AMS11B: Mathematical Methods for Economists II

Solutions of Assignment #8

Problems from supplementary note #4

**Problem 1**

Price \((p)\) as a function of output quantity \((q)\)

\[ p = 100 - 2q \]

Total cost \((c)\) as a function of quantity

\[ c = 20q + 300 \]

Profit \((\pi)\) as a function of quantity

\[
\pi(q) = \text{price} \times \text{quantity} - (\text{total cost}) \\
= p(q) \times q - c(q) = (100 - 2q) \times q - (20q + 300) \\
= -2q^2 + 80q - 300
\]

(a) Find critical point(s) of \(\pi(q)\)

\[
\pi'(q) = -4q + 80 = 0 \\
\Rightarrow q^* = 20, \quad \text{this is the profit maximizing quantity.}
\]

Second derivative test:

\[
\pi''(q) = -4 < 0 \\
\Rightarrow \pi(q) \text{ has a maximum at } q^* = 20.
\]

The profit maximizing price and the maximum profit are

\[
p^* = 100 - 2q^* = 60 \\
\pi^* = \pi(q^*) = -2(q^*)^2 + 80q^* - 300 = 500
\]

(b) Let \(\alpha\) be the marginal cost. The total cost is

\[ c = \alpha q + 300 \]

Profit as a function of quantity is

\[
\pi(q; \alpha) = (100-2q)q-(\alpha q+300) = -2q^2 + (100-\alpha)q - 300
\]

Let \(\pi^*(\alpha)\) be the maximum profit at marginal cost \(\alpha\).

We estimate the change in \(\pi^*\) when \(\alpha\) changes from \(\alpha_0 = 20\) to \(\alpha_0 + \Delta \alpha = 20.75\).

The linear Taylor approximation is

\[
\Delta \pi^* = \pi^*(\alpha_0 + \Delta \alpha) - \pi^*(\alpha_0) \approx \pi'(\alpha_0) \Delta \alpha
\]

The envelope theorem gives us
\[
\pi'(\alpha) = \frac{\partial \pi}{\partial \alpha}_{q = q^*(\alpha)} = -q^*(\alpha)
\]

In (a), we obtained that at \( \alpha_0 = 20 \), we have \( q^*(\alpha_0) = 20 \) and \( \pi^*(\alpha_0) = 500 \).

\[
\Rightarrow \quad \pi'(\alpha_0) = -q^*(\alpha_0) = -20
\]

Using this result in the linear Taylor approximation, we obtain

\[
\Delta \pi^* \approx \pi'(\alpha_0) \Delta \alpha = -20 \times 0.75 = -15
\]

\[
\Rightarrow \quad \pi^*(20.75) \approx 500 - 15 = 485
\]

(c) We maximize function \( \pi(q; \alpha) = -2q^2 + (100-\alpha)q - 300 \) exactly

Find critical point(s) of \( \pi(q) \)

\[
\pi'(q) = -4q + (100-\alpha) = 0
\]

\[
\Rightarrow \quad q^* = (100-\alpha)/4, \quad \text{this is the profit maximizing quantity}
\]

Second derivative test:

\[
\pi''(q) = -4 < 0
\]

\[
\Rightarrow \quad \pi(q) \text{ has a maximum at } q^* = (100-\alpha)/4.
\]

The maximum profit is

\[
\pi^*(\alpha) = \pi(q^*) = -2((100-\alpha)/4)^2 + (100-\alpha)(100-\alpha)/4 - 300
\]

\[
= (100-\alpha)^2/8 - 300
\]

At \( \alpha = 20.75 \)

\[
\pi^*(20.75) = (100-20.75)^2/8 - 300 = 485.07
\]

This is very close to the estimate obtained in (b), \( \pi^*(20.75) \approx 485 \).

**Problem 2**

The joint demand functions for the two products are

\[
Q_A = 100 - 3P_A + 2P_B
\]
\[
Q_B = 60 + 2P_A - 2P_B
\]

The total cost as a function of two prices is

\[
C = 20Q_A + 30Q_B + 1200
\]

\[
= 20(100 - 3P_A + 2P_B) + 30(60 + 2P_A - 2P_B) + 1200
\]

\[
= 5000 - 20P_B
\]

Profit (\( \pi \)) as a function of two prices is

\[
\pi(P_A, P_B) = P_A \times Q_A + P_B \times Q_B - C
\]
\[ P_A(100 - 3P_A + 2P_B) + P_B(60 + 2P_A - 2P_B) - (5000 - 20P_B) \]
\[ = -3P_A^2 + 4P_A P_B - 2P_B^2 + 100P_A + 80P_B - 5000 \]

(a) Find critical point(s) of \( \pi(P_A, P_B) \)

\[ \frac{\partial \pi(P_A, P_B)}{\partial P_A} = -6P_A + 4P_B + 100 = 0 \]  
\[ \frac{\partial \pi(P_A, P_B)}{\partial P_B} = 4P_A - 4P_B + 80 = 0 \]

(1) + (2) \implies -2P_A + 180 = 0 \implies P_A^* = 90

(2) \implies P_B^* = P_A^* + 20 = 110

\( (P_A^*, P_B^*) = (90, 110) \), these are the profit maximizing prices.

Second derivative test:

\[ \frac{\partial^2 \pi(P_A, P_B)}{\partial P_A^2} = -6 < 0, \quad \frac{\partial^2 \pi(P_A, P_B)}{\partial P_A \partial P_B} = 4, \quad \frac{\partial^2 \pi(P_A, P_B)}{\partial P_B^2} = -4 \]

\[ D = \frac{\partial^2 \pi(P_A, P_B)}{\partial P_A^2} \frac{\partial^2 \pi(P_A, P_B)}{\partial P_B^2} - \left( \frac{\partial^2 \pi(P_A, P_B)}{\partial P_A \partial P_B} \right)^2 = (-6)(-4) - 4^2 = 8 > 0 \]

\( \implies \pi(P_A, P_B) \) has a maximum at \( (P_A^*, P_B^*) = (90, 110) \).

