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37. \[ F(b, C, T, i) = \frac{bT}{C} + \frac{ic}{2} \]

\[ \frac{\partial F}{\partial C} = \frac{\partial}{\partial C} \left[ \frac{bT}{C} \right] + \frac{\partial}{\partial C} \left[ \frac{ic}{2} \right] = -\frac{bT}{C^2} + \frac{i}{2} \]

38. From \[ \eta = \frac{r}{D} \], we have \[ \frac{\partial r}{\partial D} = \frac{r}{D\eta} \]. Substituting into Equation (3) gives

\[ r_L = r + D \cdot \frac{r}{D\eta} + \frac{dC}{dD} \]

\[ r_L = r + \frac{r}{\eta} + \frac{dC}{dD} \]

\[ r_L = \left[ \frac{1}{\eta} + \frac{1}{\eta} \right] \frac{dC}{dD} \]

\[ r_L = \left[ \frac{\eta + 1}{\eta} \right] \frac{dC}{dD} \]

\[ r_L = \left[ \frac{1 + \eta}{\eta} \right] \frac{dC}{dD} \]

which is Equation (4).

39. \[ R = f(r, a, n) = \frac{r}{1 + a \left( \frac{n-1}{2} \right)} \]

\[ \frac{\partial R}{\partial n} = r(-1) \left[ 1 + a \left( \frac{n-1}{2} \right) \right]^{-2} \cdot a \left( \frac{n-1}{2} \right) \]

\[ = -\frac{ra}{2 \left[ 1 + a \left( \frac{n-1}{2} \right) \right]^2} \]

Problems 17.2

1. \[ c = 7x + 0.3y^2 + 2y + 900 \]

\[ \frac{\partial c}{\partial y} = 0.6y + 2 \]

When \( x = 20 \) and \( y = 30 \), then \( \frac{\partial c}{\partial y} = 0.6(30) + 2 = 20 \)

2. \[ c = 2\sqrt{x+y} + 6000 \]

\[ \frac{\partial c}{\partial x} = \frac{x}{\sqrt{x+y}} + 2\sqrt{x+y} \]

When \( x = 70 \) and \( y = 74 \), then \( \frac{\partial c}{\partial x} = \frac{70}{\sqrt{70+74}} + 2\sqrt{70+74} = \frac{179}{6} \)

3. \[ c = 0.03(x + y)^3 - 0.6(x + y)^2 + 9.5(x + y) + 7700 \]

\[ \frac{\partial c}{\partial x} = 0.09(x + y)^2 - 1.2(x + y) + 9.5 \]

When \( x = 50 \) and \( y = 80 \), then \( \frac{\partial c}{\partial x} = 0.09(130)^2 - 1.2(130) + 9.5 = 1374.5 \)

4. \[ P = 15l - 3l^2 + 5k^2 + 500 \]

\[ \frac{\partial P}{\partial k} = 15l + 10k \]

\[ \frac{\partial P}{\partial l} = 15k - 6l \]

5. \[ P = 2.314l^{0.357}k^{0.643} \]

\[ \frac{\partial P}{\partial l} = 2.314(0.357)l^{-0.643}k^{0.643} \]

\[ = 0.826098 \left( \frac{k}{l} \right)^{0.643} \]

\[ \frac{\partial P}{\partial k} = 2.314(0.643)l^{0.357}k^{-0.357} \]

\[ = 1.487902 \left( \frac{1}{k} \right)^{0.357} \]

6. \[ P = Al^\alpha k^\beta \]

a. \[ \frac{\partial P}{\partial l} = A\alpha l^{\alpha-1}k^\beta = \left( \frac{\alpha}{l} \right) Al^\alpha k^\beta = \frac{\alpha P}{l} \]

b. \[ \frac{\partial P}{\partial k} = A\beta l^\alpha k^{\beta-1} = \left( \frac{\beta}{k} \right) Al^\alpha k^\beta = \frac{\beta P}{k} \]

c. From parts (a) and (b),

\[ l \frac{\partial P}{\partial l} + k \frac{\partial P}{\partial k} = l \left( \frac{\alpha P}{l} \right) + k \left( \frac{\beta P}{l} \right) \]

\[ = \alpha P + \beta P = P(\alpha + \beta) = P(1) = P \]

