AMS11B: Mathematical Methods for Economists II

Solutions of Assignment #7

Problems from supplementary note #2

1 $f(x, y) = x^3y + 3x^2y^2 - 2xy^2 + 3x - 5y$, $(x_0, y_0) = (1, 1)$

$f(1, 1) = 1 + 3 - 2 + 3 - 5 = 0$

$f_x = 3x^2y + 6xy^2 - 2y^2 + 3$, $f_x(1, 1) = 3 + 6 - 2 + 3 = 10$

$f_y = x^3 + 6x^2y - 4xy - 5$, $f_y(1, 1) = 1 + 6 - 4 - 5 = -2$

$f_{xx} = 6xy + 6y^2$, $f_{xx}(1, 1) = 6 + 6 = 12$

$f_{xy} = 3x^2 + 12xy - 4y$, $f_{xy}(1, 1) = 3 + 12 - 4 = 11$

$f_{yy} = 6x^2 - 4x$, $f_{yy}(1, 1) = 6 - 4 = 2$

$$T_1(x, y) = f(1,1) + f_x \cdot (x-1) + f_y \cdot (y-1)$$
$$= 10(x-1) - 2(y-1)$$

$$T_2(x, y) = T_1(x, y) + \frac{1}{2} f_{xx} \cdot (x-1)^2 + f_{xy} \cdot (x-1)(y-1) + \frac{1}{2} f_{yy} \cdot (y-1)^2$$
$$= 10(x-1) - 2(y-1) + 6(x-1)^2 + 11(x-1)(y-1) + (y-1)^2$$

4. $H(x, y) = \sqrt{2x+5y} = (2x+5y)^{1/2}$, $(x_0, y_0) = (3, 2)$

$$H(3,2) = \sqrt{2\times 3 + 5\times 2} = \sqrt{16} = 4$$

$$H_x = \frac{1}{2}(2x+5y)^{-1/2}2 = (2x+5y)^{-1/2}, \quad H_x(3,2) = (16)^{-1/2} = \frac{1}{4}$$

$$H_y = \frac{1}{2}(2x+5y)^{-1/2}5 = \frac{5}{2}(2x+5y)^{-1/2}, \quad H_y(3,2) = \frac{5}{2}(16)^{-1/2} = \frac{5}{8}$$

$$H_{xx} = \frac{-1}{2}(2x+5y)^{-3/2}2 = -(2x+5y)^{-3/2}, \quad H_{xx}(3,2) = -(16)^{-3/2} = \frac{-1}{64}$$

$$H_{xy} = \frac{-1}{2}(2x+5y)^{-3/2}5 = \frac{-5}{2}(2x+5y)^{-3/2}, \quad H_{xy}(3,2) = \frac{-5}{2}(16)^{-3/2} = \frac{-5}{128}$$

$$H_{yy} = \frac{5}{2}\left(\frac{-1}{2}\right)(2x+5y)^{-3/2}5 = \frac{-25}{4}(2x+5y)^{-3/2}, \quad H_{yy}(3,2) = \frac{-25}{4}(16)^{-3/2} = \frac{-25}{256}$$

$$T_1(x,y) = H(3,2) + H_x \cdot (x-1) + H_y \cdot (y-1)$$
$$= 4 + \frac{1}{4}(x-3) + \frac{5}{8}(y-2)$$

$$T_2(x,y) = T_1(x,y) + \frac{1}{2}H_{xx} \cdot (x-3)^2 + H_{xy} \cdot (x-3)(y-2) + \frac{1}{2}H_{yy} \cdot (y-2)^2$$
$$= 4 + \frac{1}{4}(x-3) + \frac{5}{8}(y-2) - \frac{1}{128}(x-3)^2 - \frac{5}{128}(x-3)(y-2) - \frac{25}{512}(y-2)^2$$

We use $T_2(x, y)$ to approximate $H(x, y)$ near (x_0, y_0).

