12. The total cost for the first 5 years is 
\[ M(5) - M(0) = \int_0^5 M'(x)dx \]
\[ = \left[ 900x^2 + 5000 \right]_0^5 = 30(5)^3 + 5000(5) - 0 \]
\[ = 3750 + 25,000 = 28,750 \]
The total cost for the first 5 years is $28,750.

Problems 14.7

1. \[ \int_0^3 5x^2 dx = 5x_0^3 = 5(3) - 5(0) = 15 - 0 = 15 \]

2. \[ \int_1^5 (e + 3e)dx = \int_1^5 4ex dx \]
\[ = 4ex_l^5 \]
\[ = 4e(5 - 1) \]
\[ = 16e \]

3. \[ \int_1^2 5x dx = 5\cdot\frac{x^2}{2}_1^2 = 10 - \frac{5}{2} = \frac{15}{2} \]

4. \[ \int_2^8 -5x dx = -5\cdot\frac{x^2}{2}_2^8 = -160 - (-10) = -150 \]

5. \[ \int_{-3}^{1} (2x-3)dx = \left( x^2 - 3x \right)_{-3}^{1} = -2 - 18 = -20 \]

6. \[ \int_{-1}^{1} (4 - 9y) = \left( 4y - \frac{9y^2}{2} \right)_{-1}^{1} = -\frac{1}{2} - \left( -\frac{17}{2} \right) \]
\[ = \frac{16}{2} = 8 \]

7. \[ \int_1^4 (y^2 + 4y + 4)dy = \int_1^4 (y + 2)^2 dy \]
\[ = \frac{(y + 2)^3}{3}_1^4 \]
\[ = \frac{1}{3}[(4 + 2)^3 - (1 + 2)^3] \]
\[ = \frac{1}{3}[216 - 27] \]
\[ = \frac{1}{3}(189) \]
\[ = 63 \]

8. \[ \int_4^1 (2t - 3t^2)dt = (t^2 - t^3)_4^1 = 0 - (-48) = 48 \]

9. \[ \int_{-2}^{-1} (3w^2 - w - 1)dw = \left( w^3 - \frac{w^2}{2} - w \right)_{-2}^{-1} \]
\[ = -\frac{1}{2} - (-8) = \frac{15}{2} \]

10. \[ \int_8^9 dt = \int_8^9 1 dt = t_8^9 = 9 - 8 = 1 \]

11. \[ \int_1^3 3r^{-3} dr = \int_1^3 r^{-3} \]
\[ = \frac{1}{6} \left( \frac{3}{2} \right) = \frac{4}{3} \]

12. \[ \int_2^3 \frac{3}{x^2} dx = \int_2^3 x^{-2} dx \]
\[ = 3\cdot\frac{x^{-1}}{-1}_2^3 \]
\[ = \frac{3}{3} \left( \frac{-3}{2} \right) \]
\[ = -1 + \frac{3}{2} \]
\[ = \frac{1}{2} \]
13. \( \int_{-8}^{8} \frac{\sqrt[4]{3}}{x} \, dx = \int_{-8}^{8} \frac{x^{4/3}}{dx} \)
   \[ = \frac{3 \cdot 7^{4/3}}{8} \left[ 8 \right] \]
   \[ = \frac{3 \cdot 128}{7} - \frac{3(-128)}{7} \]
   \[ = \frac{768}{7} \]

14. \( \int_{1/2}^{3/2} (x^2 + x + 1) \, dx = \left( \frac{x^3}{3} + \frac{x^3}{2} + x \right) \]
   \[ = \left[ \frac{15}{3} + \frac{2}{3} + 37 \right] = \left[ \frac{15}{3} + \frac{2}{3} + 37 \right] \]
   \[ = \frac{4}{3} \cdot 37 \]

15. \( \int_{1/2}^{3/2} x^2 \, dx = \frac{1}{3} \left[ x^3 \right]_{1/2}^{3/2} \)
   \[ = \frac{1}{3} \left[ \frac{3^3}{2^3} - \frac{1^3}{2^3} \right] \]
   \[ = \frac{1}{5} \left[ (2 + 1)^3 - (-2 + 1)^3 \right] \]
   \[ = \frac{1}{5} \left( 243 + 1 \right) \]
   \[ = \frac{244}{5} \]

16. \( \int_{0}^{36} (\sqrt{x} - 2) \, dx = \left( \frac{2}{3} \cdot 3^2 - 2x \right) \]
   \[ = \left[ \frac{2}{3} \cdot 3^2 - 2x \right] \]
   \[ = 72 - 0 = 72 \]

17. \( \int_{-2}^{2} (z + 1)^4 \, dz = \left( \frac{z + 1}{5} \right)^5 \)
   \[ = \frac{1}{5} \left[ (2 + 1)^5 - (-2 + 1)^5 \right] \]
   \[ = \frac{1}{5} \left( 243 + 1 \right) \]
   \[ = \frac{244}{5} \]

18. \( \int_{1}^{3} \left( \frac{1}{x^3} - \frac{1}{x^3} \right) \, dx = \left( \frac{3}{4} \cdot 3^2 - \frac{3}{2} \right) \)
   \[ = 6 - \left( \frac{3}{4} \right) = \frac{27}{4} \]

19. \( \int_{0}^{1} 2x^2 (x^3 - 1)^3 \, dx = 2 \int_{0}^{1} (x^3 - 1)^3 \left[ 3x^2 \, dx \right] \)
   \[ = \frac{1}{6} \left( x^3 - 1 \right)^4 \]
   \[ = \frac{1}{6} \left[ 0 - \frac{1}{6} \right] = -\frac{1}{6} \]

20. \( \int_{2}^{3} (x + 2)^3 \, dx = \left( \frac{x + 2}{4} \right)^4 \)
   \[ = \frac{625}{4} - 64 = \frac{369}{4} \]

21. \( \int_{1}^{4} \frac{1}{y} \, dy = 4 \ln |y| \)
   \[ = 4(\ln 8 - \ln 1) \]
   \[ = 4(\ln 8 - 0) = 4 \ln 8 \]

22. \( \int_{-e^\pi}^{e^\pi} \frac{2}{x} \, dx = 2 \int_{-e^\pi}^{e^\pi} \left[ \ln |x| - \ln |e^\pi| \right] \)
   \[ = 2(0 - \pi) = -2\pi \]

23. \( \int_{-1}^{1} x^5 \, dx = e^5 \left[ \frac{1}{5} \right] = e^5 - 0 = e^5 \)

24. \( \int_{0}^{2} \frac{1}{x - 1} \, dx = \ln |x - 1| \)
   \[ = \ln e - \ln 1 = 1 - 0 = 1 \]

25. \( \int_{0}^{1/3} e^{x} \, dx = \left[ \frac{1}{3} \right] \)
   \[ = \frac{1}{3} \left( e^1 - e^0 \right) = \frac{1}{3} (e - 0) \]

26. \( \int_{0}^{1/3} \left( x^3 + 2x^2 \right) ^4 \, dx \)
   \[ = \int_{0}^{1/3} \left( x^3 + 2x^2 \right) ^4 \left[ \left( x^3 + 2x^2 \right) \, dx \right] \]
   \[ = \left( \frac{x^3 + 2x^2}{5} \right) \]
   \[ = \frac{243}{5} - 0 = \frac{243}{5} \]

27. \( \int_{3/4}^{3} \frac{3}{x^2} \, dx = \left[ \frac{3}{x} \right] \)
   \[ = \frac{1}{3} \left( x + 3 \right) ^{-2} \]
   \[ = \frac{1}{3} \left( \frac{1}{4 + 3} - \frac{1}{3 + 3} \right) \]
   \[ = \frac{1}{3} \left( \frac{1}{7} - \frac{1}{6} \right) = \frac{1}{14} \]
28. \[ \int_{-1/3}^{20/3} \sqrt[3]{3x + 5} \, dx = \frac{1}{3} \int_{-1/3}^{20/3} (3x + 5)^{1/3} \, dx \]
\[ = \frac{2}{9} (3x + 5)^{4/3} \bigg|_{-1/3}^{20/3} \]
\[ = \frac{2}{9} (125 - 8) = 26 \]

29. \[ \int_{1/3}^{2} \sqrt[3]{10 - 3p} \, dp = -\frac{1}{3} \int_{1/3}^{2} (10 - 3p)^{1/3} \, dp \]
\[ = -\frac{2}{9} (10 - 3p)^{4/3} \bigg|_{1/3}^{2} \]
\[ = -\frac{2}{9} (8 - 27) = \frac{38}{9} \]

30. \[ \int_{-1}^{1} q \sqrt[3]{q^2 + 3} \, dq = \frac{1}{2} \int_{-1}^{1} (q^2 + 3)^{1/3} \, [2q \, dq] \]
\[ = \frac{8}{3} - \frac{8}{3} = 0 \]

31. \[ \int_{0}^{1} x^2 \sqrt[3]{7x^3 + 1} \, dx = \frac{1}{21} \int_{0}^{1} (7x^3 + 1)^{1/3} \, [21x^2 \, dx] \]
\[ = \frac{1}{21} \left( \frac{7x^3 + 1}{3} \right)^{4/3} \bigg|_{0}^{1} \]
\[ = \frac{16}{28} - \frac{1}{28} = \frac{15}{28} \]

32. \[ \int_{0}^{\sqrt[3]{2}} \left( 2x - x \right) \, \left( \frac{x}{x^2 + 1} \right)^{2/3} \, dx \]
\[ = \int_{0}^{\sqrt[3]{2}} 2x \, dx - \frac{1}{2} \int_{0}^{\sqrt[3]{2}} (x^2 + 1)^{-2/3} \, [2x \, dx] \]
\[ = \frac{2}{3} \left( \frac{x^3}{3} \right)^{1/3} \bigg|_{0}^{\sqrt[3]{2}} \]
\[ = (2 - 0) - \frac{3}{2} (2 + 1)^{1/3} - (0 + 1)^{1/3} \]
\[ = 2 - \frac{3}{2} (3^{1/3} - 1) \]
\[ = 2 - \frac{3\sqrt[3]{3}}{2} + \frac{3}{2} \]
\[ = \frac{7 - 3\sqrt[3]{3}}{2} \]

