AMS11B: Mathematical Methods for Economists II

Solutions of Assignment #8

Problems from supplementary note #4

Problem 1

Price (p) as a function of output quantity (q)

$p = 100 - 2q$

<u>Total cost</u> (c) as a function of quantity

$c = 20q + 300$

Profit (π) as a function of quantity

$\pi(q)$ = price × quantity – (total cost)

$= p(q) \times q - c(q) = (100 - 2q) \times q - (20q + 300)$

$= -2q^2 + 80q - 300$

(a) Find critical point(s) of $\pi(q)$

$\pi'(q) = -4q + 80 = 0$

==> $q^* = 20$, this is the profit maximizing quantity.

Second derivative test:

$\pi''(q) = -4 < 0$

==> $\pi(q)$ has a maximum at $q^* = 20$.

The profit maximizing price and the maximum profit are

$p^* = 100 - 2q^* = 60$

$\pi^* = \pi(q^*) = -2(q^*)^2 + 80q^* - 300 = 500$

(b) Let α be the marginal cost. The total cost is

$c = \alpha q + 300$

Profit as a function of quantity is

$\pi(q; \alpha) = (100-2q)\times q-(\alpha q+300) = -2q^2 + (100-\alpha)q-300$

Let $\pi^*(\alpha)$ be the maximum profit at marginal cost α.

We estimate the change in π^* when α changes from $\alpha_0 = 20$ to $\alpha_0+\Delta\alpha = 20.75$.

The linear Taylor approximation is

$\Delta\pi^* = \pi^*(\alpha_0+\Delta\alpha)-\pi^*(\alpha_0) \approx \pi'(\alpha_0)\Delta\alpha$

The envelope theorem gives us

$$\pi'(\alpha) = \left.\frac{\partial \pi}{\partial \alpha}\right|_{q=q^*(\alpha)} = -q^*(\alpha)$$

In (a), we obtained that at $\alpha_0 = 20$, we have $q^*(\alpha_0) = 20$ and $\pi^*(\alpha_0) = 500$.

==> $\pi'(\alpha_0) = -q^*(\alpha_0) = -20$

Using this result in the linear Taylor approximation, we obtain

$\Delta\pi^* \approx \pi'(\alpha_0)\Delta\alpha = -20\times0.75 = -15$

==> $\pi^*(20.75) \approx 500-15 = 485$

(c) We maximize function $\pi(q; \alpha) = -2q^2 + (100-\alpha)q-300$ exactly

Find critical point(s) of $\pi(q)$

$\pi'(q) = -4q + (100-\alpha) = 0$

==> $q^* = (100-\alpha)/4$, this is the profit maximizing quantity

Second derivative test:

$\pi''(q) = -4 < 0$

==> $\pi(q)$ has a maximum at $q^* = (100-\alpha)/4$.

The maximum profit is

$\pi^*(\alpha) = \pi(q^*) = -2((100-\alpha)/4)^2 + (100-\alpha)(100-\alpha)/4 - 300$

$= (100-\alpha)^2/8 - 300$

At $\alpha = 20.75$

$\pi^*(20.75) = (100-20.75)^2/8 - 300 = 485.07$

This is very close to the estimate obtained in (b), $\pi^*(20.75) \approx 485$.

Problem 2

The joint demand functions for the two products are

$Q_A = 100 - 3P_A + 2P_B$

$Q_B = 60 + 2P_A - 2P_B$

The total cost as a function of two prices is

$C = 20Q_A + 30Q_B + 1200$

$= 20(100 - 3P_A + 2P_B) + 30(60 + 2P_A - 2P_B) + 1200$

$= 5000 - 20P_B$

Profit (π) as a function of two prices is

$\pi(P_A, P_B) = P_A\times Q_A + P_B\times Q_B - C$

$= P_A(100 - 3P_A + 2P_B) + P_B(60 + 2P_A - 2P_B) - (5000 - 20P_B)$

$= -3P_A^2 + 4P_AP_B - 2P_B^2 + 100P_A + 80P_B - 5000$

(a) Find critical point(s) of $\pi(P_A, P_B)$

$$\frac{\partial \pi(P_A, P_B)}{\partial P_A} = -6P_A + 4P_B + 100 = 0 \qquad (1)$$

$$\frac{\partial \pi(P_A, P_B)}{\partial P_B} = 4P_A - 4P_B + 80 = 0 \qquad (2)$$

(1) + (2) ==> $-2P_A + 180 = 0$ ==> $P_A^* = 90$

(2) ==> $P_B^* = P_A^* + 20 = 110$

$(P_A^*, P_B^*) = (90, 110)$, these are the profit maximizing prices.

Second derivative test:

$$\frac{\partial^2 \pi(P_A, P_B)}{\partial P_A^2} = -6 < 0, \qquad \frac{\partial^2 \pi(P_A, P_B)}{\partial P_A \partial P_B} = 4, \qquad \frac{\partial^2 \pi(P_A, P_B)}{\partial P_B^2} = -4$$

$$D = \frac{\partial^2 \pi(P_A, P_B)}{\partial P_A^2}\frac{\partial^2 \pi(P_A, P_B)}{\partial P_B^2} - \left(\frac{\partial^2 \pi(P_A, P_B)}{\partial P_A \partial P_B}\right)^2 = (-6)(-4) - 4^2 = 8 > 0$$

==> $\pi(P_A, P_B)$ has a maximum at $(P_A^*, P_B^*) = (90, 110)$.

The profit maximizing quantities and the maximum profit are

$Q_A^* = 100 - 3P_A^* + 2P_B^* = 100 - 3\times 90 + 2\times 110 = 50$

$Q_B^* = 60 + 2P_A^* - 2P_B^* = 60 + 2\times 90 - 2\times 110 = 20$

$\pi^* = \pi(P_A^*, P_B^*) = -3(P_A^*)^2 + 4P_A^*P_B^* - 2(P_B^*)^2 + 100P_A^* + 80P_B^* - 5000$

$= 3900$

(b) Let α and β be the marginal costs of products A and B.