$$\sqrt{17} = H(3.25, 2.1) \approx T_2(3.25, 2.1)$$

$$= 4 + \frac{1}{4}0.25 + \frac{5}{8}0.1 - \frac{1}{128}(0.25)^2 - \frac{5}{128}0.25 \times 0.1 - \frac{25}{512}(0.1)^2$$
$$= 4.123047$$

20. $\frac{1}{x+y+z}\left(1+\frac{\partial z}{\partial x}\right)+yz+xy\frac{\partial z}{\partial x}$

$$=\frac{\partial z}{\partial x}e^{x+y+z}+ze^{x+y+z}\left(1+\frac{\partial z}{\partial x}\right)$$

When $x = 0$, $y = 1$, and $z = 0$, then $\frac{1}{1}\left(1+\frac{\partial z}{\partial x}\right)+(1)(0)+(0)(1)\frac{\partial z}{\partial x}$

$$=\frac{\partial z}{\partial x}e^{1}+0(e^{1})\left(1+\frac{\partial z}{\partial x}\right)$$

$1+\frac{\partial z}{\partial x}=e\frac{\partial z}{\partial x},\ 1=\frac{\partial z}{\partial x}(e-1),\ \frac{\partial z}{\partial x}=\frac{1}{e-1}$

21. $c+\sqrt{c}=12+q_{\text{A}}\sqrt{9+q_{\text{B}}^2}$

a. If $q_{\text{A}}=6$ and $q_{\text{B}}=4$, then $c+\sqrt{c}=12+6(5)=42$, $\sqrt{c}=42-c$, $c=(42-c)^2=42^2-84c+c^2$,

$c^2-85c+1764=0$, $c=\frac{85\pm\sqrt{(-85)^2-4(1)(1764)}}{2}=\frac{85\pm\sqrt{169}}{2}=\frac{85\pm 13}{2}$.

Thus $c = 49$ or $c = 36$. However $c = 49$ is extraneous but $c = 36$ is not. Thus $c = 36$.

b. Differentiating with respect to q_{A}:

$\frac{\partial c}{\partial q_{\text{A}}}+\frac{1}{2\sqrt{c}}\cdot\frac{\partial c}{\partial q_{\text{A}}}=\sqrt{9+q_{\text{B}}^2}$. $\left(1+\frac{1}{2\sqrt{c}}\right)\frac{\partial c}{\partial q_{\text{A}}}=\sqrt{9+q_{\text{B}}^2}$.

When $q_{\text{A}}=6$ and $q_{\text{B}}=4$, then $c = 36$ and $\left(1+\frac{1}{12}\right)\frac{\partial c}{\partial q_{\text{A}}}=5$, $\frac{13}{12}\cdot\frac{\partial c}{\partial q_{\text{A}}}=5$, or $\frac{\partial c}{\partial q_{\text{A}}}=\frac{60}{13}$.

Differentiating with respect to q_{B}:

$$\frac{\partial c}{\partial q_{\text{B}}}+\frac{1}{2\sqrt{c}}\cdot\frac{\partial c}{\partial q_{\text{B}}}=q_{\text{A}}\cdot\frac{q_{\text{B}}}{\sqrt{9+q_{\text{B}}^2}}$$

$$\left(1+\frac{1}{2\sqrt{c}}\right)\frac{\partial c}{\partial q_{\text{B}}}=\frac{q_{\text{A}}q_{\text{B}}}{\sqrt{9+q_{\text{B}}^2}}$$

When $q_{\text{A}}=6$ and $q_{\text{B}}=4$, then $c = 36$ and

$\left(1+\frac{1}{12}\right)\frac{\partial c}{\partial q_{\text{B}}}=\frac{24}{5}$, $\frac{13}{12}\cdot\frac{\partial c}{\partial q_{\text{B}}}=\frac{24}{5}$, or $\frac{\partial c}{\partial q_{\text{B}}}=\frac{288}{65}$.