33. \[ \int_{0}^{1} \frac{2x^3 + x}{x^2 + x^4 + 1} \, dx \]
\[ = \frac{1}{2} \left( \frac{1}{x^4 + x^2 + 1} \right)^{1/2} \left[ (4x^3 + 2x) \, dx \right] \]
\[ = \frac{1}{2} \ln (x^4 + x^2 + 1) \bigg|_{0}^{1} = \frac{1}{2} [\ln 3 - \ln 1] = \frac{1}{2} \ln 3 \]

34. \[ \int_{a}^{b} (m + ny) \, dy = \left( my + \frac{ny^2}{2} \right) \bigg|_{a}^{b} \]
\[ = m(b - a) + \frac{1}{2} (b^2 - a^2) \]

35. \[ \int_{0}^{1} e^{x} - e^{-x} \, dx = \left( e^x + e^{-x} \right) \bigg|_{0}^{1} \]
\[ = \frac{1}{2} \left( e^1 + e^{-1} + (1 + 1) \right) \]
\[ = \frac{1}{2} \left( e + \frac{1}{e} + 2 \right) \]

36. \[ \int_{-2}^{1} |x| \, dx = 8 \left( \int_{-2}^{0} -x \, dx + \int_{0}^{1} x \, dx \right) \]
\[ = 8 \left( \frac{-x^2}{2} \bigg|_{-2}^{0} + \frac{x^2}{2} \bigg|_{0}^{1} \right) = 8 \left( [0 - (-2)] + \left( \frac{1}{2} - 0 \right) \right) \]

37. \[ \int_{e}^{\sqrt[3]{2}} 3(x^2 + x^3 - x^{-4}) \, dx \]
\[ = 3 \left( \frac{x^{-4}}{-1} - \frac{x^{-3}}{-3} \right) \bigg|_{e}^{\sqrt[3]{2}} \]
\[ = 3 \left( -\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{3x^3} \right) \bigg|_{e}^{\sqrt[3]{2}} \]
\[ = 3 \left( -\frac{1}{\sqrt[3]{2}} + \frac{1}{2 \cdot \sqrt[3]{2}^2} + \frac{1}{3 \cdot \sqrt[3]{2}^3} \right) - 3 \left( -\frac{1}{e} + \frac{1}{2e^2} + \frac{1}{3e^3} \right) \]
\[ = 3 \left( -\frac{5}{6 \sqrt[3]{2}} - \frac{1}{4} + \frac{1}{2e^2} + \frac{1}{3e^3} \right) \]
38. \[ \int_1^2 \left(6\sqrt{x} - \frac{1}{\sqrt{2x}}\right) \, dx \]
   \[ = 6 \int_1^2 \frac{\sqrt{x}}{2} \, dx - \frac{1}{2} \int_1^2 (2x)^{-\frac{1}{2}} \, dx \]
   \[ = \left[ 4x^{\frac{3}{2}} - (2x)^{\frac{1}{2}} \right]_1^2 = (8\sqrt{2} - 2) - (4 - \sqrt{2}) \]
   \[ = 9\sqrt{2} - 6 \]

39. \[ \int_1^3 (x+1)e^{x^2+2x} \, dx = \int_1^3 e^{x^2+2x} \, dx \]
   \[ = \frac{1}{2} x e^{x^2+2x} \bigg|_1^3 = \frac{1}{2} (e^{15} - e^1) = \frac{e^{12} - 1}{2} \]

40. \[ \frac{1}{\ln e^x} \, dx = \int_1^9 e^x \, dx = \int_1^9 1 \, dx = x \bigg|_1^9 = 95 - 1 = 94 \]

41. Using long division on the integrand
   \[ \int \frac{x^6 + 6x^4 + 3x^3 + 2x^2 + x + 5}{x^3 + 5x + 1} \, dx \]
   \[ = \int \left( x^3 + x + \frac{3x^2 + 5}{x^3 + 5x + 1} \right) \, dx \]
   \[ = \left[ \frac{x^4}{4} + \frac{x^2}{2} + \ln |x^3 + 5x + 1| \right]_0 \]
   \[ = (6 + \ln 19) - 0 = 6 + \ln 19 \]

42. \[ \int \frac{1}{1 + e^x} \, dx = \int \frac{e^x}{e^x + 1} \, dx \]
   \[ = \int_1^2 \frac{1}{e^{-x} + 1} \, dx \]
   \[ = -\ln |e^{-x} + 1| \bigg|_1^2 = -\ln \left| e^{-2} + 1 \right| - \ln \left| e^{-1} + 1 \right| \]
   \[ = (\ln 1 + 1) - \ln (1 + e^{-1}) \]

43. \[ \int_0^1 f(x) \, dx = \int_0^{1/2} 4x^2 \, dx + \int_{1/2}^2 2x \, dx \]
   \[ = \frac{4x^3}{3} \bigg|_0^{1/2} + x^2 \bigg|_{1/2}^2 = \frac{1}{6} - 0 + \left(4 - \frac{1}{4}\right) = \frac{47}{12} \]

44. \[ \int_1^3 x^3 \, dx - \int_1^3 x^2 \, dx = \left( \frac{x^4}{4} \right)_1^3 - \left( \frac{x^3}{3} \right)_1^3 \]
   \[ = \frac{9}{2} - \frac{1}{2} = \frac{81}{4} - \frac{1}{4} \]
   \[ = 43 - 20 = 44 \]

45. \[ f(x) = \int_1^3 \frac{-3}{x^2} \, dt = -3 \int_1^3 \frac{1}{t^2} \, dt = -3 \left[ \frac{1}{x} \right]_1^3 = -3 + 3 = -\frac{3}{3} \]
   \[ = \frac{1}{x} - 3 = 6 - 3e \]

46. \[ \int_0^1 x^2 \, dx + \int_0^{\sqrt{2}} \frac{1}{3\sqrt{2}} \, dx = 0 + \frac{1}{3\sqrt{2}} \left[ x \right]_0^{\sqrt{2}} \]
   \[ = \frac{1}{3\sqrt{2}} \left( \sqrt{2} - 0 \right) \]
   \[ = \frac{1}{3} \]

47. \[ \int_x^3 f(x) \, dx = \int_1^3 f(x) \, dx - \int_1^2 f(x) \, dx \]
   \[ = -\int_1^3 f(x) \, dx - \int_1^2 f(x) \, dx \]
   \[ = -2 - 5 \]
   \[ = -7 \]

48. \[ \int_1^4 f(x) \, dx = \int_1^2 f(x) \, dx + \int_2^4 f(x) \, dx \]
   \[ \int_1^4 f(x) \, dx = \int_1^3 f(x) \, dx - \int_2^3 f(x) \, dx + \int_2^4 f(x) \, dx \]
   \[ = 6 - \int_2^3 f(x) \, dx + \frac{4}{3} \int_2^4 f(x) \, dx \]
   \[ = 6 - \int_2^3 f(x) \, dx + 5 \]
   \[ = 7 - 6 = 1 \]

49. \[ \int_2^3 x^3 \, dx \] is a constant, so \[ \frac{d}{dx} \left( \int_2^3 x^3 \, dx \right) = 0. \]
   Thus
   \[ \int_2^3 \left( \frac{d}{dx} \int_2^3 x^3 \, dx \right) \, dx = \int_2^3 0 \, dx = C^3_2 - C = C = 0 \]
50. \( f(x) = \frac{e^x e^{-t} - e^{-t}}{e^x + e^{-t}} dt \\
= \int e^x \frac{1}{e^x + e^{-t}} [(e^t - e^{-t})] dt \\
= \ln(e^x + e^{-t}) \bigg|_t^x - \ln(e^e - e^{-e}) \\
= \ln(e^x + e^{-t}) - \ln(e^e - e^{-e}) \)

51. \( \int_0^T \alpha^2 dt = \alpha^2 \int_0^T - 0 = \alpha^2 T \)

52. \( \mu = \int_0^1 [6(x - x^2)] dx \\
= 6 \int_0^1 (x^2 - x^3) dx \\
= 6 \left( \frac{x^4}{4} - \frac{x^2}{2} \right) \bigg|_0^1 \\
= 6 \left( \frac{1}{4} - \frac{1}{2} \right) - 6(0 - 0) \\
= \frac{1}{2} \)

53. The total number receiving between \( a \) and \( b \) dollars equals the number \( N(a) \) receiving \( a \) or more dollars minus the number \( N(b) \) receiving \( b \) or more dollars. Thus
\[
N(a) - N(b) = \int_a^b e^{-x} dx.
\]

54. \( \int_0^{10^{-4}} x^{1/2} dx = \int_0^{10^{-4}} \frac{1}{2} x^{1/2} \bigg|_0^{10^{-4}} = 2 \sqrt{10^{-4}} - 0 \\
= 2 \left(10^{-2} \right) = 0.02 \)

55. \( \int_0^5 2000 e^{-0.06r} dr = 2000 \int_0^5 e^{-0.06r} [-0.06 dr] \\
= \frac{2000}{0.06} e^{-0.06r} \bigg|_0^5 = \frac{2000}{0.06} (e^{-0.03} - 1) \\
= \approx 88639 \)