The total cost as a function of two prices is

$C = \alpha Q_A + \beta Q_B + 1200$

$= \alpha(100 - 3P_A + 2P_B) + \beta(60 + 2P_A - 2P_B) + 1200$

Profit as a function of two prices is

$\pi(P_A, P_B; \alpha, \beta) = P_A(100 - 3P_A + 2P_B) + P_B(60 + 2P_A - 2P_B)$

$- (\alpha(100 - 3P_A + 2P_B) + \beta(60 + 2P_A - 2P_B) + 1200)$

Let $\pi^*(\alpha, \beta)$ be the maximum profit at marginal costs (α, β).

We estimate the change in π^* when (α, β) changes from $(\alpha_0, \beta_0) = (20, 30)$ to $(\alpha_0+\Delta\alpha, \beta_0+\Delta\beta) = (21, 31.5)$.

The linear Taylor approximation is

$$\Delta\pi^* = \pi^*(\alpha_0 + \Delta\alpha, \beta_0 + \Delta\beta) - \pi^*(\alpha_0, \beta_0) \approx \left.\frac{\partial\pi(\alpha,\beta)}{\partial\alpha}\right|_{(\alpha_0,\beta_0)} \Delta\alpha + \left.\frac{\partial\pi(\alpha,\beta)}{\partial\beta}\right|_{(\alpha_0,\beta_0)} \Delta\beta$$

The envelope theorem gives us

$$\frac{\partial\pi(\alpha,\beta)}{\partial\alpha} = \left.\frac{\partial\pi}{\partial\alpha}\right|_{(P_A^*(\alpha,\beta),\ P_B^*(\alpha,\beta))} = -(100 - 3P_A^*(\alpha,\beta) + 2P_B^*(\alpha,\beta))$$

$$\frac{\partial\pi(\alpha,\beta)}{\partial\beta} = \left.\frac{\partial\pi}{\partial\beta}\right|_{(P_A^*(\alpha,\beta),\ P_B^*(\alpha,\beta))} = -(60 + 2P_A^*(\alpha,\beta) - 2P_B^*(\alpha,\beta))$$

In (a), we obtained that at $(\alpha_0, \beta_0) = (20, 30)$, we have

$P_A^*(\alpha_0, \beta_0) = 90$, $\quad P_B^*(\alpha_0, \beta_0) = 110$

$\pi^*(\alpha_0, \beta_0) = 3900$

$$\left.\frac{\partial\pi(\alpha,\beta)}{\partial\alpha}\right|_{(\alpha_0,\beta_0)} = -(100 - 3P_A^*(\alpha_0,\beta_0) + 2P_B^*(\alpha_0,\beta_0)) = -50$$

$$\left.\frac{\partial\pi(\alpha,\beta)}{\partial\beta}\right|_{(\alpha_0,\beta_0)} = -(60 + 2P_A^*(\alpha_0,\beta_0) - 2P_B^*(\alpha_0,\beta_0)) = -20$$

Using this result in the linear Taylor approximation, we obtain

$$\Delta\pi^* \approx \left.\frac{\partial\pi(\alpha,\beta)}{\partial\alpha}\right|_{(\alpha_0,\beta_0)} \Delta\alpha + \left.\frac{\partial\pi(\alpha,\beta)}{\partial\beta}\right|_{(\alpha_0,\beta_0)} \Delta\beta = (-50)\cdot 1 + (-20)\cdot 1.5 = -80$$

==> $\quad \pi^*(21, 31.5) \approx 3900 - 80 = 3820$

3. $f(x, y) = \frac{5}{3}x^3 + \frac{2}{3}y^3 - \frac{15}{2}x^2 + y^2 - 4y + 7$

$$\begin{cases} f_x(x, y) = 5x^2 - 15x = 0 \\ f_y(x, y) = 2y^2 + 2y - 4 = 0 \end{cases}$$

Both equations are easily solved by factoring.
Critical points: (0, –2), (0, 1), (3, –2), (3, 1)

4. $f(x, y) = xy - x + y$

$f_x(x, y) = y - 1$

$f_y(x, y) = x + 1$

Critical point: (–1, 1)

5. $f(x, y, z) = 2x^2 + xy + y^2 + 100 - z(x + y - 200)$

$$\begin{cases} f_x(x, y, z) = 4x + y - z = 0 \\ f_y(x, y, z) = x + 2y - z = 0 \\ f_z(x, y, z) = -x - y + 200 = 0 \end{cases}$$

Solving the system gives the critical point (50, 150, 350).

6. $f(x, y, z, w) = x^2 + y^2 + z^2 + w(x + y + z - 3)$

$$\begin{cases} f_x(x, y, z, w) = 2x + w = 0 \\ f_y(x, y, z, w) = 2y + w = 0 \\ f_z(x, y, z, w) = 2z + w = 0 \\ f_w(x, y, z, w) = x + y + z - 3 = 0 \end{cases}$$

Solving the system gives the critical point (1, 1, 1, –2).

7. $f(x, y) = x^2 + 3y^2 + 4x - 9y + 3$

$$\begin{cases} f_x(x, y) = 2x + 4 = 0 \\ f_y(x, y) = 6y - 9 = 0 \end{cases}$$

Critical point $\left(-2, \frac{3}{2}\right)$

Second-Derivative Test

$f_{xx}(x, y) = 2$, $f_{yy}(x, y) = 6$, $f_{xy}(x, y) = 0$. At $\left(-2, \frac{3}{2}\right)$, $D = (2)(6) - 0^2 = 12 > 0$ and $f_{xx}(x, y) = 2 > 0$. Thus at $\left(-2, \frac{3}{2}\right)$ there is a relative minimum.

8. $f(x, y) = -2x^2 + 8x - 3y^2 + 24y + 7$

$$\begin{cases} f_x(x, y) = -4x + 8 = 0 \\ f_y(x, y) = -6y + 24 = 0 \end{cases}$$

Critical point: (2, 4)

Second-Derivative Test

$f_{xx}(x, y) = -4$, $f_{yy}(x, y) = -6$, $f_{xy}(x, y) = 0$. At (2, 4), $D = (-4)(-6) - 0^2 = 24 > 0$ and $f_{xx}(x, y) = -4 < 0$; thus there is a relative maximum at (2, 4).