Problems 17.4

1. $f_x(x,\ y)=6(1)y^2=6y^2$
$f_{xy}(x,\ y)=6(2y)=12y$
$f_y(x,\ y)=6x(2y)=12xy$
$f_{yx}(x,\ y)=12(1)y=12y$

2. $f_x(x,\ y)=6x^2y^2+12xy^3-3y$
$f_{xx}(x,\ y)=12xy^2+12y^3$

3. $f_y(x, y) = 3$

$f_{yy}(x, y) = 0$

$f_{yyx}(x, y) = 0$

4. $f_x(x, y) = \left(x^2 + xy + y^2\right)[y+1] + (xy + x + y)[2x + y]$

$= 3x^2y + 3x^2 + 2xy^2 + 4xy + y^3 + 2y^2$

$f_{xy}(x, y) = 3x^2 + 0 + 2x(2y) + 4x(1) + 3y^2 + 4y$

$= 3x^2 + 4xy + 4x + 3y^2 + 4y$

5. $f_y(x, y) = 9\left[e^{2xy}(2x)\right] = 18xe^{2xy}$

$f_{yx}(x, y) = 18\left[x\left(e^{2xy} \cdot 2y\right) + e^{2xy}(1)\right] = 18e^{2xy}(2xy + 1)$

$f_{yxy}(x, y) = 18\left[e^{2xy}(2x) + (2xy + 1)\left(e^{2xy} \cdot 2x\right)\right]$

$= 18e^{2xy}(2x)[1 + (2xy + 1)] = 18e^{2xy}(2x)[2 + 2xy]$

$= 72x(1 + xy)e^{2xy}$

6. $f_x(x, y) = \dfrac{1}{x^2 + y^2} \cdot 2x = \dfrac{2x}{x^2 + y^2}$

$f_{xx}(x, y) = \dfrac{\left(x^2 + y^2\right)[2] - (2x)[2x]}{\left(x^2 + y^2\right)^2} = \dfrac{2\left(y^2 - x^2\right)}{\left(x^2 + y^2\right)^2}$

$f_{xy}(x, y) = (2x)(-1)\left(x^2 + y^2\right)^{-2}[2y] = -\dfrac{4xy}{\left(x^2 + y^2\right)^2}$

7. $f(x, y) = (x + y)^2(xy) = \left(x^2 + 2xy + y^2\right)(xy) = x^3y + 2x^2y^2 + xy^3$

$f_x(x, y) = 3x^2y + 4xy^2 + y^3$

$f_y(x, y) = x^3 + 4x^2y + 3xy^2$

$f_{xx}(x, y) = 6xy + 4y^2$

$f_{yy}(x, y) = 4x^2 + 6xy$

8. $f_x(x, y, z) = 2xy^3z^4$

$f_{xz}(x, y, z) = 8xy^3z^3$

$f_z(x, y, z) = 4x^2y^3z^3$

$f_{zx}(x, y, z) = 8xy^3z^3$

9. $z = \ln\sqrt{x^2+y^2} = \frac{1}{2}\ln(x^2+y^2)$

$\frac{\partial z}{\partial y} = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot(2y) = \frac{y}{x^2+y^2}$

$\frac{\partial^2 z}{\partial y^2} = \frac{(x^2+y^2)(1)-y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$

10. $\frac{\partial z}{\partial x} = \frac{1}{y}\cdot\frac{1}{x^2+5}(2x) = \frac{2x}{y\left(x^2+5\right)}$

$\frac{\partial^2 z}{\partial y\partial x} = \frac{2x}{x^2+5}\left(-\frac{1}{y^2}\right) = -\frac{2x}{y^2\left(x^2+5\right)}$

11. $f_y(x, y, z) = 0$

$f_{yx}(x, y, z) = 0$

$f_{yxx}(x, y, z) = 0$

Thus $f_{yxx}(4, 3, -2) = 0$.

