converges, it converges to a solution of \( f(x) = 0 \).

Now we discuss Question #1. In the discussion below, we assume

*) \( x_{n+1} = g(x_n) \) satisfies the consistency condition, and

*) \( g(x) \) is continuously differentiable.

**Convergence** (addressing Question #1)

Let us rephrase the question.

**Question:** Under what condition does the iteration \( x_{n+1} = g(x_n) \) converge?

Let \( x^* \) be a fixed point of \( g(x) \).

(Because of the consistency condition, \( x^* \) is also a solution of \( f(x) = 0 \)). We have

\[ x_{n+1} = g(x_n) \]
\[ x^* = g(x^*) \]

\[ \Rightarrow \quad x_{n+1} - x^* = g(x_n) - g(x^*) \]

We want to relate \( |x_{n+1} - x^*| \) to \( |x_n - x^*| \).

Recall the mean value theorem:

\[ g(b) - g(a) = g'(c)(b - a), \quad c \in (a, b) \]

Applying the mean value theorem to \( g(x) \), we get

\[ g(x_n) - g(x^*) = g'(\tilde{x}_n)(x_n - x^*) \]

where \( \tilde{x}_n \) is between \( x_n \) and \( x^* \).

\[ \Rightarrow \quad |x_{n+1} - x^*| = |g'(\tilde{x}_n)| \cdot |x_n - x^*| \]
Using this equation, we discuss 3 cases.

**Case #1:** Suppose \( |g'(x)| \leq q < 1 \) for all values of \( x \).

\[
\Rightarrow |x_{n+1} - x^*| \leq q |x_n - x^*|
\]
\[
\Rightarrow |x_n - x^*| \leq q |x_{n-1} - x^*| \leq q^2 |x_{n-2} - x^*| \leq q^3 |x_{n-3} - x^*| \leq \cdots
\]
\[
\Rightarrow \lim_{n \to \infty} |x_n - x^*| = 0
\]
\[
\Rightarrow \lim_{n \to \infty} x_n = x^* \quad \text{for all values of } x_0
\]

**Conclusion for case #1:**

Suppose \( |g'(x)| \leq q < 1 \) for all values of \( x \). Then the iteration \( x_{n+1} = g(x_n) \) converges to \( x^* \) for all values of \( x_0 \).

Mappings satisfying this strong condition have a special name.

We take a short digression to introduce “contraction mapping” before discussing case #2.

**Definition:**

If \( |g'(x)| \leq q < 1 \) for all values of \( x \), then \( g(x) \) is called a contraction mapping.

**Question:** Why is it called a contraction mapping?

**Answer:** The distance between the images of two points is smaller than the distance between the two points.

\[
|g(b) - g(a)| = |g'(c)| |b - a|
\]
\[
\Rightarrow |g(b) - g(a)| \leq q |b - a|
\]

**Theorem:**

A contraction mapping has one and only one fixed point.

**Case #2:** Suppose \( |g'(x^*)| < 1 \) at fixed point \( x^* \).

Let \( q = \frac{|g'(x^*)| + 1}{2} < 1 \).

(Draw the real axis to show \( |g'(x^*)|, q \) and 1).

\[
\Rightarrow |g'(x^*)| < q < 1
\]
(recall that \(g(x)\) is continuously differentiable and thus, \(g'(x)\) is continuous).

\[\Rightarrow\] There exists \(\delta > 0\) such that \(|g'(x)| \leq q < 1\) for \(|x - x^*| \leq \delta\).

\[\text{We start with } x_0 \text{ satisfying } |x_0 - x^*| \leq \delta.\]

\[|x_1 - x^*| = |g'(\tilde{x}_0)||x_0 - x^*|\]

\(\tilde{x}_0\) is between \(x_0\) and \(x^*\).

\[\Rightarrow\] \(|\tilde{x}_0 - x^*| \leq \delta\]

\[\Rightarrow\] \(|g'(\tilde{x}_0)| \leq q\]

\[\Rightarrow\] \(|x_1 - x^*| \leq q|x_0 - x^*|\]

Noticing that \(|x_1 - x^*| \leq \delta\). So we can repeat the process

\[\Rightarrow\] \(|x_2 - x^*| = |g'(\tilde{x}_1)||x_1 - x^*| \leq q|x_1 - x^*| \leq q^2|x_0 - x^*|\]

\[\Rightarrow\] \(|x_3 - x^*| = |g'(\tilde{x}_2)||x_2 - x^*| \leq q|x_2 - x^*| \leq q^3|x_0 - x^*|\]

\[\vdots\]

\[\Rightarrow\] \(|x_n - x^*| \leq q^n|x_0 - x^*|\]

Therefore, \(\lim_{n \to \infty} x_n = x^*\) if we start with \(x_0\) satisfying \(|x_0 - x^*| \leq \delta\)

Conclusion for case #2:

Suppose \(|g'(x^*)| < 1\) at a fixed point \(x^*\). Then the iteration \(x_{n+1} = g(x_n)\) converges to \(x^*\) if \(x_0\) is sufficiently close to \(x^*\).

\(\text{(Go through sample codes in assignment #1)}\)

\(\text{(Computer illustration of divergence of Newton’s method)}\)