The profit maximizing quantities and the maximum profit are

\[ Q_A^* = 100 - 3P_A^* + 2P_B^* = 100 - 3 \times 90 + 2 \times 110 = 50 \]
\[ Q_B^* = 60 + 2P_A^* - 2P_B^* = 60 + 2 \times 90 - 2 \times 110 = 20 \]
\[ \pi^* = \pi(P_A^*, P_B^*) = -3(P_A^*)^2 + 4P_A^* P_B^* - 2(P_B^*)^2 + 100P_A^* + 80P_B^* - 5000 \]
\[ = 3900 \]

(b) Let \( \alpha \) and \( \beta \) be the marginal costs of products A and B.

The total cost as a function of two prices is

\[ C = \alpha Q_A + \beta Q_B + 1200 \]
\[ = \alpha(100 - 3P_A + 2P_B) + \beta(60 + 2P_A - 2P_B) + 1200 \]

Profit as a function of two prices is

\[ \pi(P_A, P_B; \alpha, \beta) = P_A(100 - 3P_A + 2P_B) + P_B(60 + 2P_A - 2P_B) - (\alpha(100 - 3P_A + 2P_B) + \beta(60 + 2P_A - 2P_B) + 1200) \]

Let \( \pi^*(\alpha, \beta) \) be the maximum profit at marginal costs \( (\alpha, \beta) \).
We estimate the change in \( \pi^* \) when \((\alpha, \beta)\) changes from \((\alpha_0, \beta_0) = (20, 30)\) to \((\alpha_0 + \Delta \alpha, \beta_0 + \Delta \beta) = (21, 31.5)\).

The linear Taylor approximation is

\[
\Delta \pi^* = \pi^* (\alpha_0 + \Delta \alpha, \beta_0 + \Delta \beta) - \pi^* (\alpha_0, \beta_0) \approx \frac{\partial \pi(\alpha, \beta)}{\partial \alpha} \Delta \alpha + \frac{\partial \pi(\alpha, \beta)}{\partial \beta} \Delta \beta
\]

The envelope theorem gives us

\[
\frac{\partial \pi(\alpha, \beta)}{\partial \alpha} = \frac{\partial \pi}{\partial \alpha} \bigg|_{(P^*_A(\alpha, \beta), P^*_B(\alpha, \beta))} = -100 - 3P^*_A(\alpha, \beta) + 2P^*_B(\alpha, \beta)
\]

\[
\frac{\partial \pi(\alpha, \beta)}{\partial \beta} = \frac{\partial \pi}{\partial \beta} \bigg|_{(P^*_A(\alpha, \beta), P^*_B(\alpha, \beta))} = -60 + 2P^*_A(\alpha, \beta) - 2P^*_B(\alpha, \beta)
\]

In (a), we obtained that at \((\alpha_0, \beta_0) = (20, 30)\), we have

\[
P^*_A(\alpha_0, \beta_0) = 90, \quad P^*_B(\alpha_0, \beta_0) = 110 \]

\[
\pi^*(\alpha_0, \beta_0) = 3900
\]

\[
\frac{\partial \pi(\alpha, \beta)}{\partial \alpha} \bigg|_{(\alpha_0, \beta_0)} = -100 - 3P^*_A(\alpha_0, \beta_0) + 2P^*_B(\alpha_0, \beta_0) = -50
\]

\[
\frac{\partial \pi(\alpha, \beta)}{\partial \beta} \bigg|_{(\alpha_0, \beta_0)} = -60 + 2P^*_A(\alpha_0, \beta_0) - 2P^*_B(\alpha_0, \beta_0) = -20
\]

Using this result in the linear Taylor approximation, we obtain

\[
\Delta \pi^* \approx \frac{\partial \pi(\alpha, \beta)}{\partial \alpha} \bigg|_{(\alpha_0, \beta_0)} \Delta \alpha + \frac{\partial \pi(\alpha, \beta)}{\partial \beta} \bigg|_{(\alpha_0, \beta_0)} \Delta \beta = -50 \cdot 1 + (-20) \cdot 1.5 = -80
\]

\[
\Rightarrow \pi^*(21, 31.5) \approx 3900 - 80 = 3820
\]
3. \( f(x, y) = \frac{5}{3}x^3 + \frac{2}{3}y^3 - \frac{15}{2}x^2 + y^2 - 4y + 7 \)

\[
\begin{align*}
  f_x(x, y) &= 5x^2 - 15x = 0 \\
  f_y(x, y) &= 2y^2 + 2y - 4 = 0
\end{align*}
\]

Both equations are easily solved by factoring.

Critical points: \((0, -2), (0, 1), (3, -2), (3, 1)\)

4. \( f(x, y) = xy - x + y \)
\[ f_y(x, y) = x + 1 \]

Critical point: \((-1, 1)\)

5. \( f(x, y, z) = 2x^2 + xy + y^2 + 100 - z(x + y - 200) \)
\[
\begin{align*}
  f_x(x, y, z) &= 4x + y - z = 0 \\
  f_y(x, y, z) &= x + 2y - z = 0 \\
  f_z(x, y, z) &= -x - y + 200 = 0
\end{align*}
\]

Solving the system gives the critical point \((50, 150, 350)\).

6. \( f(x, y, z, w) = x^2 + y^2 + z^2 + w(x + y + z - 3) \)
\[
\begin{align*}
  f_x(x, y, z, w) &= 2x + w = 0 \\
  f_y(x, y, z, w) &= 2y + w = 0 \\
  f_z(x, y, z, w) &= 2z + w = 0 \\
  f_w(x, y, z, w) &= x + y + z - 3 = 0
\end{align*}
\]

Solving the system gives the critical point \((1, 1, 1, -2)\).

7. \( f(x, y) = x^2 + 3y^2 + 4x - 9y + 3 \)
\[
\begin{align*}
  f_x(x, y) &= 2x + 4 = 0 \\
  f_y(x, y) &= 6y - 9 = 0
\end{align*}
\]

Critical point \((2, 3/2)\).