9. $f(x, y) = y - y^2 - 3x - 6x^2$

$$\begin{cases} f_x(x, y) = -3 - 12x = 0 \\ f_y(x, y) = 1 - 2y = 0 \end{cases}$$

Critical point $\left(-\frac{1}{4}, \frac{1}{2}\right)$

Second-Derivative Test

$f_{xx}(x, y) = -12$, $f_{yy}(x, y) = -2$, $f_{xy}(x, y) = 0$

At $\left(-\frac{1}{4}, \frac{1}{2}\right)$, $D = (-12)(-2) - 0^2 = 24 > 0$ and $f_{xx}(x, y) = -12 < 0$. Thus at $\left(-\frac{1}{4}, \frac{1}{2}\right)$ there is a relative maximum.

10. $f(x, y) = 2x^2 + \frac{3}{2}y^2 + 3xy - 10x - 9y + 2$

$$\begin{cases} f_x(x, y) = 4x + 3y - 10 = 0 \\ f_y(x, y) = 3y + 3x - 9 = 0 \end{cases}$$

Critical point: (1, 2)

Second-Derivative Test

$f_{xx}(x, y) = 4$, $f_{yy}(x, y) = 3$, $f_{xy}(x, y) = 3$.

At (1, 2), $D = (4)(3) - 3^2 = 3 > 0$ and $f_{xx}(x, y) = 4 > 0$; thus there is a relative minimum at (1, 2).

11. $f(x, y) = x^2 + 3xy + y^2 - 9x - 11y + 3$

$$\begin{cases} f_x(x, y) = 2x + 3y - 9 = 0 \\ f_y(x, y) = 3x + 2y - 11 = 0 \end{cases}$$

Critical point: (3, 1)

Second-Derivative Test

$f_{xx}(x, y) = 2$, $f_{yy} = 2$, $f_{xy} = 3$. At (3, 1), $D = (2)(2) - (3)^2 = -5 < 0$, so there is no relative extremum at (3, 1).

12. $f(x, y) = \frac{x^3}{3} + y^2 - 2x + 2y - 2xy$

$$\begin{cases} f_x(x, y) = x^2 - 2 - 2y = 0 \\ f_y(x, y) = 2y + 2 - 2x = 0 \end{cases}$$

Critical points: (2, 1), (0, –1)
Second-Derivative Test
$f_{xx}(x, y) = 2x$, $f_{yy}(x, y) = 2$, $f_{xy}(x, y) = -2$.
At (2, 1), $D = (4)(2) - (-2)^2 = 4 > 0$ and $f_{xx}(x, y) = 4 > 0$, so a relative minimum at (2, 1). At (0, –1), $D = (0)(2) - (-2)^2 = -4 < 0$; thus neither at (0, –1).

13. $f(x, y) = \frac{1}{3}\left(x^3 + 8y^3\right) - 2\left(x^2 + y^2\right) + 1$

$$\begin{cases} f_x(x, y) = x^2 - 4x = 0 \\ f_y(x, y) = 8y^2 - 4y = 0 \end{cases}$$

Critical points: (0, 0), $\left(4, \frac{1}{2}\right)$, $\left(0, \frac{1}{2}\right)$, (4, 0)
Second-Derivative Test
$f_{xx}(x, y) = 2x - 4$, $f_{yy}(x, y) = 16y - 4$, $f_{xy}(x, y) = 0$. At (0, 0), $D = (-4)(-4) - 0^2 = 16 > 0$ and $f_{xx}(x, y) = -4 < 0$; thus a relative maximum.
At $\left(4, \frac{1}{2}\right)$, $D = (4)(4) - 0^2 = 16 > 0$ and $f_{xx}(x, y) = 4 > 0$; thus a relative minimum.
At $\left(0, \frac{1}{2}\right)$, $D = (-4)(4) - 0^2 = -16 < 0$; thus neither.
At (4, 0), $D = (4)(-4) - 0^2 = -16 < 0$, thus neither.

14. $f(x, y) = x^2 + y^2 - xy + x^3$

$$\begin{cases} f_x(x, y) = 2x - y + 3x^2 = 0 \\ f_y(x, y) = 2y - x = 0 \end{cases}$$

Critical points: (0, 0), $\left(-\frac{1}{2}, -\frac{1}{4}\right)$
Second-Derivative Test
$f_{xx}(x, y) = 2 + 6x$, $f_{yy}(x, y) = 2$, $f_{xy}(x, y) = -1$. At (0, 0), $D = (2)(2) - (-1)^2 = 3 > 0$ and $f_{xx}(x, y) = 2 > 0$; thus relative minimum.
At $\left(-\frac{1}{2}, -\frac{1}{4}\right)$, $D = (-1)(2) - (-1)^2 = -3 < 0$; thus neither.

15. $f(l, k) = \frac{l^2}{2} + 2lk + 3k^2 - 69l - 164k + 17$

$$\begin{cases} f_l(l, k) = l + 2k - 69 = 0 \\ f_k(l, k) = 2l + 6k - 164 = 0 \end{cases}$$

Critical point: (43, 13)
Second-Derivative Test
$f_{ll}(l, k) = 1$, $f_{kk}(l, k) = 6$, $f_{lk}(l, k) = 2$
At (43, 13), $D = (1)(6) - 2^2 = 2 > 0$ and $f_{ll}(l, k) = 1 > 0$; thus there is a relative minimum at (43, 13).

16. $f(l, k) = l^2 + 4k^2 - 4lk$

$$\begin{cases} f_l(l, k) = 2l - 4k \\ f_k(l, k) = 8k - 4l \end{cases}$$

Critical points: (2*r*, *r*) where *r* is any real number.
Second Derivative Test
$f_{ll}(l, k) = 2$, $f_{kk}(l, k) = 8$, and $f_{lk}(l, k) = -4$.
At (2*r*, *r*), $D = (2)(8) - (-4)^2 = 0$, thus we cannot make a conclusion.

17. $f(p, q) = pq - \frac{1}{p} - \frac{1}{q}$

$$\begin{cases} f_p(p, q) = q + \frac{1}{p^2} = 0 \\ f_q(p, q) = p + \frac{1}{q^2} = 0 \end{cases}$$

Critical point: (–1, –1)
Second-Derivative Test
$f_{pp}(p, q) = -\frac{2}{p^3}$, $f_{qq}(p, q) = -\frac{2}{q^3}$, $f_{pq}(p, q) = 1$. At (–1, –1), $D = (2)(2) - 1^2 = 3 > 0$ and $f_{pp}(p, q) = 2 > 0$; thus there is a relative minimum at (–1, –1).