7. \[ \frac{\partial q_A}{\partial p_A} = -40, \ rac{\partial q_A}{\partial p_B} = 3, \ rac{\partial q_B}{\partial p_A} = 5, \ rac{\partial q_B}{\partial p_B} = -20 \]

Since \( \frac{\partial q_A}{\partial p_B} > 0 \) and \( \frac{\partial q_B}{\partial p_A} > 0 \) the products are competitive.
8. \( \frac{\partial q_A}{\partial p_A} = -1, \frac{\partial q_A}{\partial p_B} = -2, \frac{\partial q_B}{\partial p_A} = -2, \frac{\partial q_B}{\partial p_B} = -3 \)

Since \( \frac{\partial q_A}{\partial p_B} < 0 \) and \( \frac{\partial q_B}{\partial p_A} < 0 \) the products are complementary.

9. \( q_A = 100p_A^{-1}p_B^{-\frac{1}{2}} \)

\[
\begin{align*}
\frac{\partial q_A}{\partial p_A} &= 100(-1)p_A^{-2}p_B^{-\frac{1}{2}} = \frac{-100}{p_A^{\frac{1}{2}}p_B^{\frac{3}{2}}} \\
\frac{\partial q_A}{\partial p_B} &= 100\left(\frac{1}{2}\right)p_A^{-1}p_B^{-\frac{3}{2}} = \frac{-50}{p_A^{\frac{1}{2}}p_B^{\frac{5}{2}}} \\
\frac{\partial q_B}{\partial p_A} &= 500\left(\frac{1}{3}\right)p_B^{-1}p_A^{-\frac{1}{2}} = \frac{-500}{p_A^{\frac{1}{2}}p_B^{\frac{7}{2}}} \\
\frac{\partial q_B}{\partial p_B} &= 500(-1)p_B^{-2}p_A^{-\frac{1}{2}} = \frac{-500}{p_A^{\frac{1}{2}}p_B^{\frac{9}{2}}}
\end{align*}
\]

Since \( \frac{\partial q_A}{\partial p_B} < 0 \) and \( \frac{\partial q_B}{\partial p_A} < 0 \), the products are complementary.

10. \( \frac{\partial P}{\partial l} = 15.18^{-0.54}k^{0.52} \)

\( \frac{\partial P}{\partial k} = 17.16^{-0.46}k^{-0.48} \)

If \( l = 1 \) and \( k = 1 \), then \( \frac{\partial P}{\partial l} = 15.18 \) and \( \frac{\partial P}{\partial k} = 17.16 \)

11. \( \frac{\partial P}{\partial B} = 0.01A^{0.27}B^{-0.99}C^{0.01}D^{0.23}E^{0.09}F^{0.27} \)

\( \frac{\partial P}{\partial C} = 0.01A^{0.27}B^{0.01}C^{-0.99}D^{0.23}E^{0.09}F^{0.27} \)

12. \( P = \frac{kl}{3k+5l} \)

a. \( \frac{\partial P}{\partial k} = \frac{l(3k+5l)-kl(3)}{(3k+5l)^2} = \frac{5l^2}{(3k+5l)^2} \)

\( \frac{\partial P}{\partial l} = \frac{k(3k+5l)-kl(5)}{(3k+5l)^2} = \frac{3k^2}{(3k+5l)^2} \)

b. When \( k = l \), then

\[
\begin{align*}
\frac{\partial P}{\partial k} + \frac{\partial P}{\partial l} &= \frac{5l^2}{(3l+5l)^2} + \frac{3l^2}{(3l+5l)^2} \\
&= \frac{8l^2}{64l^2} \\
&= \frac{1}{8}
\end{align*}
\]

13. \( \frac{\partial z}{\partial x} = 4480.0 \). If a staff manager with an M.B.A. degree had an extra year of work experience before the degree, the manager would receive \$4480 more per year in extra compensation.