Second-Derivative Test
\[ f_{xx}(x, y) = 2, f_{yy}(x, y) = 6, f_{xy}(x, y) = 0 \]

At \((-2, 3/2)\), \(D = (2)(6) - 0^2 = 12 > 0\) and
\[ f_{xx}(x, y) = 2 > 0 \]

Thus at \((-2, 3/2)\) there is a relative minimum.

8. \( f(x, y) = -2x^2 + 8x - 3y^2 + 24y + 7 \)
\[
\begin{align*}
  f_x(x, y) &= -4x + 8 = 0 \\
  f_y(x, y) &= -6y + 24 = 0
\end{align*}
\]

Critical point: \((2, 4)\)

Second-Derivative Test
\[ f_{xx}(x, y) = -4, f_{yy}(x, y) = -6 \]
\[ f_{xy}(x, y) = 0 \]

At \((2, 4)\),
\[ D = (-4)(-6) - 0^2 = 24 > 0 \] and
\[ f_{xx}(x, y) = -4 < 0 \]

thus there is a relative maximum at \((2, 4)\).

9. \( f(x, y) = y - y^2 - 3x - 6x^2 \)
\[
\begin{align*}
  f_x(x, y) &= -3 - 12x = 0 \\
  f_y(x, y) &= 1 - 2y = 0
\end{align*}
\]

Critical point \((2, 1)\)

Second-Derivative Test
\[ f_{xx}(x, y) = -12, f_{yy}(x, y) = -2, f_{xy}(x, y) = 0 \]

At \((-1, 1)\), \(D = (-12)(-2) - 0^2 = 24 > 0\) and
\[ f_{xx}(x, y) = -12 < 0 \] . Thus at \((-1, 1)\) there is a relative maximum.

10. \( f(x, y) = 2x^2 + \frac{3}{2} y^2 + 3xy - 10x - 9y + 2 \)
\[
\begin{align*}
  f_x(x, y) &= 4x + 3y - 10 = 0 \\
  f_y(x, y) &= 3y + 3x - 9 = 0
\end{align*}
\]

Critical point: \((1, 2)\)

Second-Derivative Test
\[ f_{xx}(x, y) = 4, f_{yy}(x, y) = 3, f_{xy}(x, y) = 3 \]

At \((1, 2)\), \(D = (4)(3) - 3^2 = 3 > 0\) and
\[ f_{xx}(x, y) = 4 > 0 \]

thus there is a relative minimum at \((1, 2)\).

11. \( f(x, y) = x^2 + 3xy + y^2 - 9x - 11y + 3 \)
\[
\begin{align*}
  f_x(x, y) &= 2x + 3y - 9 = 0 \\
  f_y(x, y) &= 3x + 2y - 11 = 0
\end{align*}
\]

Critical point: \((3, 1)\)

Second-Derivative Test
\[ f_{xx}(x, y) = 2, f_{yy}(x, y) = 2, f_{xy}(x, y) = 3 \]

At \((3, 1)\),
\[ D = (2)(2) - (3)^2 = -5 < 0 \]

so there is no relative extremum at \((3, 1)\).

12. \( f(x, y) = \frac{y^3}{3} + y^2 - 2x + 2y - 2xy \)
\[
\begin{align*}
  f_x(x, y) &= x^2 - 2 - 2y = 0 \\
  f_y(x, y) &= 2y + 2 - 2x = 0
\end{align*}
\]
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13. \( f(x, y) = \frac{1}{3}(x^3 + 8y^3) - 2(x^2 + y^2) + 1 \)
   
   \[ \begin{align*}
   f_x(x, y) &= x^2 - 4x = 0 \\
   f_y(x, y) &= 8y^2 - 4y = 0 \\
   \end{align*} \]

   Critical points: \((0, 0), (4, \frac{1}{2})\), \((0, \frac{1}{2})\), \((4, 0)\)

   Second-Derivative Test
   
   At \((4, \frac{1}{2})\), \(D = (4)(4) - 0^2 = 16 > 0\) and
   
   \(f_{xx}(x, y) = 4 > 0\); thus a relative minimum.

   At \((0, \frac{1}{2})\), \(D = (-4)(4) - 0^2 = -16 < 0\); thus neither.

   At \((4, 0)\), \(D = (4)(4) - 0^2 = -16 < 0\); thus neither.

14. \( f(x, y) = x^2 + y^2 - xy + x^3 \)
   
   \[ \begin{align*}
   f_x(x, y) &= 2x - y + 3x^2 = 0 \\
   f_y(x, y) &= 2y - x = 0 \\
   \end{align*} \]

   Critical points: \((0, 0), (-\frac{1}{2}, -\frac{1}{4})\)

   Second-Derivative Test
   
   \(f_{xx}(x, y) = 2 + 6x\), \(f_{yy}(x, y) = 2\), \(f_{xy}(x, y) = -1\). At \((0, 0)\),
   
   \(D = (2)(2) - (-1)^2 = 3 > 0\) and
   
   \(f_{xx}(x, y) = 2 > 0\); thus relative minimum.

   At \((-\frac{1}{2}, -\frac{1}{4})\), \(D = (-1)(2) - (-1)^2 = -3 < 0\); thus neither.

15. \( f(l, k) = \frac{j^2}{2} + 2lk + 3k^2 - 69l - 164k + 17 \)
   
   \[ \begin{align*}
   f_l(l, k) &= l + 2k - 69 = 0 \\
   f_k(l, k) &= 2l + 6k - 164 = 0 \\
   \end{align*} \]

   Critical point: \((43, 13)\)

   Second-Derivative Test
   
   \(f_{ll}(l, k) = 1, f_{lk}(l, k) = 6, f_{kk}(l, k) = 2\)

   At \((43, 13)\), \(D = (1)(6) - 2^2 = 2 > 0\) and
   
   \(f_{ll}(l, k) = 1 > 0\); thus there is a relative minimum at \((43, 13)\).