18. $f(x, y) = (x-3)(y-3)(x+y-3)$
$= (y-3)(x^2 + xy - 6x - 3y + 9)$
$= (x-3)(xy - 3x + y^2 - 6y + 9)$

$$\begin{cases} f_x(x, y) = (y-3)(2x + y - 6) = 0 \\ f_y(x, y) = (x-3)(x + 2y - 6) = 0 \end{cases}$$

Critical points: (2, 2), (3, 3), (3, 0), (0, 3)
Second-Derivative Test
$f_{xx}(x, y) = 2(y-3)$, $f_{yy}(x, y) = 2(x-3)$, $f_{xy}(x, y) = 2x + 2y - 9$. At (2, 2),

$D = (-2)(-2) - (-1)^2 = 3 > 0$ and
$f_{xx}(x, y) = -2 < 0$; thus relative maximum.
At (3, 3), $D = (0)(0) - 3^2 = -9 < 0$; thus neither.
At (3, 0), $D = (-6)(0) - (-3)^2 = -9 < 0$; thus neither. At (0, 3), $D = (0)(-6) - (-3)^2 = -9 < 0$; thus neither.

19. $f(x, y) = \left(y^2 - 4\right)\left(e^x - 1\right)$

$$\begin{cases} f_x(x, y) = e^x\left(y^2 - 4\right) = 0 & (1) \\ f_y(x, y) = 2y\left(e^x - 1\right) = 0 & (2) \end{cases}$$

Critical points: (0, –2), (0, 2)
[Note that $y = 0$ does not give rise to a common solution of (1) and (2).]
Second-Derivative Test
$f_{xx}(x, y) = e^x\left(y^2 - 4\right)$, $f_{yy}(x, y) = 2\left(e^x - 1\right)$,
$f_{xy}(x, y) = 2ye^x$. At (0, –2),
$D = (0)(0) - (-4)^2 = -16 < 0$; thus neither. At (0, 2), $D = (0)(0) - (4)^2 = -16 < 0$; thus neither.

20. $f(x, y) = \ln(xy) + 2x^2 - xy - 6x$

$$\begin{cases} f_x(x, y) = \dfrac{1}{x} + 4x - y - 6 = 0 \\ f_y(x, y) = \dfrac{1}{y} - x = 0 \end{cases}$$

The only critical point is $\left(\dfrac{3}{2}, \dfrac{2}{3}\right)$.

$f_{xx}(x, y) = -\dfrac{1}{x^2} + 4$, $f_{yy}(x, y) = -\dfrac{1}{y^2}$,

$f_{xy}(x, y) = -1$. At $\left(\dfrac{3}{2}, \dfrac{2}{3}\right)$,

$D = \left(\dfrac{32}{9}\right)\left(\dfrac{-9}{4}\right) - (-1)^2 = -9 < 0$; thus neither.

21. $P = f(l, k) = 2.18l^2 - 0.02l^3 + 1.97k^2 - 0.03k^3$

$$\begin{cases} P_l = 4.36l - 0.06l^2 = 0 \\ P_k = 3.94k - 0.09k^2 = 0 \end{cases}$$

Critical points: (0, 0), $\left(0, \dfrac{394}{9}\right)$, $\left(\dfrac{218}{3}, 0\right)$, $\left(\dfrac{218}{3}, \dfrac{394}{9}\right)$

Second-Derivative Test
$P_{ll} = 4.36 - 0.12l$, $P_{kk} = 3.94 - 0.18k$, $P_{lk} = 0$.
At (0, 0), $D = (4.36)(3.94) - 0^2 > 0$ and
$P_{ll} = 4.36 > 0$; thus relative minimum.
At $\left(0, \dfrac{394}{9}\right)$, $D = (4.36)(-3.94) - 0^2 < 0$; thus no extremum.
At $\left(\dfrac{218}{3}, 0\right)$, $D = (-4.36)(3.94) - 0^2 < 0$; thus no extremum.
At $\left(\dfrac{218}{3}, \dfrac{394}{9}\right)$, $D = (-4.36)(-3.94) - 0^2 > 0$
and $P_{ll} = -4.36 < 0$; thus $l = \dfrac{218}{3} \approx 72.67$,
$k = \dfrac{394}{9} \approx 43.78$ gives a relative maximum.

22. $Q = 18c + 20d - 2c^2 - 4d^2 - cd$

$$\begin{cases} Q_c = 18 - 4c - d = 0 \\ Q_d = 20 - 8d - c = 0 \end{cases}$$

Critical point: $c = 4$, $d = 2$
$Q_{cc} = -4$, $Q_{dd} = -8$, $Q_{cd} = -1$
When $c = 4$ and $d = 2$, then
$D = (-4)(-8) - (-1)^2 > 0$ and $Q_{cc} = -4 < 0$; thus relative maximum at $c = 4$, $d = 2$.

23. Profit per lb for A = $p_A - 60$.
Profit per lb for B = $p_B - 70$.
Total Profit $= P = \left(p_A - 60\right)q_A + \left(p_B - 70\right)q_B$
$P = \left(p_A - 60\right)\left[5\left(p_B - p_A\right)\right]$
$+ \left(p_B - 70\right)\left[500 + 5\left(p_A - 2p_B\right)\right]$
Thus

$$\begin{cases} \dfrac{\partial P}{\partial p_A} = -10\left(p_A - p_B + 5\right) = 0 \\ \dfrac{\partial P}{\partial p_B} = 10\left(p_A - 2p_B + 90\right) = 0 \end{cases}$$

Critical point: $p_A = 80$, $p_B = 85$

$\dfrac{\partial^2 P}{\partial p_A^2} = -10$, $\dfrac{\partial^2 P}{\partial p_B^2} = -20$, $\dfrac{\partial^2 P}{\partial p_B \partial p_A} = 10$.

When $p_A = 80$ and $p_B = 85$, then
$D = (-10)(-20) - (10)^2 = 100 > 0$ and
$\dfrac{\partial^2 P}{\partial p_A^2} = -10 < 0$; thus relative maximum at $p_A = 80$, $p_B = 85$.