56. \( \int_0^1 (e^{-a\tau} - e^{-b\tau}) d\tau \\
= \frac{1}{-a} \int_0^1 e^{-a\tau} [-a d\tau] - \frac{1}{-b} \int_0^1 e^{-b\tau} [-b d\tau] \\
= \left( \frac{e^{-a\tau}}{a} + \frac{e^{-b\tau}}{b} \right) \bigg|_0^1 \\
= \left( \frac{e^{-a\tau}}{a} + \frac{e^{-b\tau}}{b} \right) \left( -\frac{1}{a} + \frac{1}{b} \right) \\
= \frac{1 - e^{-at}}{a} - \frac{1 - e^{-bt}}{b} \)

57. \( \int_{10}^{29} 1000 \sqrt{110 - t} dt \\
= -1000 \int_{10}^{29} \sqrt{110 - t} [-dt] \\
= -1000 \left( \frac{110 - t}{{\frac{3}{2}}} \right)^2 \bigg|_{10}^{29} \\
= -2000 \left( \frac{110 - 10}{{\frac{3}{2}}} \right)^2 \bigg|_{10}^{29} \\
= -2000 \left[ (110 - 29)^{3/2} - (110 - 10)^{3/2} \right] \\
= 180,667 \)

For the entire population, \( a = 0 \) and \( b = 110. \)
\( \int_{0}^{1} \frac{20000 e^{-0.05\tau}}{0.05} d\tau = 60,000 \cdot \int_0^1 e^{-0.05\tau} (0.05 d\tau) \\
= 60,000 e^{-0.05\tau} \bigg|_0^1 = 60,000 (e^{-0.05} - 1) \)
59. \[ \int_{65}^{75} (0.2q + 8) \, dq = \left[ 0.1q^2 + 8q \right]_{65}^{75} \]
\[ = 1162.5 - 942.5 = 220 \]

60. \[ \int_{90}^{180} (0.004q^2 - 0.5q + 50) \, dq \]
\[ = \frac{0.004}{3} q^3 - 0.25q^2 + 50q \bigg|_{90}^{180} \]
\[ = 8676 - 3447 \]
\[ = 5229 \]

61. \[ \int_{500}^{800} \frac{2000}{\sqrt{300q}} \, dq = \int_{500}^{800} \frac{2000}{10\sqrt{3}q} \, dq \]
\[ = \frac{200}{\sqrt{3}} \int_{500}^{800} q^{-1/2} \, dq = \frac{200}{\sqrt{3}} \frac{q^{1/2}}{\frac{1}{2}} \bigg|_{500}^{800} \]
\[ = \frac{400}{\sqrt{3}} \sqrt{q} \bigg|_{500}^{800} = \frac{400}{\sqrt{3}} (\sqrt{800} - \sqrt{500}) = 1367.99 \]

62. \[ \int_{5}^{10} (100 + 50q - 3q^2) \, dq \]
\[ = (100q + 25q^2 - q^3) \bigg|_{5}^{10} \]
\[ = (1000 + 2500 - 1000) - (500 + 625 - 125) \]
\[ = 1500 \]

63. \[ \int_{0}^{12} (8t + 10) \, dt = \left[ 4t^2 + 10t \right]_{0}^{12} = 696 - 0 = 696 \]
\[ \int_{6}^{12} (8t + 10) \, dt = \left[ 4t^2 + 10t \right]_{6}^{12} = 696 - 204 = 492 \]

64. \[ \int_{0}^{700} \frac{81 \times 10^6}{(300 + t)^4} \, dt = \left( 81 \times 10^6 \right) \int_{0}^{700} (300 + t)^{-4} \, dt \]
\[ = \left( 81 \times 10^6 \right) \frac{(300 + t)^{-3}}{-3} \bigg|_{0}^{700} \]
\[ = -\left( 27 \times 10^6 \right) \frac{1}{(300 + t)^3} \bigg|_{0}^{700} \]
\[ = -\left( 27 \times 10^6 \right) \left( \frac{1}{1000^3} - \frac{1}{300^3} \right) \]
\[ = -\left( 27 \times 10^6 \right) \left( \frac{1}{10^9} - \frac{1}{27 \times 10^6} \right) \]
\[ = -\frac{27}{10^3} + 1 = -\frac{27}{1000} + 1 = \frac{973}{1000} = 0.973 \]
65. \[ G = \int_{-R}^{R} i \, dx = ix \bigg|_{-R}^{R} = iR - (-iR) = 2Ri \]

66. \[ E = \int_{-R}^{R} \frac{i}{2k} \left[ e^{-k(R-x)} + e^{-k(x+R)} \right] \, dx \]
\[ = \frac{i}{2k} \left[ \int_{-R}^{R} e^{-k(R-x)} \, dx + \int_{-R}^{R} e^{-k(x+R)} \, dx \right] \]
\[ = \frac{i}{2k} \left[ \int_{-R}^{R} e^{-k(R-x)} \, dx - \int_{R}^{R} e^{-k(R+R-x)} \, dx \right] \]
\[ = \frac{i}{2k} \left[ e^{-k(R-x)} - e^{-k(R+x)} \right] \bigg|_{-R}^{R} \]
\[ = \frac{i}{2k} \left[ (1 - e^{-k(2R)}) - (e^{-k(2R)} - 1) \right] \]
\[ = \frac{i}{2k} \left[ 2 - 2e^{-2kR} \right] = \frac{i}{k} \left( 1 - e^{-2kR} \right) \]

67. \[ A = \int_{0}^{R} (m + x)(1 - (m + x)) \, dx = \int_{0}^{R} (m + x - m^2 - 2mx - x^2) \, dx \]
\[ = \int_{0}^{R} (m + x - m^2 - 2mx - x^2) \, dx \]
\[ = \left[ mx^2 + \frac{x^3}{2} - mx^2 - mx^2 - \frac{x^3}{3} \right]_{0}^{R} \]
\[ = \left[ x^3 - mx - \frac{x^2}{2} \right]_{0}^{R} \]
\[ = \left[ mR + \frac{R^2}{2} - m^2R - m^2R - R^3 \right] - 0 \]
\[ = \frac{R^2 - m^2R - R^3}{2} - 0 \]
\[ = \frac{R \left[ m + \frac{R^2}{2} - m^2R - R^3 \right]}{1 - m - \frac{R^2}{3}} \]

68. \[ \int_{2.5}^{3.5} (1 + 2x + 3x^2) \, dx = \left( x + x^2 + x^3 \right) \bigg|_{2.5}^{3.5} \]
\[ = 58.625 - 24.375 \]
\[ = 34.25 \]

69. \[ \int_{0}^{4} \frac{1}{(4x + 4)^2} \, dx = \frac{1}{4} \int_{0}^{4} (4x + 4)^{-2} \, dx \]
\[ = \frac{1}{4} \left[ \frac{1}{x} \right]_{0}^{4} \]
\[ = -\frac{1}{4} \left[ \frac{1}{16} \right]_{0}^{4} \]
\[ = -\frac{1}{16} \left[ \frac{1}{5} - 1 \right] = \frac{1}{20} = 0.05 \]

70. \[ \int_{0}^{3} e^{4} \, dt = \int_{0}^{3} e^{4} \, dt = \left[ \frac{1}{3} e^{3} \right]_{0}^{3} \]
\[ = \frac{1}{3} e^{3} - 0 = 6.36 \]
Chapter 14: Integration

ISM: Introductory Mathematical Analysis

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3. \( f(x) = x^3, \ n = 5, \ a = 0, \ b = 1 \)

   Trapezoidal
   \[
   h = \frac{b-a}{n} = \frac{1-0}{5} = \frac{1}{5} = 0.2 \\
   f(0) = 0.0000 \\
   2f(0.2) = 0.0160 \\
   2f(0.4) = 0.1280 \\
   2f(0.6) = 0.4320 \\
   2f(0.8) = 1.0240 \\
   f(1) = 1.0000 \\
   \]

   \[
   \int_0^1 x^3 \, dx = \frac{0.2}{2} (2.6000) = 0.260 \\
   \]

   Actual value: \( \int_0^1 x^3 \, dx = \frac{x^4}{4} \bigg|_0^1 = \frac{1}{4} = 0.250 \)

4. \( f(x) = x^2, \ n = 4, \ a = 0, \ b = 1 \)

   Simpson's
   \[
   h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25 \\
   f(0) = 0.0000 \\
   4f(0.25) = 0.2500 \\
   2f(0.50) = 0.5000 \\
   4f(0.75) = 2.2500 \\
   f(1) = 1.0000 \\
   \]

   \[
   \int_0^1 x^2 \, dx = \frac{0.25}{3} (4.0000) = \frac{1}{3} = 0.333 \\
   \]

   Actual value: \( \int_0^1 x^2 \, dx = \frac{x^3}{3} \bigg|_0^1 = \frac{1}{3} = 0.333 \)

5. \( f(x) = \frac{1}{x^2}, \ n = 4, \ a = 1, \ b = 4 \)

   Simpson's
   \[
   h = \frac{b-a}{n} = \frac{4-1}{4} = 0.75 \\
   f(1) = 1.0000 \\
   4f(1.75) = 1.3061 \\
   2f(2.50) = 0.3200 \\
   4f(3.25) = 0.3787 \\
   f(4) = 0.0625 \\
   \]

   \[
   \int_1^4 \frac{1}{x^2} \, dx = \frac{0.75}{3} (3.0673) = 0.767 \\
   \]

   Actual value:
   \[
   \int_1^4 \frac{1}{x^2} \, dx = -\frac{1}{x} \bigg|_1^4 = -\frac{1}{4} - (-1) = 0.750 \\
   \]

6. \( f(x) = \frac{1}{x}, \ n = 6, \ a = 1, \ b = 4 \)