16. \( f(l, k) = l^2 + 4k^2 - 4lk \)
   
   \[ \begin{align*}
   f_l(l, k) &= 2l - 4k \\
   f_k(l, k) &= 8k - 4l \\
   \end{align*} \]

   Critical points: \((2r, r)\) where \(r\) is any real number.

   Second Derivative Test
   
   \(f_{ll}(l, k) = 2, f_{lk}(l, k) = 8, f_{kk}(l, k) = -4\).

   At \((2r, r)\), \(D = (2)(8) - (-4)^2 = 0\), thus we cannot make a conclusion.

17. \( f(p, q) = pq - \frac{1}{p} - \frac{1}{q} \)
   
   \[ \begin{align*}
   f_p(p, q) &= q + \frac{1}{p^2} = 0 \\
   f_q(p, q) &= p + \frac{1}{q^2} = 0 \\
   \end{align*} \]

   Critical point: \((-1, -1)\)

   Second-Derivative Test
   
   \(f_{pp}(p, q) = -\frac{2}{p^3}, f_{qq}(p, q) = -\frac{2}{q^3}\), \(f_{pq}(p, q) = 1\). At \((-1, -1)\),
   
   \(D = (2)(2) - (-1)^2 = 3 > 0\) and \(f_{pp}(p, q) = 2 > 0\); thus there is a relative minimum at \((-1, -1)\).

18. \( f(x, y) = (x - 3)(y - 3)(x + y - 3) \)
   
   \[ \begin{align*}
   &= (y - 3)(x^2 + xy - 6x - 3y + 9) \\
   &= (x - 3)(xy - 3x + y^2 - 6y + 9) \\
   \end{align*} \]

   \[ \begin{align*}
   f_x(x, y) &= (y - 3)(2x + 3) = 0 \\
   f_y(x, y) &= (x - 3)(x + 2y - 3) = 0 \\
   \end{align*} \]

   Critical points: \((2, 2), (3, 3), (3, 0), (0, 3)\)

   Second-Derivative Test
   
   \(f_{xx}(x, y) = 2(y - 3), f_{yy}(x, y) = 2(x - 3), f_{xy}(x, y) = 2x + 2y - 9\). At \((2, 2)\),
21. \( P = f(l,k) = 2.181l^2 - 0.021l^3 + 1.97k^2 - 0.03k^3 \)

\[
\begin{align*}
P_l &= 4.36l - 0.06l^2 = 0 \\
P_k &= 3.94k - 0.09k^2 = 0
\end{align*}
\]

Critical points: \((0, 0), \left(0, \frac{394}{9}\right), \left(\frac{218}{3}, 0\right), \left(\frac{218}{3}, \frac{394}{9}\right)\) 

Second-Derivative Test
\[ P_{ll} = 4.36 - 0.12l, \quad P_{kk} = 3.94 - 0.18k, \quad P_{lk} = 0. \]

At \((0, 0), \quad D = (4.36)(3.94) - 0^2 > 0 \quad \text{and} \quad P_{ll} = 4.36 > 0 \quad \text{relative minimum.} \]

At \(0, \frac{394}{9}\), \(D = (4.36)(-3.94) - 0^2 < 0 \); thus no extremum.

At \(\frac{218}{3}, 0\), \(D = (-4.36)(3.94) - 0^2 < 0 \); thus no extremum.

At \(\frac{218}{3}, \frac{394}{9}\), \(D = (-4.36)(-3.94) - 2^2 > 0 \)

and \(P_{ll} = -4.36 < 0 \); then \(l = \frac{218}{3} = 72.67\), 

\[ k = \frac{394}{9} = 43.78 \quad \text{gives a relative maximum.} \]

22. \( Q = 18c + 20d - 2c^2 - 4d^2 - cd \)

\[
\begin{align*}
Q_c &= 18 - 4c - d = 0 \\
Q_d &= 20 - 8d - c = 0
\end{align*}
\]

Critical point: \(c = 4, \quad d = 2 \quad \text{and} \quad Q_{cc} = -4, \quad Q_{dd} = -8, \quad Q_{cd} = -1 \)

When \(c = 4 \quad \text{and} \quad d = 2 \), then 

\(D = (-4)(-8) - (-1)^2 > 0 \quad \text{and} \quad Q_{cc} = -4 < 0 \); 

thus relative maximum at \(c = 4, \quad d = 2 \).
24. Profit per lb for \( A = p_A - a \).
Profit per lb for \( B = p_B - b \).
Total Profit \( P = (p_A - a)q_A + (p_B - b)q_B \)
\[
P = (p_A - a)[5(p_B - p_A)] + (p_B - b)[500 + 5(p_A - 2p_B)]
\]
\[
\frac{\partial P}{\partial p_A} = -5(2p_A - 2p_B + b - a) = 0
\]
\[
\frac{\partial P}{\partial p_B} = 5(2p_A - 4p_B + 2b - a + 100) = 0
\]
Critical point: \( p_A = 50 + \frac{a}{2} \), \( p_B = 50 + \frac{b}{2} \)
\[
\frac{\partial^2 P}{\partial p_A^2} = -10, \quad \frac{\partial^2 P}{\partial p_B^2} = -20, \quad \frac{\partial^2 P}{\partial p_B \partial p_A} = 10
\]
When \( p_A = 50 + \frac{a}{2} \) and \( p_B = 50 + \frac{b}{2} \), then \( D = (-10)(-20) - (10)^2 = 100 > 0 \) and \( \frac{\partial^2 P}{\partial p_A^2} = -10 < 0 \); thus a relative maximum at \( p_A = 50 + \frac{a}{2} \), \( p_B = 50 + \frac{b}{2} \).