24. Profit per lb for $A = p_A - a$.
Profit per lb for $B = p_B - b$.
Total Profit $= P = (p_A - a)q_A + (p_B - b)q_B$

$$P = (p_A - a)[5(p_B - p_A)] + (p_B - b)[500 + 5(p_A - 2p_B)]$$

$$\begin{cases} \dfrac{\partial P}{\partial p_A} = -5(2p_A - 2p_B + b - a) = 0 \\ \dfrac{\partial P}{\partial p_B} = 5(2p_A - 4p_B + 2b - a + 100) = 0 \end{cases}$$

Critical point: $p_A = 50 + \frac{a}{2}$, $p_B = 50 + \frac{b}{2}$

$$\frac{\partial^2 P}{\partial p_A^2} = -10, \ \frac{\partial^2 P}{\partial p_B^2} = -20, \ \frac{\partial^2 P}{\partial p_B \partial p_A} = 10$$

When $p_A = 50 + \frac{a}{2}$ and $p_B = 50 + \frac{b}{2}$, then $D = (-10)(-20) - (10)^2 = 100 > 0$ and $\frac{\partial^2 P}{\partial p_A^2} = -10 < 0$; thus a relative maximum at $p_A = 50 + \frac{a}{2}$, $p_B = 50 + \frac{b}{2}$.

25. $p_A = 100 - q_A$, $p_B = 84 - q_B$, $c = 600 + 4(q_A + q_B)$.
Revenue from market A $= r_A = p_A q_A = (100 - q_A)q_A$. Revenue from market B $= r_B = p_B q_B = (84 - q_B)q_B$.
Total Profit = Total Revenue – Total Cost

$$P = (100 - q_A)q_A + (84 - q_B)q_B - [600 + 4(q_A + q_B)]$$

$$\begin{cases} \dfrac{\partial P}{\partial q_A} = 96 - 2q_A = 0 \\ \dfrac{\partial P}{\partial q_B} = 80 - 2q_B = 0 \end{cases}$$

Critical point: $q_A = 48$, $q_B = 40$

$$\frac{\partial^2 P}{\partial q_A^2} = -2, \ \frac{\partial^2 P}{\partial q_B^2} = -2, \ \frac{\partial^2 P}{\partial q_B \partial q_A} = 0.$$

At $q_A = 48$ and $q_B = 40$, then $D = (-2)(-2) - 0^2 = 4 > 0$ and $\frac{\partial^2 P}{\partial q_A^2} = -2 < 0$; thus relative maximum at $q_A = 48$, $q_B = 40$. When $q_A = 48$ and $q_B = 40$, then selling prices are $p_A = 52$, $p_B = 44$, and profit = 3304.

26. $q_A = 16 - p_A + p_B$, $q_B = 24 + 2p_A - 4p_B$
Revenue from A = $p_A q_A$. Revenue from B = $p_B q_B$.
Total cost of producing q_A units of A and q_B units of B is $2q_A + 4q_B$.
Total Profit = Total Revenue – Total Cost

$$P = p_A q_A + p_B q_B - (2q_A + 4q_B)$$

$$P = 16p_A - p_A^2 + p_A p_B + 24p_B + 2p_A p_B - 4p_B^2 - 32 + 2p_A - 2p_B - 96 - 8p_A + 16p_B$$
$$= -p_A^2 - 4p_B^2 + 3p_A p_B + 10p_A + 38p_B - 128$$

$$\begin{cases} \dfrac{\partial P}{\partial p_A} = -2p_A + 3p_B + 10 \\ \dfrac{\partial P}{\partial p_B} = 3p_A - 8p_B + 38 \end{cases}$$

Critical point: $p_A = \frac{194}{7}$, $p_B = \frac{106}{7}$

$\frac{\partial^2 P}{\partial p_A^2} = -2$, $\frac{\partial^2 P}{\partial p_B^2} = -8$, $\frac{\partial^2 P}{\partial p_B \partial p_A} = 3$

At $p_A = \frac{194}{7}$, $p_B = \frac{106}{7}$, then $D = (-2)(-8) - 3^2 = 7 > 0$ and $\frac{\partial^2 P}{\partial p_A^2} = -2 < 0$; thus relative maximum at

$p_A = \frac{194}{7}$, $p_B = \frac{106}{7}$.

$q_A = 16 - \frac{194}{7} + \frac{106}{7} = \frac{24}{7}$

$q_B = 24 + 2\left(\frac{194}{7}\right) - 4\left(\frac{106}{7}\right) = \frac{132}{7}$

So, $\frac{24}{7} \approx 3$ of A and $\frac{132}{7} \approx 19$ of B should be sold.

27. $c = \frac{3}{2}q_A^2 + 3q_B^2$, $p_A = 60 - q_A^2$, $p_B = 72 - 2q_B^2$

Total Profit = Total Revenue – Total Cost

$P = (p_A q_A + p_B q_B) - c$

$P = 60q_A - q_A^3 + 72q_B - 2q_B^3 - \left(\frac{3}{2}q_A^2 + 3q_B^2\right)$

$$\begin{cases} \dfrac{\partial P}{\partial q_A} = 60 - 3q_A - 3q_A^2 = 3(5 + q_A)(4 - q_A) \\ \dfrac{\partial P}{\partial q_B} = 72 - 6q_B - 6q_B^2 = 6(4 + q_B)(3 - q_B) \end{cases}$$

Since we want $q_A \ge 0$ and $q_B \ge 0$, the critical point occurs when $q_A = 4$ and $q_B = 3$.

$\frac{\partial^2 P}{\partial q_A^2} = -3 - 6q_A$, $\frac{\partial^2 P}{\partial q_B^2} = -6 - 12q_B$, $\frac{\partial^2 P}{\partial q_B \partial q_A} = 0$. When $q_A = 4$ and $q_B = 3$, then $D = (-27)(-42) - 0^2 > 0$

and $\frac{\partial^2 P}{\partial q_A^2} = -27 < 0$; thus relative maximum at $q_A = 4$, $q_B = 3$.