14. \( S_g = 7S_e^{\frac{1}{3}}S_l^{\frac{1}{2}} \)

\[
\begin{align*}
\frac{\partial S_g}{\partial S_e} &= 7\left(\frac{1}{3}\right)S_e^{\frac{2}{3}}S_l^{\frac{1}{2}} = 7\left(\frac{1}{3}\right)\sqrt[3]{S_e} \\
\frac{\partial S_g}{\partial S_l} &= 7\left(\frac{1}{2}\right)S_e^{\frac{1}{2}}S_l^{\frac{1}{2}} = 7\left(\frac{1}{2}\right)\sqrt[2]{S_e} \\
\end{align*}
\]

If \( S_e = 125 \) and \( S_l = 100 \), then

\[
\begin{align*}
\frac{\partial S_g}{\partial S_e} &= \left(\frac{7}{3}\right)rac{10}{15} = \frac{14}{15} \quad \text{and} \\
\frac{\partial S_g}{\partial S_l} &= \left(\frac{7}{2}\right)\frac{5}{10} = \frac{7}{4}
\end{align*}
\]

Thus if \( S_g \) increases to 126 and \( S_l \) remains at 100, then \( S_g \) increases by approximately \( \frac{14}{15} \); if \( S_l \) increases to 101 and \( S_e \) remains at 125, then \( S_g \) increases by approximately \( \frac{7}{4} \).

15. a. \( \frac{\partial R}{\partial w} = -0.105 \); \( \frac{\partial R}{\partial s} = -0.846 \)

b. One for which \( w = w_0 \) and \( s = s_0 \) since increasing \( w \) by 1 while holding \( s \) fixed decreases the reading ease score.

16. \( \omega = b^{-1}L^{-1} \sqrt{\frac{\tau}{\pi p}} = \frac{1}{bl}\sqrt{\frac{\tau}{\pi p}} \)

\[
\begin{align*}
\frac{\partial \omega}{\partial b} &= (-1)b^{-2}L^{-1} \sqrt{\frac{\tau}{\pi p}} = -\frac{1}{b^2L} \sqrt{\frac{\tau}{\pi p}} \\
\frac{\partial \omega}{\partial L} &= b^{-1}(-1)L^{-2} \sqrt{\frac{\tau}{\pi p}} = -\frac{1}{bL^2} \sqrt{\frac{\tau}{\pi p}}
\end{align*}
\]

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\[
\frac{\partial \omega}{\partial \rho} = \frac{1}{bL} \sqrt{\frac{\tau}{\pi}} \left( \frac{1}{2} \right) \rho^{-\frac{3}{2}} = -\frac{1}{2bL} \sqrt{\frac{\tau}{\pi}} \\
\frac{\partial \omega}{\partial \tau} = \frac{1}{bL} \sqrt{\frac{1}{\pi \rho}} \left( \frac{1}{2} \right) \tau^{-\frac{1}{2}} = \frac{1}{2bL} \sqrt{\frac{\tau}{\pi \rho}}
\]

17. \( \frac{\partial g}{\partial x} = \frac{1}{V_F} > 0 \) for \( V_F > 0 \). Thus if \( x \) increases and \( V_F \) and \( V_S \) are fixed, then \( g \) increases.

18. \( q_A = e^{-(p_A+p_B)} \) and \( q_B = \frac{16}{p_A^2 p_B^2} = 16 p_A^{-2} p_B^{-2} \)

a. \( \frac{\partial q_A}{\partial p_B} = -e^{-(p_A+p_B)} < 0 \)
\( \frac{\partial q_B}{\partial p_A} = -32 p_A^{-3} p_B^{-2} < 0 \)
Since both are < 0, A and B are complementary.

b. Note that \( p_A \) and \( p_B \) are in units of thousands of dollars. When \( p_A = 1 \) and \( p_A = 2 \), then
\( \frac{\partial q_A}{\partial p_B} = -e^{-} = -\frac{1}{e} \).
A decrease in the price of B of $20 is a decrease in \( p_B \) of \( \frac{20}{2000} = 0.01 \). Thus the change in \( q_B \) is approximately \( -\frac{1}{e}(-0.01) = \frac{0.01}{e} \). So demand increases by approximately \( \frac{0.01}{e} \) unit.