25. \( p_A = 100 - q_A \), \( p_B = 84 - q_B \), \( c = 600 + 4(q_A + q_B) \).
Revenue from market \( A = r_A = p_Aq_A = (100 - q_A)q_A \). Revenue from market \( B = r_B = p_Bq_B = (84 - q_B)q_B \).
Total Profit = Total Revenue – Total Cost
\[
P = (100 - q_A)q_A + (84 - q_B)q_B - [600 + 4(q_A + q_B)]
\]
\[
\frac{\partial P}{\partial q_A} = 96 - 2q_A = 0
\]
\[
\frac{\partial P}{\partial q_B} = 80 - 2q_B = 0
\]
Critical point: \( q_A = 48 \), \( q_B = 40 \)
\[
\frac{\partial^2 P}{\partial q_A^2} = -2, \quad \frac{\partial^2 P}{\partial q_B^2} = -2, \quad \frac{\partial^2 P}{\partial q_B \partial q_A} = 0
\]
At \( q_A = 48 \) and \( q_B = 40 \), then \( D = (-2)(-2) - 0^2 = 4 > 0 \) and \( \frac{\partial^2 P}{\partial q_A^2} = -2 < 0 \); thus relative maximum at \( q_A = 48 \), \( q_B = 40 \). When \( q_A = 48 \) and \( q_B = 40 \), then selling prices are \( p_A = 52 \), \( p_B = 44 \), and profit = 3304.

26. \( q_A = 16 - p_A + p_B \), \( q_B = 24 + 2p_A - 4p_B \)
Revenue from \( A = p_Aq_A \), Revenue from \( B = p_Bq_B \).
Total cost of producing \( q_A \) units of \( A \) and \( q_B \) units of \( B \) is \( 2q_A + 4q_B \).
Total Profit = Total Revenue – Total Cost
\[
P = p_Aq_A + p_Bq_B - (2q_A + 4q_B)
\]
\[
\]
\[
= -p_A^2 - 4p_B^2 + 3p_Ap_B + 10p_A + 38p_B - 128
\]
\[
\begin{align*}
\frac{\partial P}{\partial p_A} &= -2p_A + 3p_B + 10 \\
\frac{\partial P}{\partial p_B} &= 3p_A - 8p_B + 38
\end{align*}
\]

Critical point: \( p_A = \frac{194}{7}, \quad p_B = \frac{106}{7} \)

\[
\begin{align*}
\frac{\partial^2 P}{\partial p_A^2} &= -2, \quad \frac{\partial^2 P}{\partial p_B^2} = -8, \quad \frac{\partial^2 P}{\partial p_B \partial p_A} = 3
\end{align*}
\]

At \( p_A = \frac{194}{7}, \quad p_B = \frac{106}{7} \), then \( D = (-2)(-8) - 3^2 = 7 > 0 \) and \( \frac{\partial^2 P}{\partial p_A^2} = -2 < 0 \); thus relative maximum at \( p_A = \frac{194}{7}, \quad p_B = \frac{106}{7} \).

\[
\begin{align*}
q_A &= 16 - \frac{194}{7} + \frac{106}{7} = \frac{24}{7} \\
q_B &= 24 + 2 \left( \frac{194}{7} \right) - 4 \left( \frac{106}{7} \right) = \frac{132}{7}
\end{align*}
\]

So, \( \frac{24}{7} = 3 \) of \( A \) and \( \frac{132}{7} = 19 \) of \( B \) should be sold.

27. \( c = \frac{3}{2}q_A^2 + 3q_B^2, \quad p_A = 60 - q_A^2, \quad p_B = 72 - 2q_B^2 \)

Total Profit = Total Revenue – Total Cost
\[
P = (p_Aq_A + p_Bq_B) - c
\]

\[
\begin{align*}
P &= 60q_A - q_A^3 + 72q_B - 2q_B^3 - \left( \frac{3}{2}q_A^2 + 3q_B^2 \right) \\
\frac{\partial P}{\partial q_A} &= 60 - 3q_A - 3q_A^2 = 3(5 + q_A)(4 - q_A) \\
\frac{\partial P}{\partial q_B} &= 72 - 6q_B - 6q_B^2 = 6(4 + q_B)(3 - q_B)
\end{align*}
\]

Since we want \( q_A \geq 0 \) and \( q_B \geq 0 \), the critical point occurs when \( q_A = 4 \) and \( q_B = 3 \).

\[
\begin{align*}
\frac{\partial^2 P}{\partial q_A^2} &= -3 - 6q_A, \quad \frac{\partial^2 P}{\partial q_B^2} = -6 - 12q_B, \quad \frac{\partial^2 P}{\partial q_B \partial q_A} = 0
\end{align*}
\]

and \( \frac{\partial^2 P}{\partial q_A^2} = -27 < 0 \); thus relative maximum at \( q_A = 4, \quad q_B = 3 \).

28. \( c = 2(q_A + q_B + q_Aq_B) \)

Total Profit = Total Revenue – Total Cost
\[
P = (p_Aq_A + p_Bq_B) - c
\]

\[
\begin{align*}
P &= p_A(20 - 2p_A) + p_B(10 - p_B) - [20 - 2p_A + 10 - p_B + (20 - 2p_A)(10 - p_B)] \\
&= -2p_A^2 - 2p_B^2 - 2p_Ap_B + 42p_A + 31p_B + 230
\end{align*}
\]
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\[
\begin{align*}
\frac{\partial P}{\partial p_A} &= -4p_A - 2p_B + 42 \\
\frac{\partial P}{\partial p_B} &= -2p_A - 2p_B + 31 \\
\end{align*}
\]

Critical point: \( p_A = \frac{11}{2}, \ p_B = 10 \)

\[
\frac{\partial^2 P}{\partial p_A^2} = 4, \quad \frac{\partial^2 P}{\partial p_B^2} = -2, \quad \frac{\partial^2 P}{\partial p_B \partial p_A} = -2
\]

When \( p_A = \frac{11}{2}, \ p_B = 10 \), then \( D = (-4)(-2) - (-2)^2 = 4 > 0 \), and \( \frac{\partial^2 P}{\partial p_A^2} = -4 < 0 \), so the maximum profit occurs when \( p_A = 5.5 \) and \( p_B = 10 \). At these prices, \( q_A = 9, \ q_B = 0 \), and the total profit is 40.5.