28. $c = 2(q_A + q_B + q_A q_B)$,

Total Profit = Total Revenue – Total Cost

$$\begin{aligned} P &= (p_A q_A + p_B q_B) - c \\ &= p_A(20 - 2p_A) + p_B(10 - p_B) - [20 - 2p_A + 10 - p_B + (20 - 2p_A)(10 - p_B)] \\ &= -2p_A^2 - p_B^2 - 2p_A p_B + 42p_A + 31p_B + 230 \end{aligned}$$

$$\begin{cases} \dfrac{\partial P}{\partial p_A} = -4p_A - 2p_B + 42 \\ \dfrac{\partial P}{\partial p_B} = -2p_A - 2p_B + 31 \end{cases}$$

Critical point: $p_A = \frac{11}{2}$, $p_B = 10$

$$\frac{\partial^2 P}{\partial p_A^2} = -4, \frac{\partial^2 P}{\partial p_B^2} = -2, \frac{\partial^2 P}{\partial p_B \partial p_A} = -2$$

When $p_A = \frac{11}{2}$, $p_B = 10$, then $D = (-4)(-2) - (-2)^2 = 4 > 0$, and $\frac{\partial^2 p}{\partial p_A^2} = -4 < 0$, so the maximum profit occurs when $p_A = 5.5$ and $p_B = 10$. At these prices, $q_A = 9$, $q_B = 0$, and the total profit is 40.5.

29. Refer to the diagram in the text.
$xyz = 6$
$C = 3xy + 2[1(xz)] + 2[0.5(yz)]$
Note that $z = \frac{6}{xy}$. Thus

$$C = 3xy + 2xz + yz = 3xy + 2x\left(\frac{6}{xy}\right) + y\left(\frac{6}{xy}\right) = 3xy + \frac{12}{y} + \frac{6}{x}$$

$$\begin{cases} \dfrac{\partial C}{\partial x} = 3y - \dfrac{6}{x^2} = 0 \\ \dfrac{dC}{dy} = 3x - \dfrac{12}{y^2} = 0 \end{cases}$$

A critical point occurs at $x = 1$ and $y = 2$. Thus $z = 3$.

$$\frac{\partial^2 C}{\partial x^2} = \frac{12}{x^3}, \frac{\partial^2 C}{\partial y^2} = \frac{24}{y^3}, \frac{\partial^2 C}{\partial x \partial y} = 3.$$

When $x = 1$ and $y = 2$, then $d = (12)(3) - (3)^2 = 27 > 0$ and $\frac{\partial^2 C}{\partial x^2} = 12 > 0$. Thus we have a minimum. The dimensions should be 1 ft by 2 ft by 3 ft.

30. $p = 92 - q_A - q_B$, $c_A = 10q_A$, $c_B = 0.5q_B^2$
Since Profit = Total Revenue – Total Cost, then
Profit of A $= pq_A - c_A$ and
Profit of B $= pq_B - c_B$.
Thus profit P of monopoly is

$$P = pq_A - c_A + pq_B - c_B$$
$$= p(q_A + q_B) - c_A - c_B$$
$$= (92 - q_A - q_B)(q_A + q_B) - 10q_A - 0.5q_B^2$$
$$= 82q_A + 92q_B - q_A^2 - 2q_Aq_B - 1.5q_B^2$$

$$\begin{cases} \frac{\partial P}{\partial q_A} = 82 - 2q_A - 2q_B = 0 \\ \frac{\partial P}{\partial q_B} = 92 - 2q_A - 3q_B = 0 \end{cases}$$

Critical point: $q_A = 31,\ q_B = 10$

$\frac{\partial^2 P}{\partial q_A^2} = -2,\ \frac{\partial^2 P}{\partial q_B^2} = -3,\ \frac{\partial^2 P}{\partial q_B \partial q_A} = -2$

When $q_A = 31$ and $q_B = 10$, then

$D = (-2)(-3) - (-2)^2 = 2 > 0$ and

$\frac{\partial^2 P}{\partial q_A^2} = -2 < 0$; thus relative maximum at

$q_A = 31,\ q_B = 10$.

31. $y = 2 - x$

$f(x, y) = x^2 + 3(2 - x)^2 + 9$

Setting the derivative equal to 0 gives
$2x + 6(2 - x)(-1) = 0$.
$2x - 12 + 6x = 0$, $8x - 12 = 0$,
$8x = 12$, or $x = \frac{3}{2}$. The second-derivative is
$8 > 0$, so we have a relative minimum. If $x = \frac{3}{2}$, then $y = \frac{1}{2}$. Thus there is a relative minimum at $\left(\frac{3}{2}, \frac{1}{2}\right)$.

32. $y = \frac{x - 10}{4}$

$f(x, y) = x^2 + 4\left(\frac{x - 10}{4}\right)^2 + 6$

Setting the derivative equal to 0 gives
$2x + 4(2)\left(\frac{x - 10}{4}\right)\left(\frac{1}{4}\right) = 0$, from which $x = 2$.

The second-derivative is $\frac{5}{2} > 0$,so we have a relative minimum If $x = 2$, then $y = -2$. Thus at $(2, -2)$ there is a relative minimum

33. $c = q_A^2 + 3q_B^2 + 2q_A q_B + aq_A + bq_B + d$

We are given that $(q_A, q_B) = (3, 1)$ is a critical point.

$$\begin{cases} \frac{\partial c}{\partial q_A} = 2q_A + 2q_B + a = 0 \\ \frac{\partial c}{\partial q_B} = 6q_B + 2q_A + b = 0 \end{cases}$$

Substituting the given values for q_A and q_B into both equations gives $a = -8$ and $b = -12$. Since
$c = 15$ when $q_A = 3$ and $q_B = 1$, from the joint-cost function we have

$15 = 3^2 + 3\left(1^2\right) + 2(3)(1) + (-8)(3) + (-12) + d$,

$15 = -18 + d$, $33 = d$. Thus $a = -8$, $b = -12$, $d = 33$.