12. $f_x(x, y, z) = z^2\left(6x-4y^3\right)$

$f_{xy}(x, y, z) = z^2\left(-12y^2\right) = -12y^2z^2$

$f_{xyz}(x, y, z) = -24y^2z$. Thus

$f_{xyz}(1, 2, 3) = -24(4)(3) = -288$.

13. $f_k(l, k) = 18l^3k^5 - 14l^2k^6$

$f_{kl}(l, k) = 54l^2k^5 - 28lk^6$

$f_{klk}(l, k) = 270l^2k^4 - 168lk^5$

Thus $f_{klk}(2, 1) = 270(4)(1) - 168(2)(1) = 744.$

14. $f_x(x, y) = 3x^2y^2 + 2xy - 2xy^2$

$f_{xx}(x, y) = 6xy^2 + 2y - 2y^2$

$f_{xxy}(x, y) = 12xy + 2 - 4y$

$f_{xy}(x, y) = 6x^2y + 2x - 4xy$

$f_{xyx}(x, y) = 12xy + 2 - 4y$

Thus

$f_{xxy}(2, 3) = f_{xyx}(2, 3) = 12(2)(3) + 2 - 4(3) = 62.$

15. $f_x(x, y) = y^2e^x + \frac{1}{x}$

$f_{xy}(x, y) = 2ye^x$

$f_{xyy}(x, y) = 2e^x$

Thus $f_{xyy}(1, 1) = 2e$.

16. $f_x(x, y) = 3x^2 - 6y^2 + 2x$

$f_{xy}(x, y) = -12y$

Thus $f_{xy}(1, -1) = 12$.

17. $\frac{\partial c}{\partial q_B} = \frac{1}{3}\left(3q_A^2 + q_B^3 + 4\right)^{-\frac{2}{3}}\left(3q_B^2\right)$

$= q_B^2\left(3q_A^2 + q_B^3 + 4\right)^{-\frac{2}{3}}$

$\frac{\partial^2 c}{\partial q_A\partial q_B} = -\frac{2}{3}q_B^2\left(3q_A^2 + q_B^3 + 4\right)^{-\frac{5}{3}}(6q_A)$

$= -4q_Aq_B^2\left(3q_A^2 + q_B^3 + 4\right)^{-\frac{5}{3}}$

When $p_A = 25$ and $p_B = 4$, then

$q_A = 10 - 25 + 16 = 1$ and $q_B = 20 + 25 - 44 = 1$,

and $\frac{\partial^2 c}{\partial q_A\partial q_B} = -4(8)^{-\frac{5}{3}} = -\frac{4}{32} = -\frac{1}{8}$.

18. $f_x(x, y) = 4x^3y^4 + 9x^2y^2 - 7$

$f_{xy}(x, y) = 16x^3y^3 + 18x^2y$

$f_{xx}(x, y) = 12x^2y^4 + 18xy^2$

$f_{xyx}(x, y) = 48x^2y^3 + 36xy$

$f_{xxy}(x, y) = 48x^2y^3 + 36xy$

Thus $f_{xyx}(x, y) = f_{xxy}(xy)$.

19. $f_x(x, y) = (2x+y)e^{x^2+xy+y^2}$

$f_y(x, y) = (x+2y)e^{x^2+xy+y^2}$

$f_{xy}(x, y)$

$= (2x+y)(x+2y)e^{x^2+xy+y^2} + e^{x^2+xy+y^2}$

$f_{yx}(x, y)$

$= (x+2y)(2x+y)e^{x^2+xy+y^2} + e^{x^2+xy+y^2}$

Thus $f_{xy}\left(x, y\right) = f_{yx}(x, y)$.