29. Refer to the diagram in the text.

\[
xyz = 6 \\
C = 3xy + 2[1(xz)] + 2[0.5(yz)]
\]

Note that \( z = \frac{6}{xy} \). Thus

\[
C = 3xy + 2xz + yz = 3xy + 2x \left( \frac{6}{xy} \right) + y \left( \frac{6}{xy} \right) = 3xy + \frac{12}{y} + \frac{6}{x}
\]

\[
\begin{align*}
\frac{\partial C}{\partial x} &= 3y - \frac{6}{x^2} = 0 \\
\frac{\partial C}{\partial y} &= 3x - \frac{12}{y^2} = 0
\end{align*}
\]

A critical point occurs at \( x = 1 \) and \( y = 2 \). Thus \( z = 3 \).

\[
\frac{\partial^2 C}{\partial x^2} = \frac{12}{x^3}, \quad \frac{\partial^2 C}{\partial y^2} = \frac{24}{y^3}, \quad \frac{\partial^2 C}{\partial x \partial y} = 3.
\]

When \( x = 1 \) and \( y = 2 \), then \( d = (12)(3) - (3)^2 = 27 > 0 \) and \( \frac{\partial^2 C}{\partial x^2} = 12 > 0 \). Thus we have a minimum. The dimensions should be 1 ft by 2 ft by 3 ft.

30. \( p = 92 - q_A - q_B, \ c_A = 10q_A, \ c_B = 0.5q_B^2 \)

Since Profit = Total Revenue – Total Cost, then

Profit of A = \( pq_A - c_A \) and

Profit of B = \( pq_B - c_B \).

Thus profit \( P \) of monopoly is

\[
P = (92 - q_A - q_B)(q_A + q_B) - 10q_A - 0.5q_B^2
\]

= \( 82q_A + 92q_B - q_A^2 - 2q_Aq_B - 1.5q_B^2 \)
\[
\frac{\partial P}{\partial q_A} = 82 - 2q_A - 2q_B = 0 \\
\frac{\partial P}{\partial q_B} = 92 - 2q_A - 3q_B = 0
\]

Critical point: \( q_A = 31, q_B = 10 \)
\[
\frac{\partial^2 P}{\partial q_A^2} = -2, \quad \frac{\partial^2 P}{\partial q_B^2} = -3, \quad \frac{\partial^2 P}{\partial q_B \partial q_A} = -2
\]

When \( q_A = 31 \) and \( q_B = 10 \), then
\( D = (-2)(-3) - (-2)^2 = 2 > 0 \) and
\( \frac{\partial^2 P}{\partial q_A^2} = -2 < 0 \); thus relative maximum at \( q_A = 31, q_B = 10 \).

31. \( y = 2 - x \)

\( f(x, y) = x^2 + 3(2 - x)^2 + 9 \)

Setting the derivative equal to 0 gives
\( 2x + 6(2 - x)(-1) = 0 \).
\( 2x - 12 + 6x = 0, 8x - 12 = 0 \),
\( 8x = 12, \) or \( x = \frac{3}{2} \).
The second-derivative is
\( 8 > 0 \), so we have a relative minimum. If \( x = \frac{3}{2} \), then \( y = \frac{1}{2} \). Thus there is a relative minimum at \( \left( \frac{3}{2}, \frac{1}{2} \right) \).

32. \( y = \frac{x - 10}{4} \)

\( f(x, y) = x^2 + 4\left(\frac{x - 10}{4}\right)^2 + 6 \)

Setting the derivative equal to 0 gives
\( 2x + 4(2)\left(\frac{x - 10}{4}\right)\left(\frac{1}{4}\right) = 0 \), from which \( x = 2 \).

The second-derivative is \( \frac{5}{2} > 0 \), so we have a relative minimum. If \( x = 2 \), then \( y = -2 \). Thus at \((2, -2)\) there is a relative minimum.

33. \( c = q_A^2 + 3q_B^2 + 2q_Aq_B + aq_A + bq_B + d \)

We are given that \( (q_A, q_B) = (3, 1) \) is a critical point.
\[
\frac{\partial c}{\partial q_A} = 2q_A + 2q_B + a = 0 \\
\frac{\partial c}{\partial q_B} = 6q_B + 2q_A + b = 0
\]

Substituting the given values for \( q_A \) and \( q_B \) into both equations gives \( a = -8 \) and \( b = -12 \).

Since \( c = 15 \) when \( q_A = 3 \) and \( q_B = 1 \), from the joint-cost function we have
\( 15 = 3^2 + 3(1^2) + 2(3)(1) + (-8)(3) + (-12) + d \),
\( 15 = -18 + d, 33 = d \). Thus \( a = -8, b = -12, d = 33 \).

34. \( D(a, b) > 0 \)

\( f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2 > 0 \)
\( f_{xx}(a, b)f_{yy}(a, b) > (f_{xy}(a, b))^2 \geq 0 \)

a. Since the product \( f_{xx}(a, b)f_{yy}(a, b) \) is positive, \( f_{xx}(a, b) \) and \( f_{yy}(a, b) \) must have the same sign. That is \( f_{xx}(a, b) < 0 \) if and only if \( f_{yy}(a, b) < 0 \).

b. Since the product \( f_{xx}(a, b)f_{yy}(a, b) \) is positive, \( f_{xx}(a, b) \) and \( f_{yy}(a, b) \) must have the same sign. That is \( f_{xx}(a, b) > 0 \) if and only if \( f_{yy}(a, b) > 0 \).
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35. a. Profit = Total Revenue – Total Cost
   \[ P = p_A q_A + p_B q_B - \text{total cost} \]
   \[ = \left(35 - 2q_A^2 + q_B\right)q_A + (20 - q_B + q_A)q_B - \left(-8 - 2q_A^3 + 3q_A q_B + 30q_A + 12q_B + \frac{1}{2} q_A^2\right) \]
   \[ P = 5q_A - \frac{1}{2} q_A^2 - q_A q_B + 8q_B - q_B^2 + 8 \]
   \[ \frac{\partial P}{\partial q_A} = 5 - q_A - q_B = 0 \]
   \[ \frac{\partial P}{\partial q_B} = -q_A + 8 - 2q_B = 0 \]
   Critical point: \( q_A = 2, q_B = 3 \)
   \[ \frac{\partial^2 P}{\partial q_A^2} = -1, \frac{\partial^2 P}{\partial q_B^2} = -2, \frac{\partial^2 P}{\partial q_B \partial q_A} = -1 \]
   At \( q_A = 2 \) and \( q_B = 3 \), then \( D = -1(-2) - (-1)^2 = 1 > 0 \) and \( \frac{\partial^2 P}{\partial q_A^2} = -1 < 0 \); thus there is a relative maximum profit for 2 units of A and 3 units of B.

b. Substituting \( q_A = 2 \) and \( q_B = 3 \) into the formulas for \( p_A, p_B, \) and \( P \) gives a selling price for A of 30, a selling price for B of 19, and a relative maximum profit of 25.