34. $D(a, b) > 0$

$f_{xx}(a, b) f_{yy}(a, b) - (f_{xy}(a, b))^2 > 0$

$f_{xx}(a, b) f_{yy}(a, b) > (f_{xy}(a, b))^2 \geq 0$

a. Since the product $f_{xx}(a, b) f_{yy}(a, b)$ is positive, $f_{xx}(a, b)$ and $f_{yy}(a, b)$ must have the same sign. That is $f_{xx}(a, b) < 0$ if and only if $f_{yy}(a, b) < 0$.

b. Since the product $f_{xx}(a, b) f_{yy}(a, b)$ is positive, $f_{xx}(a, b)$ and $f_{yy}(a, b)$ must have the same sign. That is $f_{xx}(a, b) > 0$ if and only if $f_{yy}(a, b) > 0$.

35. a. Profit = Total Revenue – Total Cost

$P = p_A q_A + p_B q_B -$ total cost

$= \left(35 - 2q_A^2 + q_B\right)q_A + (20 - q_B + q_A)q_B - \left(-8 - 2q_A^3 + 3q_A q_B + 30q_A + 12q_B + \frac{1}{2}q_A^2\right)$

$P = 5q_A - \frac{1}{2}q_A^2 - q_A q_B + 8q_B - q_B^2 + 8$

$$\begin{cases} \frac{\partial P}{\partial q_A} = 5 - q_A - q_B = 0 \\ \frac{\partial P}{\partial q_B} = -q_A + 8 - 2q_B = 0 \end{cases}$$

Critical point: $q_A = 2,\ q_B = 3$

$\frac{\partial^2 P}{\partial q_A^2} = -1,\ \frac{\partial^2 P}{\partial q_B^2} = -2,\ \frac{\partial^2 P}{\partial q_B \partial q_A} = -1$

At $q_A = 2$ and $q_B = 3$, then $D = (-1)(-2) - (-1)^2 = 1 > 0$ and $\frac{\partial^2 P}{\partial q_A^2} = -1 < 0$; thus there is a relative maximum profit for 2 units of A and 3 units of B.

b. Substituting $q_A = 2$ and $q_B = 3$ into the formulas for p_A, p_B, and P gives a selling price for A of 30, a selling price for B of 19, and a relative maximum profit of 25.

36. $P = 300\left[\frac{7x}{2+x} + \frac{4y}{5+y}\right] - x - y$

$$\begin{cases} \frac{\partial P}{\partial x} = 300 \cdot \frac{(2+x)(7) - 7x}{(2+x)^2} - 1 = \frac{4200}{(2+x)^2} - 1 = 0 \\ \frac{\partial P}{\partial y} = 300 \cdot \frac{(5+y)(4) - 4y}{(5+y)^2} - 1 = \frac{6000}{(5+y)^2} - 1 = 0 \end{cases}$$

$4200 = (2+x)^2$

$\pm\sqrt{4200} = 2 + x$

$x = -2 \pm \sqrt{4200} = -2 \pm 10\sqrt{42}$

$6000 = (5+y)^2$

$\pm\sqrt{6000} = 5 + y$

$y = -5 \pm \sqrt{6000} = -5 \pm 20\sqrt{15}$

The values of *x* and *y* must be nonnegative.

Critical point: $x = 10\sqrt{42} - 2,\quad y = 20\sqrt{15} - 5$

$\frac{\partial^2 P}{\partial x^2} = -\frac{8400}{(2+x)^3},\ \frac{\partial^2 P}{\partial y^2} = -\frac{12{,}000}{(5+y)^3},\ \frac{\partial^2 P}{\partial y \partial x} = 0$

At $x = 10\sqrt{42} - 2$ and $y = 20\sqrt{15} - 5$, then $D \approx (0.031)(0.026) - 0^2 > 0$ and $\frac{\partial^2 P}{\partial x^2} \approx -0.031 < 0.$

Thus relative maximum profit at $x = 10\sqrt{42} - 2 \approx 62.81,\quad y = 20\sqrt{15} - 5 \approx 72.46.$

37. a. $P = 5T\left(1 - e^{-x}\right) - 20x - 0.1T^2$

b. $\frac{\partial P}{\partial T} = 5\left(1 - e^{-x}\right) - 0.2T$

$\frac{\partial P}{\partial x} = 5Te^{-x} - 20$

At the point $(T, x) = (20, \ln 5)$,

$$\frac{\partial P}{\partial T} = 5\left(1 - e^{-\ln 5}\right) - 0.2(20) = 5\left(1 - \frac{1}{5}\right) - 4 = 0$$

$$\frac{\partial P}{\partial x} = 5(20)e^{-\ln 5} - 20 = 100\left(\frac{1}{5}\right) - 20 = 0$$

Thus (20, ln 5) is a critical point. In a similar fashion we verify that $\left(5, \ln\frac{5}{4}\right)$ is a critical point.

c. $\frac{\partial^2 P}{\partial T^2} = -0.2$, $\frac{\partial^2 P}{\partial x^2} = -5Te^{-x}$, $\frac{\partial^2 P}{\partial T \partial x} = 5e^{-x}$

At (20, ln 5),

$$D = (-0.2)\left[-5(20)e^{-\ln 5}\right] - \left(5e^{-\ln 5}\right)^2 = 20\left(\frac{1}{5}\right) - \left[5\left(\frac{1}{5}\right)\right]^2 = 3 > 0,$$

and $\frac{\partial^2 P}{\partial T^2} = -0.2 < 0$. Thus we get a relative maximum at (20, ln 5).

At $\left(5, \ln\frac{5}{4}\right)$,

$D = (-0.2)\left[-5(5)e^{-\ln\left(\frac{5}{4}\right)}\right] - \left[5e^{-\ln\left(\frac{5}{4}\right)}\right]^2 = 5\left(\frac{4}{5}\right) - \left[5\left(\frac{4}{5}\right)\right]^2 = -12 < 0$, so there is no relative extremum at $\left(5, \ln\frac{5}{4}\right)$.

Problems 17.7

1. $f(x, y) = x^2 + 4y^2 + 6,\ 2x - 8y = 20$

$F(x, y, \lambda) = x^2 + 4y^2 + 6 - \lambda(2x - 8y - 20)$

$$\begin{cases} F_x = 2x - 2\lambda = 0 & (1) \\ F_y = 8y + 8\lambda = 0 & (2) \\ F_\lambda = -2x + 8y + 20 = 0 & (3) \end{cases}$$

From (1), $x = \lambda$; from (2), $y = -\lambda$. Substituting $x = \lambda$ and $y = -\lambda$ into (3) gives $-2\lambda - 8\lambda + 20 = 0$, $-10\lambda = -20$, so $\lambda = 2$. Thus $x = 2$ and $y = -2$. Critical point of F: (2, –2, 2). Critical point of f: (2, –2).