   Trapezoidal
   \[
   h = \frac{b-a}{n} = \frac{4-1}{6} = 0.5 \\
   f(1) = 1.0000 \\
   2f(1.5) = 1.3333 \\
   2f(2) = 1.0000 \\
   2f(2.5) = 0.8000 \\
   2f(3) = 0.6667 \\
   2f(3.5) = 0.5714 \\
   f(4) = 0.2500 \\
   \]

   \[
   \int_1^4 \frac{1}{x} \, dx = \frac{0.5}{2} (5.6214) = 1.405 \\
   \]

   Actual value:
   \[
   \int_1^4 \frac{1}{x} \, dx = \ln |x| \bigg|_1^4 = \ln 4 - \ln 1 = \ln 4 - 0 = \ln 4 \\
   \approx 1.386 \\
   \]

7. \( f(x) = \frac{x}{x+1}, \ n = 4, \ a = 0, \ b = 2 \)

   Trapezoidal
   \[
   h = \frac{b-a}{n} = \frac{2-0}{4} = 0.5 \\
   f(0) = 0.0000 \\
   2f(0.5) = 0.6667 \\
   2f(1) = 1.0000 \\
   2f(1.5) = 1.2000 \\
   f(2) = 0.6667 \\
   \]

   \[
   \int_0^2 \frac{x}{x+1} \, dx = \frac{0.5}{2} (3.5334) = 0.883 \\
   \]

   Thus
   \[
   \int_0^2 \frac{x}{x+1} \, dx = \frac{0.5}{2} (3.5334) = 0.883 \\
   \]

8. \( f(x) = \frac{1}{x}, \ n = 6, \ a = 1, \ b = 4 \)

   Simpson’s
   \[
   h = \frac{b-a}{n} = \frac{4-1}{6} = 0.5 \\
   \]

   \[
   \int_1^4 \frac{1}{x} \, dx = \frac{0.5}{2} (3.5334) = 0.883 \\
   \]
Chapter 14: Integration

ISM: Introductory Mathematical Analysis

11. \( a = 1, b = 5, h = 1 \)

\[
\begin{align*}
    f(1) &= 0.4 \\
    4f(1.5) &= 0.4 \\
    2f(2) &= 0.5 \\
    4f(2.5) &= 2.4 \\
    2f(3) &= 2.4 \\
    4f(3.5) &= 3.2 \\
    f(4) &= 1.0 \\
    f(5) &= 0.5 \\
\end{align*}
\]

\[
\int_{1}^{5} f(x) \, dx = \frac{1}{3}(8.9) = 3.0
\]

The area is about 3.0 square units.

12. \( a = 2, b = 5, h = 0.5 \)

\[
\begin{align*}
    f(2) &= 0 \\
    4f(2.5) &= 24 \\
    2f(3) &= 20 \\
    4f(3.5) &= 44 \\
    2f(4) &= 28 \\
    4f(4.5) &= 60 \\
    f(5) &= 16
\end{align*}
\]

\[
\int_{2}^{5} f(x) \, dx = \frac{0.5}{3}(192) = 32
\]

The area is about 32 square units.

13. \( a = 2, b = 5, h = 0.5 \)

\[
\begin{align*}
    f(2) &= 0.4 \\
    4f(2.5) &= 2.4 \\
    2f(3) &= 2.4 \\
    4f(3.5) &= 3.2 \\
    f(4) &= 1.0 \\
    f(5) &= 0.5
\end{align*}
\]

\[
\int_{2}^{5} f(x) \, dx = \frac{0.5}{3}(192) = 32
\]

The area is about 32 square units.

14. \( a = 2, b = 5, h = 0.5 \)

\[
\begin{align*}
    f(2) &= 0.4 \\
    4f(2.5) &= 2.4 \\
    2f(3) &= 2.4 \\
    4f(3.5) &= 3.2 \\
    f(4) &= 1.0 \\
    f(5) &= 0.5
\end{align*}
\]

\[
\int_{2}^{5} f(x) \, dx = \frac{0.5}{3}(192) = 32
\]

The area is about 32 square units.
15. \( f(x) = \sqrt{1-x^2} \), \( a = 0 \), \( b = 1 \), \( n = 4 \)

\[
h = \frac{1 - 0}{4} = 0.25
\]

Simpson’s

\[
f(0) = 1.0000
\]

\[
4f(0.25) = 3.8730
\]

\[
2f(0.50) = 1.7321
\]

\[
4f(0.75) = 2.6458
\]

\[
f(1) = 0.0000
\]

\[
\int_0^1 \sqrt{1-x^2} \, dx = \frac{0.25}{3} (9.2509) = 0.771
\]

16. \[
\int_0^{80} \frac{dr}{dq} \, dq = r(80) - r(0) = r(80)
\]

[since \( r(0) = 0 \)]

Using Simpson’s rule with \( h = 10 \) and \( f(q) = \frac{dr}{dq} \):

\[
f(0) = 10 = 10
\]

\[
4f(10) = 4(9) = 36
\]

\[
2f(20) = 2(8.5) = 17
\]

\[
4f(30) = 4(8) = 32
\]

\[
2f(40) = 2(8.5) = 17
\]

\[
4f(50) = 4(7.5) = 30
\]

\[
2f(60) = 2(7) = 14
\]

\[
4f(70) = 4(6.5) = 26
\]

\[
f(80) = 7 = 7
\]

\[
\int_0^{80} \frac{dr}{dq} \, dq = \frac{10}{3} (189) = 630
\]

The total revenue is about $630.

17. The distance along the fence is \( x \).

The distance across the pool is \( f(x) \).

\( a = 0 \), \( b = 8 \), and \( n = 8 \).

\[
h = \frac{8 - 0}{8} = 1
\]

Area = \[
\frac{h}{3} \left[ 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + 2f(x_6) + 4f(x_7) \right]
\]

\[
= \frac{1}{3} \left[ 4(3) + 2(4) + 4(3) + 2(3) + 4(2) + 2(2) + 4(2) \right]
\]

\[
= \frac{58}{3}
\]

Yes; Lesley’s calculation is correct.
18. a. \( MC = \frac{dc}{dq} \)

\[
\int_{0}^{100} \frac{dc}{dq} \, dq = c(100) - c(0)
\]

= (total cost of 100 units) − (fixed costs)

= total variable costs of 100 units

Using the trapezoidal rule with \( h = 20 \) and

\[ f(q) = \frac{dc}{dq} \] to estimate the integral:

\[
\begin{align*}
  f(0) &= 260 \\
  2f(20) &= 500 \\
  2f(40) &= 480 \\
  2f(60) &= 400 \\
  2f(80) &= 480 \\
  f(100) &= 250
\end{align*}
\]

\[
\int_{0}^{100} \frac{dc}{dq} \, dq = 20 \times (2370) = 23,700
\]

b. \( MR = \frac{dr}{dq} \)

\[
\int_{0}^{100} \frac{dr}{dq} \, dq = r(100) - r(0) = r(100)
\]

[since \( r(0) = 0 \)]

= total revenue from sale of 100 units

Using the trapezoidal rule with \( h = 20 \) and

\[ g(q) = \frac{dr}{dq} \] to estimate the integral:

\[
\begin{align*}
  g(0) &= 410 \\
  2g(20) &= 700 \\
  2g(40) &= 600 \\
  2g(60) &= 500 \\
  2g(80) &= 540 \\
  g(100) &= 250
\end{align*}
\]

\[
\int_{0}^{100} \frac{dr}{dq} \, dq = 20 \times (3000) = 30,000
\]

c. At \( q = 100 \): total revenue = 30,000

\[
\text{total cost} = (\text{total var. costs}) + (\text{fixed costs})
\]

= 23,700 + 2000 = 25,700

Thus maximum profit

= (total revenue) − (total costs)

= 30,000 − 25,700 = $4300.

Problems 14.9

In Problems 1–24, answers are assumed to be expressed in square units.

1. \( y = 5x + 2, \ x = 1, \ x = 4 \)

\[
\text{Area} = \int_{1}^{4} (5x + 2) \, dx = \left[ \frac{5x^2}{2} + 2x \right]_{1}^{4} = 48 - \frac{9}{2} = \frac{87}{2}
\]

2. \( y = x + 5, \ x = 2, \ x = 4 \)

\[
\text{Area} = \int_{2}^{4} (x + 5) \, dx = \left[ \frac{x^2}{2} + 5x \right]_{2}^{4} = 28 - 12 = 16
\]

3. \( y = 3x^2, \ x = 1, \ x = 3 \)

\[
\text{Area} = \int_{1}^{3} 3x^2 \, dx = x^3 \bigg|_{1}^{3} = 27 - 1 = 26
\]
4. \( y = x^2, x = 2, x = 3 \)
\[
\text{Area} = \int_2^3 x^2 \, dx = \frac{x^3}{3} \bigg|_2^3 = 9 - \frac{8}{3} = \frac{19}{3}
\]

5. \( y = x + x^2 + x^3, x = 1 \)
\[
\text{Area} = \int_0^1 (x + x^2 + x^3) \, dx = \left( \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} \right) \bigg|_0^1 = \frac{13}{12} - 0 = \frac{13}{12}
\]

6. \( y = x^2 - 2x, x = -3, x = -1 \)
\[
\text{Area} = \int_{-3}^{-1} (x^2 - 2x) \, dx = \left( \frac{x^3}{3} - x^2 \right) \bigg|_{-3}^{-1} = \frac{4}{3} - (-18) = \frac{50}{3}
\]

7. \( y = 3x^2 - 4x, x = -2, x = -1 \)
\[
\text{Area} = \int_{-2}^{-1} (3x^2 - 4x) \, dx = (x^3 - 2x^2) \bigg|_{-2}^{-1} = -3 - (-16) = 13
\]

8. \( y = 2 - x - x^2 \)
\[
\text{Area} = \int_{-2}^{1} (2 - x - x^2) \, dx = \left( 2x - \frac{x^2}{2} - \frac{x^3}{3} \right) \bigg|_{-2}^{1} = \frac{7}{6} - \left( \frac{-10}{3} \right) = \frac{9}{2}
\]