18. a. $\dfrac{dw}{dt}=\dfrac{\partial w}{\partial x}\dfrac{dx}{dt}+\dfrac{\partial w}{\partial y}\dfrac{dy}{dt}$

b. Since $\dfrac{dy}{dt}=1$, from (a), $\dfrac{dw}{dt}=\dfrac{\partial w}{\partial x}\dfrac{dx}{dt}+\dfrac{\partial w}{\partial y}$

19. a. $\dfrac{\partial w}{\partial t}=\dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial t}+\dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial t}$

b. $w=2x^2\ln|3x-5y|,\ \ x=s\sqrt{t^2+2}$ and $y=t-3e^{2-s}$.

$$\frac{\partial w}{\partial t}=\left[4x\ln|3x-5y|+\frac{2x^2(3)}{3x-5y}\right]\frac{s(2t)}{2\sqrt{t^2+2}}+\left[\frac{2x^2}{3x-5y}(-5)\right](1)$$

When $s = 1$ and $t = 0$, then $x=\sqrt{2}$ and $y = -3e$.

$$\frac{\partial w}{\partial t}=\left[4\sqrt{2}\ln\left|3\sqrt{2}-5(-3e)\right|+\frac{2(2)(3)}{3\sqrt{2}-5(-3e)}\right](0)+\left[\frac{2(2)}{3\sqrt{2}-5(-3e)}(-5)\right]$$
$$=-\frac{20}{3\sqrt{2}+15e}$$

20. $p = aP - whL$, where $P = f(l, k)$ and $l = Lg(h)$.

$$\frac{\partial p}{\partial L}=a\frac{\partial P}{\partial L}-wh=a\left[\frac{\partial P}{\partial l}\frac{\partial l}{\partial L}+\frac{\partial P}{\partial k}\frac{\partial k}{\partial L}\right]-wh$$
$$=a\left[\frac{\partial P}{\partial l}g(h)+\frac{\partial P}{\partial k}\cdot 0\right]-wh=a\frac{\partial P}{\partial l}g(h)-wh$$
$$\frac{\partial p}{\partial h}=a\frac{\partial P}{\partial h}-wL=a\left[\frac{\partial P}{\partial l}\frac{\partial l}{\partial h}+\frac{\partial P}{\partial k}\frac{\partial k}{\partial h}\right]-wL$$
$$=a\left[\frac{\partial P}{\partial l}Lg'(h)+\frac{\partial P}{\partial k}\cdot 0\right]-wL$$
$$=a\frac{\partial P}{\partial l}Lg'(h)-wL$$

Problems 17.6

1. $f(x,\ y)=x^2-3y^2-8x+9y+3xy$

$$\begin{cases} f_x(x,\ y)=2x-8+3y=0 \\ f_y(x,\ y)=-6y+9+3x=0 \end{cases}$$

Solving the system gives the critical point (1, 2).

2. $f(x,\ y)=x^2+4y^2-6x+16y$

$$\begin{cases} f_x(x,\ y)=2x-6=0 \\ f_y(x,\ y)=8y+16=0 \end{cases}$$

Critical point: (3, –2)

3. $f(x, y) = \frac{5}{3}x^3 + \frac{2}{3}y^3 - \frac{15}{2}x^2 + y^2 - 4y + 7$

$$\begin{cases} f_x(x, y) = 5x^2 - 15x = 0 \\ f_y(x, y) = 2y^2 + 2y - 4 = 0 \end{cases}$$

Both equations are easily solved by factoring.
Critical points: (0, –2), (0, 1), (3, –2), (3, 1)

4. $f(x, y) = xy - x + y$
$f_x(x, y) = y - 1$
$f_y(x, y) = x + 1$
Critical point: (–1, 1)

5. $f(x, y, z) = 2x^2 + xy + y^2 + 100 - z(x + y - 200)$

$$\begin{cases} f_x(x, y, z) = 4x + y - z = 0 \\ f_y(x, y, z) = x + 2y - z = 0 \\ f_z(x, y, z) = -x - y + 200 = 0 \end{cases}$$

Solving the system gives the critical point (50, 150, 350).