2. $f(x, y) = 3x^2 - 2y^2 + 9,\ x + y = 1$

$F(x, y, \lambda) = 3x^2 - 2y^2 + 9 - \lambda(x + y - 1)$

$$\begin{cases} F_x = 6x - \lambda = 0 & (1) \\ F_y = -4y - \lambda = 0 & (2) \\ F_\lambda = -x - y + 1 = 0 & (3) \end{cases}$$

From (1), $x = \frac{\lambda}{6}$; from (2), $y = -\frac{\lambda}{4}$.

Substituting $x = \frac{\lambda}{6}$ and $y = -\frac{\lambda}{4}$ into (3) gives $-\frac{\lambda}{6} + \frac{\lambda}{4} + 1 = 0$, from which $\lambda = -12$. Thus $x = -2$ and $y = 3$.

Critical point of F: (–2, 3, –12). Critical point of f: (–2, 3).

3. $f(x, y, z) = x^2 + y^2 + z^2$, $2x + y - z = 9$

$F(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda(2x + y - z - 9)$

$$\begin{cases} F_x = 2x - 2\lambda = 0 & (1) \\ F_y = 2y - \lambda = 0 & (2) \\ F_z = 2z + \lambda = 0 & (3) \\ F_\lambda = -2x - y + z + 9 = 0 & (4) \end{cases}$$

From (1), $x = \lambda$; from (2), $y = \frac{\lambda}{2}$; from (3), $z = -\frac{\lambda}{2}$. Substituting into (4) gives $-2\lambda - \frac{\lambda}{2} + \left(\frac{-\lambda}{2}\right) + 9 = 0$, $-6\lambda + 18 = 0$, so $\lambda = 3$. Thus $x = 3$, $y = \frac{3}{2}$, $z = -\frac{3}{2}$. Critical point of F: $\left(3, \frac{3}{2}, -\frac{3}{2}, 3\right)$. Critical point of f: $\left(3, \frac{3}{2}, -\frac{3}{2}\right)$.

4. $f(x, y, z) = x + y + z$, $xyz = 8$

$F(x, y, z, \lambda) = x + y + z - \lambda(xyz - 8)$

$$\begin{cases} F_x = 1 - \lambda yz = 0 & (1) \\ F_y = 1 - \lambda xz = 0 & (2) \\ F_z = 1 - \lambda xy = 0 & (3) \\ F_\lambda = -xzy + 8 = 0 & (4) \end{cases}$$

From (1) and (2), $\lambda yz = \lambda xz$, so $y = x$. From (2) and (3), $\lambda xz = \lambda xy$, so $y = z$. Therefore $x = y = z$, so from (4), $x = y = z = 2$. Hence, Critical point of f is (2, 2, 2). Note that it is not necessary to determine λ.

5. $f(x, y, z) = 2x^2 + xy + y^2 + z$, $x + 2y + 4z = 3$

$F(x, y, z, \lambda)$

$= 2x^2 + xy + y^2 + z - \lambda(x + 2y + 4z - 3)$

$$\begin{cases} F_x = 4x + y - \lambda = 0 \\ F_y = x + 2y - 2\lambda = 0 \\ F_z = 1 - 4\lambda = 0 \\ F_\lambda = -x - 2y - 4z - 3 = 0 \end{cases}$$

From the third equation we have $\lambda = \frac{1}{4}$.

Substituting this value into the first two equations and then eliminating y gives $x = 0$ and $y = \frac{1}{4}$.

Finally, solving for z in the last equation gives $z = -\frac{7}{8}$.

Critical point of F: $\left(0, \frac{1}{4}, -\frac{7}{8}, \frac{1}{4}\right)$

Critical point of f: $\left(0, \frac{1}{4}, -\frac{7}{8}\right)$

6. $f(x, y, z) = xyz^2$, $x - y + z = 20$ $\left(xyz^2 \neq 0\right)$

$f(x, y, z, \lambda) = xyz^2 - \lambda(x - y + z - 20)$

$$\begin{cases} F_x = yz^2 - \lambda = 0 & (1) \\ F_y = xz^2 + \lambda = 0 & (2) \\ F_z = 2xyz - \lambda = 0 & (3) \\ F_\lambda = -x + y - z + 20 = 0 & (4) \end{cases}$$

From (1) and (2), $y = -x$, From (1) and (3), $z = 2x$. Hence from (4), $x = 5$, so $y = -5$ and $z = 10$. Critical point of f is (5, –5, 10). Note that it is not necessary to determine λ.

7. $f(x, y, z) = xyz$, $x + y + z = 1$ $(xyz \neq 0)$

$F(x, y, z, \lambda) = xyz - \lambda(x + y + z - 1)$

$$\begin{cases} F_x = yz - \lambda = 0 & (1) \\ F_y = xz - \lambda = 0 & (2) \\ F_z = xy - \lambda = 0 & (3) \\ F_\lambda = -x - y - z + 1 = 0 & (4) \end{cases}$$

From (1) and (2), $y = x$. From (1) and (3), $x = z$. Hence from (4) $x = y = z = \frac{1}{3}$. Critical point of f is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Note that it is not necessary to determine λ.

8. $f(x, y, z) = x^2 + y^2 + z^2$, $x + y + z = 3$

$F(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda(x + y + z - 3)$

$$\begin{cases} F_x = 2x - \lambda = 0 & (1) \\ F_y = 2y - \lambda = 0 & (2) \\ F_z = 2z - \lambda = 0 & (3) \\ F_\lambda = -x - y - z + 3 = 0 & (4) \end{cases}$$

From (1)–(3), $x = y = z = \frac{\lambda}{2}$. Substituting into (4), $-\frac{\lambda}{2} - \frac{\lambda}{2} - \frac{\lambda}{2} + 3 = 0$, so $\lambda = 2$. Thus $x = 1$, $y = 1$, $z = 1$. Critical point of F: $(1, 1, 1, 2)$. Critical point of f: $(1, 1, 1)$.