9. \( y = \frac{4}{x}, x = 1, x = 2 \)
\[
\text{Area} = \int_{1}^{2} \frac{4}{x} \, dx = 4 \ln |x| \bigg|_{1}^{2} = 4 \ln(2) - 0 = 4 \ln 2 = \ln 16
\]
10. \( y = 2 - x - x^3, x = -3, x = 0 \)

\[
\text{Area} = \int_{-3}^{0} (2 - x - x^3) \, dx = \left[ 2x - \frac{x^2}{2} - \frac{x^4}{4} \right]_{-3}^{0} = 0 - \left( -\frac{123}{4} \right) = \frac{123}{4}
\]

11. \( y = e^x, x = 1, x = 3 \)

\[
\text{Area} = \int_{1}^{3} e^x \, dx = e^3 - e
\]

12. \( y = \frac{1}{(x-1)^2}, x = 2, x = 3 \)

\[
\text{Area} = \int_{2}^{3} \frac{1}{(x-1)^2} \, dx = \int_{2}^{3} (x-1)^{-2} \, dx = \left[ -\frac{1}{x-1} \right]_{2}^{3} = -\frac{1}{2} - (-1) = \frac{1}{2}
\]

13. \( y = \frac{1}{x}, x = 1, x = e \)

\[
\text{Area} = \int_{1}^{e} \frac{1}{x} \, dx = \ln|x| \bigg|_{1}^{e} = \ln e - \ln 1 = 1 - 0 = 1
\]

14. \( y = \sqrt{x+9}, x = -9, x = 0 \)

\[
\text{Area} = \int_{-9}^{0} \sqrt{x+9} \, dx = \int_{-9}^{0} (x+9)^{\frac{1}{2}} \, dx
\]

\[
= \left[ \frac{2(x+9)^{\frac{3}{2}}}{3} \right]_{-9}^{0} = \frac{2(9)^{\frac{3}{2}}}{3} - \frac{2(0)^{\frac{3}{2}}}{3} = 18 - 0 = 18
\]

15. \( y = x^2 - 4x, x = 2, x = 6 \)

\[
\text{Area} = \int_{2}^{6} (x^2 - 4x) \, dx = \int_{2}^{4} (x^2 - 4x) \, dx + \int_{4}^{6} (x^2 - 4x) \, dx
\]

\[
= \left[ \frac{x^3}{3} - 2x^2 \right]_{2}^{4} + \left[ \frac{x^3}{3} - 2x^2 \right]_{4}^{6} = \frac{32}{3} - \frac{16}{3} + \left[ 0 - \frac{32}{3} \right] = 16
\]
16. \( y = \sqrt{2x-1} \), \( x = 1 \), \( x = 5 \)

Area = \( \int_1^5 \sqrt{2x-1} \, dx \)

\[ = \frac{1}{2} \int_1^5 (2x-1)^{\frac{3}{2}} \, [2 \, dx] \]

\[ = \frac{(2x-1)^{\frac{5}{2}}}{\frac{5}{2}} \bigg|_1^5 = \frac{9}{3} - \frac{1}{3} = \frac{26}{3} \]

17. \( y = x^3 + 3x^2 \), \( x = -2 \), \( x = 2 \)

Area = \( \int_{-2}^2 \left( x^3 + 3x^2 \right) \, dx = \left( \frac{x^4}{4} + x^3 \right) \bigg|_{-2}^2 \)

\[ = 12 - (-4) = 16 \]

18. \( y = \sqrt[3]{x} \), \( x = 2 \)

Area = \( \int_0^2 \sqrt[3]{x} \, dx = \int_0^2 x^{\frac{1}{3}} \, dx = \frac{3x^{\frac{4}{3}}}{4} \bigg|_0^2 = \frac{3(2)^{\frac{4}{3}}}{4} - 0 \)

\[ = \frac{3(2\sqrt{2})}{4} = \frac{3 \sqrt{2}}{2} \]

19. \( y = e^x + 1 \), \( x = 0 \), \( x = 1 \)

Area = \( \int_0^1 (e^x + 1) \, dx = (e^x + x) \bigg|_0^1 = (e^1 + 1) - 1 = e \)

\[
\int_0^1 (e^x + 1) \, dx = (e^x + x) \bigg|_0^1 = (e^1 + 1) - 1 = e
\]

20. \( y = |x| \), \( x = -2 \), \( x = 2 \)

Area = \( \int_{-2}^2 |x| \, dx = \int_{-2}^0 (-x) \, dx + \int_0^2 x \, dx \)

\[ = -\int_{-2}^0 x \, dx + \int_0^2 x \, dx \]

\[ = [0 - (-2)] + [2 - 0] = 4 \]

21. \( y = x + \frac{2}{x} \), \( x = 1 \), \( x = 2 \)

Area = \( \int_1^2 \left( x + \frac{2}{x} \right) \, dx = \left( \frac{x^2}{2} + 2 \ln |x| \right) \bigg|_1^2 \)

\[ = (2 + 2 \ln 2) - \frac{1}{2} = \frac{3}{2} + 2 \ln 2 = \frac{3}{2} + \ln 4 \]
22. \( y = x^3, x = -2, x = 4 \)

\[
\text{Area} = \int_{-2}^{0} -x^3 \, dx + \int_{0}^{4} x^3 \, dx = -\left[\frac{x^4}{4}\right]_{-2}^{0} + \left[\frac{x^4}{4}\right]_{0}^{4} = [0 - (-4)] + [64 - 0] = 68
\]

23. \( y = \sqrt{x-2}, x = 2, x = 6 \)

\[
\text{Area} = \int_{2}^{6} \sqrt{x-2} \, dx = \left[\frac{2(x-2)^{3/2}}{3}\right]_{2}^{6} = \frac{16}{3} - 0 = \frac{16}{3}
\]

24. \( y = x^2 + 1, x = 0, x = 4 \)

\[
\text{Area} = \int_{0}^{4} (x^2 + 1) \, dx = \left[\frac{x^3}{3} + x\right]_{0}^{4} = \frac{76}{3} - 0 = \frac{76}{3}
\]

25. \( f(x) = \begin{cases} 3x^2 & \text{if } 0 \leq x < 2 \\ 16-2x & \text{if } x \geq 2 \end{cases} \)

\[
\text{Area} = \int_{0}^{2} 3x^2 \, dx + \int_{2}^{3} (16-2x) \, dx = x^3\bigg|_{0}^{2} + \left[16x - x^2\right]_{2}^{3} = [8 - 0] + [39 - 28] = 19 \text{ sq units}
\]

26. \( y = \frac{1}{b-a} \)

\[
\text{Area} = \int_{a}^{b} \frac{1}{b-a} \, dx = \left[\frac{x}{b-a}\right]_{a}^{b} = \frac{b-a}{b-a} = b-a \text{ sq units}
\]

27. a. \( P(0 \leq x \leq 1) = \int_{0}^{1} \frac{11}{8} \, dx = \frac{11}{16} \cdot 1 = \frac{1}{16} - 0 = \frac{1}{16} \)

b. \( P(2 \leq x \leq 4) = \int_{2}^{4} \frac{1}{8} \, dx = \frac{1}{16} \cdot 2 = \frac{1}{4} - \frac{3}{4} \)

c. \( P(x \geq 3) = \int_{3}^{4} \frac{1}{8} \, dx = \frac{1}{16} \cdot 1 = \frac{9}{16} = \frac{7}{16} \)
28. a. \( P(1 \leq x \leq 2) = \int_{1}^{2} \frac{1}{3} (1-x)^2 \, dx \)

= \( \frac{1}{3} \left[ -x \right]_{1}^{2} (1-x)^2 \, dx = \frac{1}{3} (1-x)^3 \bigg|_{1}^{2} \)

= \( -\frac{1}{3} (2-1) = \frac{1}{9} \)

b. \( P \left( 1 \leq x \leq \frac{5}{2} \right) = \int_{1}^{\frac{5}{2}} \frac{1}{3} (1-x)^2 \, dx \)

= \( -\frac{1}{9} (1-x)^3 \bigg|_{1}^{\frac{5}{2}} = -\frac{1}{9} \left( \frac{27}{8} - 0 \right) = \frac{3}{8} \)

c. \( P(x \leq 1) = \int_{0}^{1} \frac{1}{3} (1-x)^2 \, dx = \frac{1}{9} (1-x)^3 \bigg|_{0}^{1} \)

= \( -\frac{1}{9} (0-1) = \frac{1}{9} \)

d. \( \int_{0}^{1} f(x) \, dx = \int_{0}^{1} f(x) \, dx + \int_{1}^{3} f(x) \, dx \)

= \( \frac{1}{9} + P(x \geq 1) \)

Thus, \( P(x \geq 1) = \frac{8}{9} \)

29. a. \( P(3 \leq x \leq 7) = \int_{3}^{7} \frac{1}{x} \, dx = \ln|x| \bigg|_{3}^{7} \)

= \( \ln 7 - \ln 3 = \ln \frac{7}{3} \)

b. \( P(x \leq 5) = \int_{e}^{5} \frac{1}{x} \, dx = \ln|x| \bigg|_{e}^{5} \)

= \( \ln 5 - \ln e = \ln 5 - 1 \)

c. \( P(x \geq 4) = \int_{4}^{e^2} \frac{1}{x} \, dx = \ln|x| \bigg|_{4}^{e^2} \)

= \( \ln e^2 - \ln 4 = 2 - \ln 4 \)

d. \( P(e \leq x \leq e^2) = \int_{e}^{e^2} \frac{1}{x} \, dx \)

= \( \ln |x| \bigg|_{e}^{e^2} = \ln e^2 - \ln e \)

= \( 2 - 1 = 1 \)