36. \[ P = 300 \left[ \frac{7x}{2 + x} + \frac{4y}{5 + y} \right] - x - y \]
   \[ \frac{\partial P}{\partial x} = 300 \cdot \frac{(2 + x)(7) - 7x}{(2 + x)^2} - 1 = \frac{4200}{(2 + x)^2} - 1 = 0 \]
   \[ \frac{\partial P}{\partial y} = 300 \cdot \frac{(5 + y)(4) - 4y}{(5 + y)^2} - 1 = \frac{6000}{(5 + y)^2} - 1 = 0 \]
   \[ 4200 = (2 + x)^2 \]
   \[ 6000 = (5 + y)^2 \]
   \[ \pm \sqrt{4200} = 2 + x \quad \pm \sqrt{6000} = 5 + y \]
   \[ x = -2 \pm \sqrt{4200} = -2 \pm 10\sqrt{42} \quad y = -5 \pm \sqrt{6000} = -5 \pm 20\sqrt{15} \]
   The values of \( x \) and \( y \) must be nonnegative.
   Critical point: \( x = 10\sqrt{42} - 2, \quad y = 20\sqrt{15} - 5 \)
   \[ \frac{\partial^2 P}{\partial x^2} = -\frac{8400}{(2 + x)^3}, \quad \frac{\partial^2 P}{\partial y^2} = -\frac{12000}{(5 + y)^3}, \quad \frac{\partial^2 P}{\partial y \partial x} = 0 \]
   At \( x = 10\sqrt{42} - 2 \) and \( y = 20\sqrt{15} - 5 \), then \( D = (0.031)(0.026) - 0^2 > 0 \) and \( \frac{\partial^2 P}{\partial x^2} = -0.031 < 0 \).
   Thus relative maximum profit at \( x = 10\sqrt{42} - 2 = 62.81, \quad y = 20\sqrt{15} - 5 = 72.46. \)

37. a. \[ P = 5T \left(1 - e^{-x} \right) - 20x - 0.1T^2 \]
   \[ \frac{\partial P}{\partial T} = 5 \left(1 - e^{-x} \right) - 0.2T \]
   \[ \frac{\partial P}{\partial x} = 5Te^{-x} - 20 \]
At the point \((T, x) = (20, \ln 5)\),
\[
\frac{\partial P}{\partial T} = 5 \left(1 - e^{-\ln 5}\right) - 0.2(20) = 5 \left(1 - \frac{1}{5}\right) - 4 = 0
\]
\[
\frac{\partial P}{\partial x} = 5(20)e^{-\ln 5} - 20 = 100 \left(\frac{1}{5}\right) - 20 = 0
\]
Thus \((20, \ln 5)\) is a critical point. In a similar fashion we verify that \((5, \ln \frac{5}{4})\) is a critical point.

c. \[
\frac{\partial^2 P}{\partial T^2} = -0.2, \quad \frac{\partial^2 P}{\partial x^2} = -5Te^{-x}, \quad \frac{\partial^2 P}{\partial T \partial x} = 5e^{-x}
\]
At \((20, \ln 5)\),
\[
D = (-0.2) \left[ -5(20)e^{-\ln 5} \right] - \left(5e^{-\ln 5}\right)^2 = 20 \left(\frac{1}{5}\right) - \left[5 \left(\frac{1}{5}\right)\right]^2 = 3 > 0,
\]
and \[
\frac{\partial^2 P}{\partial T^2} = -0.2 < 0 . \text{ Thus we get a relative maximum at } (20, \ln 5).
\]
At \(\left(5, \ln \frac{5}{4}\right)\),
\[
D = (-0.2) \left[ -5(5)e^{-\ln\left(\frac{5}{4}\right)} \right] - \left(5e^{-\ln\left(\frac{5}{4}\right)}\right)^2 = 5 \left(\frac{4}{5}\right) - \left[5 \left(\frac{4}{5}\right)\right]^2 = -12 < 0 , \text{ so there is no relative extremum at } \left(5, \ln \frac{5}{4}\right).
\]

Problems 17.7

1. \(f(x, y) = x^2 + 4y^2 + 6, \quad 2x - 8y = 20\)

\[
F(x, y, \lambda) = x^2 + 4y^2 + 6 - \lambda(2x - 8y - 20)
\]
\[
\begin{cases}
F_x = 2x - 2\lambda = 0 \\ F_y = 8y + 8\lambda = 0 \\ F_\lambda = -2x + 8y + 20 = 0
\end{cases}
\]
From (1), \(x = \frac{\lambda}{2} ; \text{ from (2), } y = -\lambda . \) Substituting \(x = \frac{\lambda}{2} \text{ and } y = -\lambda \) into (3) gives \(-2\lambda - 8\lambda + 20 = 0 , \)
\(-10\lambda = 20 , \text{ so } \lambda = 2 . \) Thus \(x = 2 \text{ and } y = -2 . \) Critical point of \(F:\)
\(2, -2, 2) . \) Critical point of \(f: (2, -2) . \)