19. a. \( \frac{\partial q_A}{\partial p_A} = 10 \sqrt{p_B} \left( -\frac{1}{2} p_A^{-\frac{3}{2}} \right) \)
\( \frac{\partial q_A}{\partial p_B} = \frac{10}{\sqrt{p_A}} \left( \frac{1}{2} p_B^{-\frac{1}{2}} \right) \)
When \( p_A = 9 \) and \( p_B = 16 \), then \( \frac{\partial q_A}{\partial p_A} = 10(4) \left( -\frac{1}{2} \cdot \frac{1}{27} \right) = -\frac{20}{27} \) and \( \frac{\partial q_A}{\partial p_B} = 10 \left( \frac{1}{2} \cdot \frac{1}{4} \right) = \frac{5}{12} \).

b. From (a), when \( p_A = 9 \) and \( p_B = 16 \), then \( \frac{\partial q_A}{\partial p_B} = \frac{5}{12} \). Hence each $1 reduction in \( p_B \) decreases \( q_A \) by approximately \( \frac{5}{12} \) unit. Thus a $2 reduction in \( p_B \) (from $16 to $14) decreases the demand for A by approximately \( \frac{5}{12}(2) = \frac{5}{6} \) unit.

20. \( c = \frac{q_A^2 \left( \frac{3}{A_B} + q_A \right)^{\frac{1}{2}}}{17} + q_A q_B^\frac{1}{2} + 600 \)

a. \( \frac{\partial c}{\partial q_A} = \frac{1}{17} \left[ 2 q_A \left( \frac{3}{A_B} + q_A \right)^{\frac{1}{2}} + 2 q_A \left( \frac{3}{A_B} + q_A \right)^{\frac{1}{2}} \right] + \frac{1}{2} q_B \)

\( = \frac{1}{17} \left[ 2 q_A \left( \frac{3}{A_B} + q_A \right)^{\frac{1}{2}} \right] + \frac{1}{2} q_B \)

b. When \( q_A = 17 \) and \( q_B = 8 \), then

\( \frac{\partial c}{\partial q_A} = \frac{1}{17} \left[ \frac{1}{2} (17)^2 \left( \frac{1}{23} \right) \right] + 2 = \left[ \frac{1}{2} (17)^2 \left( \frac{1}{23} \right) + 2 \right] = 48.37 \).

c. From (b), if \( q_A \) is reduced by one unit (from 17 to 16) while \( q_B \) remains at 8, then the cost will decrease by approximately $48.37.

21. a. \( \frac{\partial R}{\partial E_r} = 2.5945 - 0.1608 E_r - 0.0277 I_r \)

If \( E_r = 18.8 \) and \( I_r = 10 \), then \( \frac{\partial R}{\partial E_r} = -0.70564 \). Since \( \frac{\partial R}{\partial E_r} < 0 \), such a candidate should not be so advised.

b. \( \frac{\partial R}{\partial N} = 0.8579 - 0.0122 N \)

If \( \frac{\partial R}{\partial N} < 0 \), then \( N > 70.3 \approx 70\% \)

22. \( S = \frac{AT + 450}{\sqrt{A + T^2}} \). Note: \( A \) is expressed in hundreds of dollars.

a. \( \frac{\partial S}{\partial T} = \frac{\left( A + T^2 \right)^{\frac{1}{2}} (A) - (AT + 450) \left[ \frac{1}{2} \left( A + T^2 \right) \right]}{\left( \sqrt{A + T^2} \right)^2} \)

\( = \frac{\left( A + T^2 \right)^{\frac{1}{2}} \left[ (A + T^2) A - (AT + 450)T \right]}{A + T^2} = \frac{A^2 - 450T}{(A + T^2)^{\frac{3}{2}}} \)

as was to be shown.
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23. \( q_A = 1000 - 50p_A + 2p_B \)

\( \eta_{p_A} = \left( \frac{P_A}{q_A} \right) \frac{\partial q_A}{\partial p_A} = \left( \frac{P_A}{q_A} \right) (-50) \)

\( \eta_{p_B} = \left( \frac{P_B}{q_A} \right) \frac{\partial q_A}{\partial p_B} = \left( \frac{P_B}{q_A} \right) (2) \)

When \( p_A = 2 \) and \( p_B = 10 \), then \( q_A = 920 \), from which \( \eta_{p_A} = -\frac{5}{46} \) and \( \eta_{p_B} = \frac{1}{46} \).