30. a. \( \int_{1}^{r} \frac{1}{x^2} \, dx = -\frac{1}{x} \bigg|_{1}^{r} = -\frac{1}{r} + 1 = 1 - \frac{1}{r} \)
35. Intersection points:
\[ x^2 - x = 2x, \quad x^2 - 3x = 0, \quad x(x-3) = 0 \Rightarrow x = 0 \text{ or } x = 3 \]
Area = \[ \int_0^3 (y_{\text{upper}} - y_{\text{lower}}) \, dx + \int_3^4 (y_{\text{upper}} - y_{\text{lower}}) \, dx \]
= \[ \int_0^3 [2x - (x^2 - x)] \, dx + \int_3^4 [(x^2 - x) - 2x] \, dx \]

36. Intersection points: \( x(x-3)^2 = 2x, \quad x(x-3)^2 - 2x = 0, \quad x[3(x-3)^2 - 2] = 0, \quad x(x^2 - 6x + 7) = 0 \Rightarrow x = 0, \quad 3 \pm \sqrt{2} \)
(from the quadratic formula)
Area = \[ \int_0^{3+\sqrt{2}} (y_{\text{upper}} - y_{\text{lower}}) \, dx + \int_{3-\sqrt{2}}^{3+\sqrt{2}} (y_{\text{upper}} - y_{\text{lower}}) \, dx \]
= \[ \int_0^{3+\sqrt{2}} [x(x-3)^2 - 2x] \, dx + \int_{3-\sqrt{2}}^{3+\sqrt{2}} [2x - x(x-3)^2] \, dx \]

37. The graphs of \( y = 1 - x^2 \) and \( y = x - 1 \) intersect when \( 1 - x^2 = x - 1, \quad 0 = x^2 + x - 2, \quad 0 = (x-1)(x+2) \Rightarrow x = 1 \text{ or } x = -2 \). When \( x = 1 \), then \( y = 0 \). We use horizontal elements, where \( y \) ranges from 0 to 1. Solving \( y = x - 1 \) for \( x \) gives \( x = y + 1 \), and solving \( y = 1 - x^2 \) for \( x \) gives \( x^2 = 1 - y \). We must choose \( x = \sqrt{1 - y} \) because \( x \) is not negative over the given region.
Area = \[ \int_0^1 (x_{\text{right}} - x_{\text{left}}) \, dy = \int_0^1 [(y+1) - \sqrt{1-y}] \, dy \]

38. The graphs of \( y = 2x \) and \( y = -2x - 8 \) intersect when \( 2x = -2x - 8, \quad 4x = -8, \quad x = -2 \). When \( x = -2 \), then \( y = -4 \). We use horizontal elements, where \( y \) ranges from -4 to 4. Solving \( y = 2x \) for \( x \) gives \( x = \frac{y}{2} \); solving \( y = -2x - 8 \) for \( x \) gives \( 2x = -y - 8, \quad x = \frac{-y - 8}{2} \).
Area = \[ \int_{-4}^4 (x_{\text{right}} - x_{\text{left}}) \, dy = \int_{-4}^4 \left[ \frac{y}{2} - \left( \frac{-y - 8}{2} \right) \right] \, dy \]

39. The graphs of \( y = x^2 - 5 \) and \( y = 7 - 2x^2 \) intersect when \( x^2 - 5 = 7 - 2x^2, \quad 3x^2 = 12, \quad x^2 = 4, \quad \text{so } x = \pm \sqrt{4} = \pm 2 \). We use vertical elements.
Area = \[ \int_1^2 (y_{\text{upper}} - y_{\text{lower}}) \, dx \]
= \[ \int_1^2 [(7 - 2x^2) - (x^2 - 5)] \, dx \]
40. The curves \( y^2 = x \) and \( 2y = 3 - x \) (or \( x = 3 - 2y \)) intersect when \( y^2 = 3 - 2y, \ y^2 + 2y - 3 = 0, \) \((y + 3)(y - 1) = 0 \Rightarrow y = -3 \) or 1. We use horizontal elements.

\[
\text{Area} = \int_{1}^{0} (x_{\text{RIGHT}} - x_{\text{LEFT}}) \, dy
\]
\[
= \int_{1}^{0} (3 - 2y - y^2) \, dy
\]

In Problems 41–58, the answers are assumed to be expressed in square units.

41. \( y = x^2, \ y = 2x \)

Region appears below.

Intersection: \( x^2 = 2x, \ x^2 - 2x = 0, \ x(x - 2) = 0, \) so \( x = 0 \) or 2.

\[
\text{Area} = \int_{0}^{2} (2x - x^2) \, dx = \left[ x^2 - \frac{x^3}{3} \right]_{0}^{2}
\]
\[
= \left( 4 - \frac{8}{3} \right) - 0 = \frac{4}{3}
\]

42. \( y = x, \ y = -x + 3, \ y = 0 \). Region appears below.

Intersection: \( x = -x + 3, \ 2x = 3, \ x = \frac{3}{2} \)

\[
\text{Area} = \int_{0}^{3/2} x \, dx + \int_{3/2}^{3} (-x + 3) \, dx
\]
\[
= \left[ \frac{x^2}{2} \right]_{0}^{3/2} + \left[ -\frac{x^2}{2} + 3x \right]_{3/2}^{3}
\]
\[
= \left[ \frac{9}{8} - 0 \right] + \left[ \left( -\frac{9}{2} + 9 \right) - \left( -\frac{9}{8} + \frac{9}{2} \right) \right] = \frac{9}{4}
\]

43. \( y = 10 - x^2, \ y = 4 \). Region appears below.

Intersection: \( 10 - x^2 = 4, \ x^2 = 6, \) so \( x = \pm \sqrt{6} \)

\[
\text{Area} = \int_{-\sqrt{6}}^{\sqrt{6}} [(10 - x^2) - 4] \, dx
\]
\[
= \int_{-\sqrt{6}}^{\sqrt{6}} (6 - x^2) \, dx
\]
\[
= \left[ 6x - \frac{x^3}{3} \right]_{-\sqrt{6}}^{\sqrt{6}}
\]
\[
= \left( 6\sqrt{6} - \frac{6\sqrt{6}}{3} \right) - \left( -6\sqrt{6} + \frac{6\sqrt{6}}{3} \right) = 8\sqrt{6}
\]

44. \( y^2 = x + 1, \ x = 1 \). Region appears below.

Intersection: \( y^2 = 2, \ y = \pm \sqrt{2} \)

\[
\text{Area} = \int_{-\sqrt{2}}^{\sqrt{2}} \left[ 1 - (y^2 - 1) \right] \, dy = \left[ 2y - \frac{y^3}{3} \right]_{-\sqrt{2}}^{\sqrt{2}}
\]
\[
= \left( 2\sqrt{2} - \frac{2\sqrt{2}}{3} \right) - \left( -2\sqrt{2} + \frac{2\sqrt{2}}{3} \right) = \frac{8\sqrt{2}}{3}
\]
45. \( x = 8 + 2y, \ x = 0, \ y = -1, \ y = 3. \) Region appears below.

\[
\text{Area} = \int_{-1}^{3} (8 + 2y) \, dy = \left[ 8y + y^2 \right]_{-1}^{3} = (24 + 9) - (-8 + 1) = 40
\]

46. \( y = x - 6, \ y^2 = x. \) Region appears below.

Intersection: \( y^2 = y + 6, \ y^2 - y - 6 = 0, \)

\( (y + 2)(y - 3) = 0, \) so \( y = -2, 3. \)

\[
\text{Area} = \int_{-2}^{3} \left[ (y + 6) - (y^2) \right] \, dy
\]

\[
= \left[ \frac{y^3}{2} + 6y - \frac{y^3}{3} \right]_{-2}^{3}
\]

\[
= \left( \frac{9}{2} + 18 - 9 \right) - \left( 2 - 12 + \frac{8}{3} \right) = 125/6
\]

47. \( y^2 = 4x, \ y = 2x - 4. \) Region appears below.

Intersection: \( y^2 = 4 \left( \frac{y}{2} + 2 \right), \ y^2 - 2y - 8 = 0, \)

\( (y + 2)(y - 4) = 0, \) so \( y = -2 \) or \( 4. \)

\[
\text{Area} = \int_{-2}^{4} \left[ \frac{y}{2} + 2 \right] - \frac{y^2}{4} \, dy
\]

\[
= \left[ \frac{y^2}{4} + 2y - \frac{y^3}{12} \right]_{-2}^{4}
\]

\[
= \left( 4 + 8 - \frac{16}{3} \right) - \left( 1 - 4 + \frac{2}{3} \right) = 9
\]

48. \( y = x^3, \ y = x + 6, \ x = 0 \)

Region appears below.

Intersection: \( x^3 = x + 6, \ x^3 - x - 6 = 0, \)

\( (x - 2)(x^2 + 2x + 3) = 0 \Rightarrow x = 2 \)

\( x^3 = 0 \Rightarrow x = 0 \)

\[
\text{Area} = \int_{0}^{2} [(x + 6) - x^3] \, dx
\]

\[
= \left( \frac{x^2}{2} + 6x - \frac{x^4}{4} \right)_{0}^{2}
\]

\[
= (2 + 12 - 4) - (0) = 10
\]

49. \( 2y = 4x - x^2, \ 2y = x - 4. \) Region appears below.