2. \(f(x, y) = 3x^2 - 2y^2 + 9, \quad x + y = 1\)

\[
F(x, y, \lambda) = 3x^2 - 2y^2 + 9 - \lambda(x + y - 1)
\]
\[
\begin{cases}
F_x = 6x - \lambda = 0 \\ F_y = -4y - \lambda = 0 \\ F_\lambda = -x - y + 1 = 0
\end{cases}
\]
From (1), \(x = \frac{\lambda}{6} ; \text{ from (2), } y = -\frac{\lambda}{4} . \)

Substituting \(x = \frac{\lambda}{6} \text{ and } y = -\frac{\lambda}{4} \) into (3) gives \(-\frac{\lambda}{6} + \frac{\lambda}{4} + 1 = 0 , \text{ from which } \lambda = -12 . \) Thus \(x = 2 \text{ and } y = 3 . \)
Critical point of \(F: (2, 3, -12) . \) Critical point of \(f: (2, 3) . \)
3. \( f(x, y, z) = x^2 + y^2 + z^2, \ 2x + y - z = 9 \)

\[
F(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda(2x + y - z - 9)
\]

\[
\begin{align*}
F_x &= 2x - 2\lambda = 0 \\
F_y &= 2y - \lambda = 0 \\
F_z &= 2z + \lambda = 0 \\
F_\lambda &= -2x - y + z + 9 = 0
\end{align*}
\]

From (1), \( x = \lambda \); from (2), \( y = \frac{\lambda}{2} \); from (3), \( z = \frac{\lambda}{2} \). Substituting into (4) gives

\[-2\lambda - \frac{\lambda}{2} + \left(\frac{-\lambda}{2}\right) + 9 = 0, \ -6\lambda + 18 = 0, \ \text{so} \ \lambda = 3. \ \text{Thus} \ x = 3, \ y = \frac{3}{2}, \ z = -\frac{3}{2}. \ \text{Critical point of} \ F: \left(3, \frac{3}{2}, -\frac{3}{2}\right). \ \text{Critical point of} \ f: \left(3, \frac{3}{2}, -\frac{3}{2}\right).\]

4. \( f(x, y, z) = x + y + z, \ xyz = 8 \)

\[
F(x, y, z, \lambda) = x + y + z - \lambda(xyz - 8)
\]

\[
\begin{align*}
F_x &= 1 - \lambda yz = 0 \\
F_y &= 1 - \lambda xz = 0 \\
F_z &= 1 - \lambda xy = 0 \\
F_\lambda &= -xyz + 8 = 0
\end{align*}
\]

From (1) and (2), \( \lambda yz = \lambda xz \), so \( y = x \). From (2) and (3), \( \lambda xz = \lambda xy \), so \( y = x \). Therefore
\( x = y = z \), so from (4), \( x = y = z = 2 \). Hence, Critical point of \( f \) is \( (2, 2, 2) \). Note that it is not necessary to determine \( \lambda \).

5. \( f(x, y, z) = 2x^2 + xy + y^2 + z, \ x + 2y + 4z = 3 \)

\[
F(x, y, z, \lambda) = 2x^2 + xy + y^2 + z - \lambda(x + 2y + 4z - 3)
\]

\[
\begin{align*}
F_x &= 4x + y - \lambda = 0 \\
F_y &= x + 2y - 2\lambda = 0 \\
F_z &= 1 - 4\lambda = 0 \\
F_\lambda &= -x - 2y - 4z + 3 = 0
\end{align*}
\]

From the third equation we have \( \lambda = \frac{1}{4} \).

Substituting this value into the first two equations and then eliminating \( y \) gives \( x = 0 \) and \( y = \frac{1}{4} \). Finally, solving for \( z \) in the last equation gives \( z = -\frac{7}{8} \).

Critical point of \( F \): \( \left(0, \frac{1}{4}, -\frac{7}{8}, \frac{1}{4}\right) \)

Critical point of \( f \): \( \left(0, \frac{1}{4}, -\frac{7}{8}\right) \)

6. \( f(x, y, z) = xyz^2, \ x - y + z = 20 \ (xyz \neq 0) \)

\[
f(x, y, z, \lambda) = xyz - \lambda(x - y + z - 20)
\]

\[
\begin{align*}
F_x &= xyz^2 - \lambda = 0 \\
F_y &= xz^2 + \lambda = 0 \\
F_z &= 2xyz - \lambda = 0 \\
F_\lambda &= -x + y - z + 20 = 0
\end{align*}
\]

From (1) and (2), \( y = x \), From (1) and (3), \( z = 2x \). Hence from (4), \( x = 5 \), so \( y = -5 \) and \( z = 10 \). Critical point of \( f \) is \( (5, -5, 10) \). Note that it is not necessary to determine \( \lambda \).

7. \( f(x, y, z) = xyz, \ x + y + z = 1 \ (xyz \neq 0) \)

\[
f(x, y, z, \lambda) = xyz - \lambda(x + y + z - 1)
\]

\[
\begin{align*}
F_x &= yz - \lambda = 0 \\
F_y &= xz - \lambda = 0 \\
F_z &= xy - \lambda = 0 \\
F_\lambda &= -x - y - z + 1 = 0
\end{align*}
\]

From (1) and (2), \( y = x \). From (1) and (3), \( x = z \).

Hence from (4) \( x = y = z = \frac{1}{3} \). Critical point of \( f \) is \( \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \). Note that it is not necessary to determine \( \lambda \).

8. \( f(x, y, z) = x^2 + y^2 + z^2, \ x + y + z = 3 \)

\[
f(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda(x + y + z - 3)
\]

\[
\begin{align*}
F_x &= 2x - \lambda = 0 \\
F_y &= 2y - \lambda = 0 \\
F_z &= 2z - \lambda = 0 \\
F_\lambda &= -x - y - z + 3 = 0
\end{align*}
\]

From (1)–(3), \( x = y = z = \frac{\lambda}{2} \). Substituting into (4), \( \frac{\lambda}{2} + \frac{\lambda}{2} + \frac{\lambda}{2} = 3 \), so \( \lambda = 2 \). Thus \( x = 1, \ y = 1, \ z = 1 \). Critical point of \( F \): \( (1, 1, 1) \). Critical point of \( f \): \( (1, 1, 1) \).