6. $f(x, y, z, w) = x^2 + y^2 + z^2 + w(x + y + z - 3)$

$$\begin{cases} f_x(x, y, z, w) = 2x + w = 0 \\ f_y(x, y, z, w) = 2y + w = 0 \\ f_z(x, y, z, w) = 2z + w = 0 \\ f_w(x, y, z, w) = x + y + z - 3 = 0 \end{cases}$$

Solving the system gives the critical point (1, 1, 1, –2).

7. $f(x, y) = x^2 + 3y^2 + 4x - 9y + 3$

$$\begin{cases} f_x(x, y) = 2x + 4 = 0 \\ f_y(x, y) = 6y - 9 = 0 \end{cases}$$

Critical point $\left(-2, \frac{3}{2}\right)$
Second-Derivative Test
$f_{xx}(x, y) = 2$, $f_{yy}(x, y) = 6$, $f_{xy}(x, y) = 0$. At $\left(-2, \frac{3}{2}\right)$, $D = (2)(6) - 0^2 = 12 > 0$ and $f_{xx}(x, y) = 2 > 0$. Thus at $\left(-2, \frac{3}{2}\right)$ there is a relative minimum.

8. $f(x, y) = -2x^2 + 8x - 3y^2 + 24y + 7$

$$\begin{cases} f_x(x, y) = -4x + 8 = 0 \\ f_y(x, y) = -6y + 24 = 0 \end{cases}$$

Critical point: (2, 4)
Second-Derivative Test
$f_{xx}(x, y) = -4$, $f_{yy}(x, y) = -6$, $f_{xy}(x, y) = 0$. At (2, 4), $D = (-4)(-6) - 0^2 = 24 > 0$ and $f_{xx}(x, y) = -4 < 0$; thus there is a relative maximum at (2, 4).

9. $f(x, y) = y - y^2 - 3x - 6x^2$

$$\begin{cases} f_x(x, y) = -3 - 12x = 0 \\ f_y(x, y) = 1 - 2y = 0 \end{cases}$$

Critical point $\left(-\frac{1}{4}, \frac{1}{2}\right)$
Second-Derivative Test
$f_{xx}(x, y) = -12$, $f_{yy}(x, y) = -2$, $f_{xy}(x, y) = 0$
At $\left(-\frac{1}{4}, \frac{1}{2}\right)$, $D = (-12)(-2) - 0^2 = 24 > 0$ and $f_{xx}(x, y) = -12 < 0$. Thus at $\left(-\frac{1}{4}, \frac{1}{2}\right)$ there is a relative maximum.

10. $f(x, y) = 2x^2 + \frac{3}{2}y^2 + 3xy - 10x - 9y + 2$

$$\begin{cases} f_x(x, y) = 4x + 3y - 10 = 0 \\ f_y(x, y) = 3y + 3x - 9 = 0 \end{cases}$$

Critical point: (1, 2)
Second-Derivative Test
$f_{xx}(x, y) = 4$, $f_{yy}(x, y) = 3$, $f_{xy}(x, y) = 3$.
At (1, 2), $D = (4)(3) - 3^2 = 3 > 0$ and $f_{xx}(x, y) = 4 > 0$; thus there is a relative minimum at (1, 2).

11. $f(x, y) = x^2 + 3xy + y^2 - 9x - 11y + 3$

$$\begin{cases} f_x(x, y) = 2x + 3y - 9 = 0 \\ f_y(x, y) = 3x + 2y - 11 = 0 \end{cases}$$

Critical point: (3, 1)
Second-Derivative Test
$f_{xx}(x, y) = 2$, $f_{yy} = 2$, $f_{xy} = 3$. At (3, 1), $D = (2)(2) - (3)^2 = -5 < 0$, so there is no relative extremum at (3, 1).

12. $f(x, y) = \frac{x^3}{3} + y^2 - 2x + 2y - 2xy$

$$\begin{cases} f_x(x, y) = x^2 - 2 - 2y = 0 \\ f_y(x, y) = 2y + 2 - 2x = 0 \end{cases}$$