Intersection: \( x - 4 = 4x - x^2, \ x^2 - 3x - 4 = 0, \)

\( (x + 1)(x - 4) = 0, \) so \( x = -1 \) or \( 4. \) Note that the \( y \)-values of the curves are given by \( y = \frac{4x - x^2}{2} \) and \( y = \frac{x - 4}{2}. \)
Area = \int_{-1}^{4} \left[ \frac{4x-x^2}{2} - \frac{x-4}{2} \right] dx
= \int_{-1}^{4} \left( \frac{3x^2 - x^3 + 2x}{2} \right) dx
= \left[ \frac{3x^2 - \frac{x^3}{6} + 2x}{2} \right]_{-1}^{4}
= \left( \frac{12 - \frac{64}{6} + 8}{2} - \left( \frac{3}{4} - \frac{6}{6} - 2 \right) \right)
= \frac{125}{12}

50. \ y = \sqrt{x}, \ y = x^2. \ Region \ appears \ below.
Intersection: \ x^2 = \sqrt{x}, \ x^4 = x, \ x^4 - x = 0,
\ x(x^3 - 1) = 0, \ so \ x = 0, 1.
Area = \int_{0}^{1} (\sqrt{x} - x^2) dx = \left[ \frac{2x^{\frac{3}{2}}}{3} - \frac{x^3}{3} \right]_{0}^{1}
= \left( \frac{2\cdot1^{\frac{3}{2}}}{3} - \frac{1^3}{3} \right) - 0 = \frac{1}{3}

51. \ y = 8 - x^2, \ y = x^2, \ x = -1, x = 1. \ Region \ appears \ below.
Intersection: \ x^2 = 8 - x^2, \ 2x^2 = 8, \ x^2 = 4, \ so \ x = \pm 2.

Area = \int_{-1}^{1} \left[ (8-x^2) - x^2 \right] dx = \int_{-1}^{1} (8 - 2x^2) dx
= \left[ \frac{8x - \frac{2x^3}{3}}{1} \right]_{-1}^{1}
= \left( \frac{8 - \frac{2}{3}}{3} - \left( -8 + \frac{2}{3} \right) \right) = \frac{44}{3}

52. \ y = x^3 + x, \ y = 0 (x-axis), \ x = -1, x = 2
Region appears below.
Area = \int_{-1}^{2} -(x^3 + x) dx + \int_{0}^{2} (x^3 + x) dx
= \left[ \frac{-x^4 - \frac{x^2}{2}}{1} \right]_{-1}^{0} + \left[ \frac{x^4 + \frac{x^2}{2}}{1} \right]_{0}^{2}
= \left[ 0 - \left( \frac{-1}{4} - \frac{1}{2} \right) \right] + [(4 + 2) - 0]
= \frac{27}{4}

53. \ y = x^3 - 1, \ y = x - 1. \ Region \ appears \ below.
Intersection: \ x^3 - 1 = x - 1, \ x^3 - x = 0,
\ x(x^2 - 1) = 0,
\ x(x + 1)(x - 1) = 0, \ so \ x = 0 \ or \ x = \pm 1.
Area = \int_{-1}^{0} [x^3 - 1 - (x-1)] \, dx + \int_{0}^{1} [x-1 - (x^3-1)] \, dx

= \int_{-1}^{0} (x^3 - x) \, dx + \int_{0}^{1} (x-x^3) \, dx

= \left[ \frac{x^4}{4} - \frac{x^2}{2} \right]_{-1}^{0} + \left[ \frac{x^2}{2} - \frac{x^4}{4} \right]_{0}^{1}

= \left[ 0 - \left( \frac{1}{4} - \frac{1}{2} \right) \right] + \left[ \frac{1}{2} - \frac{1}{4} - 0 \right] = \frac{1}{2}

54. \ y = x^3, \ y = \sqrt{x} \ . \ Region \ appears \ below. \ Intersection: \ x^3 = \sqrt{x}, \ x^6 = x, \ x^6 - x = 0 \ , \ x(x^5 - 1) = 0 \ , \ x = 0, \ 1

Area = \int_{0}^{1} (\sqrt{x} - x^3) \, dx = \left[ \frac{2x^{\frac{7}{4}}}{3} - \frac{x^{\frac{4}{2}}}{4} \right]_{0}^{\frac{1}{3}}

= \left( \frac{2}{3} - \frac{1}{4} \right) - 0 = \frac{5}{12}

55. \ 4x + 4y + 17 = 0, \ y = \frac{1}{x} \ . \ Region \ appears \ below. \ Intersection: \ \frac{-17 - 4x}{4} = \frac{1}{x}, \ -17x - 4x^2 = 4 \ , \ 4x^2 + 17x + 4 = 0 \ ,

(4x + 1)(x + 4) = 0, \ so \ x = -\frac{1}{4} \ or \ -4.

Area = \int_{-4}^{-1/4} \left[ \frac{1}{x} + \left( \frac{-17 - 4x}{4} \right) \right] \, dx = \left[ \ln|x| + \frac{17}{4}x + \frac{x^2}{2} \right]_{-4}^{-1/4}

= \left( \ln \frac{1}{4} - \frac{17}{16} + \frac{1}{32} \right) - \left( \ln 4 - 17 + 8 \right) = \frac{255}{32} - 4 \ln 2
56. \( y^2 = -x - 2 \), \( x - y = 5 \), \( y = -1 \), \( y = 1 \).

Region appears below.

Intersection: \( y^2 = -x - 2 \) intersects \( y = \pm 1 \) when \( x = -3 \); \( x - y = 5 \) intersects \( y = 1 \) when \( x = 6 \); 
\( x - y = 5 \) intersects \( y = -1 \) when \( x = 4 \)

Area \( = \int_{-1}^{1} [(y+5)-(-y^2-2)]dy = \int_{-1}^{1} (y+7+y^2)dy = \left[ \frac{y^2}{2} + 7y + \frac{y^3}{3} \right]_{-1}^{1} \)

\( = \left( \frac{1}{2} + 7 + \frac{1}{3} \right) - \left( \frac{1}{2} - 7 - \frac{1}{3} \right) = \frac{44}{3} \)


57. \( y = x - 1 \), \( y = 5 - 2x \). Region appears below.

Intersection: \( x - 1 = 5 - 2x \), \( 3x = 6 \), so \( x = 2 \).

Area \( = \int_{0}^{2} [(5-2x)-(x-1)]dx + \int_{2}^{4} [(x-1)-(5-2x)]dx = \int_{0}^{2} (6-3x)dx + \int_{2}^{4} (3x-6)dx \)

\( = -\frac{1}{3} \int_{0}^{2} (6-3x)[-3 \, dx] + \frac{1}{3} \int_{2}^{4} (3x-6)[3 \, dx] = -\frac{(6-3x)^2}{6} \bigg|_{0}^{2} + \frac{(3x-6)^2}{6} \bigg|_{2}^{4} \)

\( = -[0 - 6] + [6 - 0] = 6 + 6 = 12 \)
58. \( y = x^2 - 4x + 4, \ y = 10 - x^2 \). Region appears below.

Intersection:  
\[ x^2 - 4x + 4 = 10 - x^2, \ 2x^2 - 4x - 6 = 0, \ x^2 - 2x - 3 = 0, \ (x-3)(x+1) = 0 \text{, so } x = 3, -1. \]

Area = \[ \int_2^3 [ (10-x^2) - (x^2-4x+4)] \, dx + \int_3^4 [ (x^2-4x+4) - (10-x^2)] \, dx \]

\[ = \int_2^3 (6+4x-2x^2) \, dx + \int_3^4 (2x^2-4x-6) \, dx = 2 \left( \int_2^3 (3+2x-x^2) \, dx + \int_3^4 (x^2-2x-3) \, dx \right) \]

\[ = 2 \left( \left[ 3x + x^2 - \frac{x^3}{3} \right]_2^3 + \left[ \frac{x^3}{3} - x^2 - 3x \right]_3^4 \right) = 2 \left[ \left[ \frac{9 - \frac{22}{3}}{3} \right] + \left[ \frac{-20}{3} - (-9) \right] \right] = 2(4) = 8 \]

59. Area between curve and diag.  
Area under diagonal  
\[
\frac{\text{Numerator} \int_0^1 x - \left( \frac{14}{15} x^2 + \frac{1}{15} x \right) \, dx}{\int_0^1 x \, dx}
\]

Numerator = \[ \int_0^1 \left[ \frac{14}{15} x - \frac{14}{15} x^2 \right] \, dx = \frac{14}{15} \int_0^1 \left( x - x^2 \right) \, dx = \frac{14}{15} \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{14}{15} \left[ \frac{1}{2} - \frac{1}{3} - 0 \right] = \frac{14}{15} \cdot \frac{1}{6} = \frac{7}{45} \]

Denominator = \[ \int_0^1 x \, dx = \frac{x^2}{2} \left|_0^1 \right. = \frac{1}{2} \]

Coefficient of inequality = \[ \frac{7}{45} \cdot \frac{2}{1} = \frac{14}{45} \]

60. Area between curve and diag.  
Area under diagonal  
\[
\frac{\text{Numerator} \int_0^1 x - \left( \frac{11}{12} x^2 + \frac{1}{12} x \right) \, dx}{\int_0^1 x \, dx}
\]

Numerator = \[ \int_0^1 \left( x - x^2 \right) \, dx = \frac{11}{12} \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{11}{12} \left[ \frac{1}{2} - \frac{1}{3} - 0 \right] = \frac{11}{12} \cdot \frac{1}{6} = \frac{11}{72} \]

Denominator = \[ \int_0^1 x \, dx = \frac{1}{2} \] (see Problem 35).

Coefficient of inequality = \[ \frac{11}{72} \cdot \frac{2}{1} = \frac{11}{36} \]
61. \( y^2 = 3x, \ y = mx \)

Intersection: \((mx)^2 = 3x, \ m^2x^2 = 3x\)

\[ m^2x^2 - 3x = 0, \ x(m^2x - 3) = 0, \ x = 0 \text{ or } \frac{3}{m^2}. \]

If \( x = 0 \), then \( y = 0 \); if \( x = \frac{3}{m^2} \), then \( y = \frac{3}{m} \).