24. \( q_A = 60 - 3p_A - 2p_B \)

\( \eta_{p_A} = \left( \frac{P_A}{q_A} \right) \frac{\partial q_A}{\partial p_A} = \left( \frac{P_A}{q_A} \right) (-3) \)

\( \eta_{p_B} = \left( \frac{P_B}{q_A} \right) \frac{\partial q_A}{\partial p_B} = \left( \frac{P_B}{q_A} \right) (-2) \)

When \( p_A = 5 \) and \( p_B = 3 \), then \( q_A = 39 \), from which \( \eta_{p_A} = -\frac{5}{13} \) and \( \eta_{p_B} = \frac{2}{13} \).

25. \( q_A = \frac{100}{p_A \sqrt{p_B}} \)

\( \eta_{p_A} = \left( \frac{P_A}{q_A} \right) \frac{\partial q_A}{\partial p_A} = \left( \frac{P_A}{q_A} \right) \left( -\frac{100}{p_A \sqrt{p_B}} \right) \)

\( \eta_{p_B} = \left( \frac{P_B}{q_A} \right) \frac{\partial q_A}{\partial p_B} = \left( \frac{P_B}{q_A} \right) \left( -\frac{50}{p_A \sqrt{p_B}^3} \right) \)

When \( p_A = 1 \) and \( p_B = 4 \), then \( q_A = 50 \). This gives \( \eta_{p_A} = -1 \) and \( \eta_{p_B} = \frac{1}{2} \).

Problems 17.3

1. \( 4x + 0 + 10z \frac{\partial z}{\partial x} = 0 \)

\( \frac{\partial z}{\partial x} = -\frac{4x}{10z} = -\frac{2x}{5z} \)

2. \( 2z \frac{\partial z}{\partial x} - 10x + 0 = 0 \)

\( \frac{\partial z}{\partial x} = \frac{10x}{2z} = \frac{5x}{2z} \)

3. \( 6z \frac{\partial z}{\partial y} - 0 - 14y = 0 \)

\( \frac{\partial z}{\partial y} = \frac{14y}{6z} = \frac{7y}{3z} \)

4. \( 0 + 2y + 6z^2 \frac{\partial z}{\partial y} = 0 \)

\( \frac{\partial z}{\partial y} = \frac{-2y}{6z^2} = \frac{-y}{3z^2} \)

5. \( x^2 - 2y - z^2 + y \left( x^2 z^2 \right) = 20 \)

\( 2x - 0 - 2z \frac{\partial z}{\partial x} + y \left[ x^2 \cdot 2z \frac{\partial z}{\partial x} + z^2 \cdot 2x \right] = 0 \)

\( \left( 2x^2 yz - 2z \right) \frac{\partial z}{\partial x} = -2x - 2xyz^2 \)

\( \frac{\partial z}{\partial x} = \frac{-2x(1 + yz^2)}{2z(1 - x^2 y)} = \frac{x(1 + yz^2)}{z(1 - x^2 y)} \)

6. \( 3z^2 \frac{\partial z}{\partial x} + 2x^2 \left( 2z \frac{\partial z}{\partial x} \right) + 2z^2(2z) - y = 0 \)

\( (3z^2 + 4x^2 z) \frac{\partial z}{\partial x} = y - 4xz^2 \)

\( \frac{\partial z}{\partial x} = \frac{y - 4xz^2}{3z^2 + 4x^2 z} \)

7. \( 0 + e^y + e^x \frac{\partial z}{\partial x} = 0 \)

\( \frac{\partial z}{\partial y} = \frac{e^y}{e^x} = -e^{y-z} \)