With horizontal elements,

\[
\text{Area} = \int_0^{3/m} \left( \frac{y}{m} - \frac{y^2}{3} \right) dy = \left( \frac{y^2}{2m} - \frac{y^3}{9} \right)_0^{\frac{3}{m}} = \frac{9}{2m^3} - \frac{3}{m^3} = \frac{3}{2m^3} \text{ square units}.
\]

Note: With vertical elements,

\[
\text{Area} = \int_0^{3/m} \left( \sqrt{\frac{3}{m} - mx} \right) dx.
\]

62. \( y = x^2 - 1, \ y = 2x + 2 \)

Intersection: \( x^2 - 1 = 2x + 2 \),

\( x^2 - 2x - 3 = 0, \ (x - 3)(x + 1) \), so \( x = 3 \) and \(-1\). The area is

\[
\int_{-1}^{3} \left[ 2x + 2 - (x^2 - 1) \right] dx = \left[ -x^3 + x^2 + 3x \right]_{-1}^{3} = \frac{32}{3}
\]

63. \( y = x^2 \) and \( y = k \) intersect when \( x^2 = k, \ x = \pm \sqrt{k} \). Equating areas gives

\[
\int_{-\sqrt{k}}^{\sqrt{k}} (k - x^2) dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} (4 - x^2) dx = \frac{1}{2} \left[ 4x - \frac{x^3}{3} \right]_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{16}{3} \]

\( k^{\frac{3}{2}} = \frac{16}{3} \Rightarrow k = \frac{256}{27} = 2.52 \)

64. 0.23 sq units

65. 4.76 sq units

66. Two integrals are involved.

Answer: 36.65 sq units

67. Two integrals are involved.

Answer: 7.26 sq units

68. Three integrals are involved.

Answer: 358.18 sq units
Chapter 14: Integration

ISM: Introductory Mathematical Analysis

Problems 14.10

1. \( D: p = 22 - 0.8q \)
   \( S: p = 6 + 1.2q \)

   Equilibrium pt. = \((q_0, p_0) = (8, 15.6)\)

   
   \[
   \text{CS} = \int_0^{q_0} [f(q) - p_0] dq
   \]
   
   \[
   = \int_0^8 [(22 - 0.8q) - 15.6] dq = \int_0^8 (6.4 - 0.8q) dq
   \]
   
   \[
   = \left(6.4q - 0.4q^2\right)_0^8 = (51.2 - 25.6) - 0 = 25.6
   \]

   \[
   \text{PS} = \int_0^{q_0} [p_0 - g(q)] dq
   \]
   
   \[
   = \int_0^8 [15.6 - (6 + 1.2q)] dq = \int_0^8 (9.6 - 1.2q) dq
   \]
   
   \[
   = \left(9.6q - 0.6q^2\right)_0^8 = (76.8 - 38.4) - 0 = 38.4
   \]

2. \( D: p = 2200 - q^2 \)
   \( S: p = 400 + q^2 \)

   Equilibrium point = \((q_0, p_0) = (30, 1300)\)

   
   \[
   \text{CS} = \int_0^{q_0} [(2200 - q^2) - 1300] dq
   \]
   
   \[
   = \int_0^{30} (900 - q^2) dq = \left(900q - \frac{q^3}{3}\right)_0^{30}
   \]
   
   \[
   = (27,000 - 9000) - 0 = 18,000
   \]

   \[
   \text{PS} = \int_0^{q_0} [1300 - (400 + q^2)] dq
   \]
   
   \[
   = \int_0^{30} (900 - q^2) dq
   \]
   
   \[
   = \left(900q - \frac{q^3}{3}\right)_0^{30}
   \]
   
   \[
   = (27,000 - 9000) - 0 = 18,000
   \]

3. \( D: p = \frac{5q}{q + 5} \)
   \( S: p = \frac{q}{10} + 4.5 \)

   Equilibrium pt. = \((q_0, p_0) = (5, 5)\)

   
   \[
   \text{CS} = \int_0^{q_0} [f(q) - p_0] dq
   \]
   
   \[
   = \int_0^5 \left[\frac{50}{q + 5} - 5\right] dq = \left(50\ln(q + 5) - 5q\right)_0^5
   \]
   
   \[
   = [50\ln(10) - 25] - [50\ln(5)]
   \]
   
   \[
   = 50\ln(2) - 25 = 50\ln(2) - 25
   \]

   \[
   \text{PS} = \int_0^{q_0} [p_0 - g(q)] dq
   \]
   
   \[
   = \int_0^5 \left[\frac{5q}{10} - 4.5\right] dq = \int_0^5 \left(0.5 - \frac{q}{10}\right) dq
   \]
   
   \[
   = \left(0.5q - \frac{q^2}{20}\right)_0^5
   \]
   
   \[
   = (2.5 - 1.25) - 0 = 1.25
   \]

4. \( D: p = 900 - q^2 \)
   \( S: p = 10q + 300 \)

   Equilibrium point = \((q_0, p_0) = (20, 500)\)

   
   \[
   \text{CS} = \int_0^{q_0} [(900 - q^2) - 500] dq
   \]
   
   \[
   = \int_0^{20} (400 - q^2) dq
   \]
   
   \[
   = \left(400q - \frac{q^3}{3}\right)_0^{20}
   \]
   
   \[
   = (8000 - 3333.33) - 0
   \]
   
   \[
   = 16,000
   \]

   \[
   \text{PS} = \int_0^{q_0} [500 - (10q + 300)] dq
   \]
   
   \[
   = \int_0^{20} (200 - 10q) dq
   \]
   
   \[
   = (200q - 5q^2)_0^{20}
   \]
   
   \[
   = (4000 - 2000) - 0
   \]
   
   \[
   = 2000
   \]

5. \( D: q = 100(10 - 2p) \)
   \( S: q = 50(2p - 1) \)

   Equilibrium pt. = \((q_0, p_0) = (300, 3.5)\)

   We use horizontal strips and integrate with respect to \(p\).

   
   \[
   \text{CS} = \int_{3.5}^{225} 100(10 - 2p) dp = 100[(10 - p)^2]_{3.5}^{225}
   \]
   
   \[
   = 100[(50 - 25) - (35 - 12.5)]
   \]
   
   \[
   = 100(25) = 2500
   \]

   \[
   \text{PS} = \int_{0.5}^{3.5} 50(2p - 1) dp = 50(p^2 - p)_{0.5}^{3.5}
   \]
   
   \[
   = 50[(12.25 - 3.5) - (0.25 - 0.5)]
   \]
   
   \[
   = 450
   \]
6. \( D: q = \sqrt{100 - p} \)
\[ S: q = \frac{p}{2} - 10 \]
Equilibrium pt. \( (q_0, p_0) = (8, 36) \)
Integrating with respect to \( p \),
\[ CS = \int_{36}^{100} \sqrt{100 - p} \, dp \]
\[ = -\frac{2}{3} (100 - p)^{\frac{3}{2}} \bigg|_{36}^{100} \]
\[ = 0 - \left( -\frac{2}{3} \cdot 512 \right) = \frac{1024}{3} \]
\[ PS = \int_{36}^{100} \left( \frac{p}{2} - 10 \right) \, dp \]
\[ = \left[ \frac{p^2}{4} - 10p \right]_{36}^{100} = (324 - 360) - (100 - 200) \]
\[ = 64 \]

7. We integrate with respect to \( p \). From the demand equation, when \( q = 0 \), then \( p = 100 \).
\[ CS = \int_{84}^{100} 10\sqrt{100 - p} \, dp \]
\[ = \int_{84}^{100} 10(100 - p)^{\frac{1}{2}} \, [\!-dp] \]
\[ = -\frac{20}{3} (100 - p)^{\frac{3}{2}} \bigg|_{84}^{100} \]
\[ = -\frac{20}{3} \left[ 0 - 16^{\frac{3}{2}} \right] = -\frac{20}{3} (-64) \]
\[ = 426 \frac{2}{3} = \$426.67 \]

8. At equilibrium, \( p = \frac{400 - p^2}{60}, \) \( 60p = 400 - p^2 + 300, p^2 + 60p - 700 = 0, \)
\( (p + 70)(p - 10) = 0 \Rightarrow p = 10 \) and
\( q = 400 - 10^2 = 300, \) so equilibrium pt. is \( (q_0, p_0) = (300, 10) \).
\[ PS = \int_{0}^{300} \left[ 10 - \left( \frac{q}{60} + 5 \right) \right] \, dq \]
\[ = \left( 5q - \frac{q^2}{120} \right)_{0}^{300} = (1500 - 750) - 0 = 750 \]
For CS we integrate with respect to \( p \). From the demand equation, \( q = 0 \Rightarrow p = 20 \).
\[ CS = \int_{10}^{20} \left( 400 - p^2 \right) \, dp = \left( 400p - \frac{p^3}{3} \right)_{10}^{20} \]
\[ = \left( 8000 - \frac{8000}{3} \right) - \left( 4000 - \frac{1000}{3} \right) = 1666 \frac{2}{3} \]

9. At equilibrium, \( 2^{10 - q} = 2^{q + 2} \Rightarrow 10 - q = q + 2 \Rightarrow q = 4, \) so
\( p = 2^{10 - q} = 64 \)
\[ CS = \int_{0}^{4} (2^{10 - q} - 64) \, dq \]
\[ = \left( \frac{2^{10 - q}}{\ln 2} - 64q \right)_{0}^{4} \]
\[ = \left( \frac{2^6 - 256}{\ln 2} - \frac{2^6 - 0}{\ln 2} \right) \]
\[ = 1128.987 \text{ hundred} \]
\[ = \$113,000 \]

10. a. \( (10 + 10)(30 + 20) = 1000, (20)(50) = 1000, 1000 = 1000 \)
\( 30 - 4(10) + 10 = 0, 30 - 40 + 10 = 0, 0 = 0 \)