AMS 10 Mathematical Methods for Engineers I

Review for the Final Exam: Part 2

In Part 2, we highlight some of the important items from Section 2.8 to Section 6.3 in the textbook.

Subspaces
- Col A
- Nul A

A basis for a subspace
  - Two conditions for a basis
  - How to find a basis for the subspace spanned by a set of vectors?

Dimension of a subspace
  - How to find dim (Col A)?
  - How to find dim (Nul A)?

Rank of a matrix
  - How to find rank A?

Question 20:
How to check if \( \vec{b} \) is in Col A?

Method:
To answer this question, we look at if \( A \vec{x} = \vec{b} \) is consistent.

Question 21:
How to check if \( \vec{b} \) is in Nul A?

Method:
To answer this question, we look at if \( A \vec{b} = \vec{0} \)

Question 22:
How to find a basis for Nul A?

Method:
To answer this question, we find the solution set of homogeneous linear system \( A \bar{x} = \bar{0} \) in parametric form.

**Question 23:**

How to find a basis for \( \text{Col} \ A \)?

**Method:**

To answer this question, we find the pivot columns of \( A \) (not the pivot columns of the reduced echelon form of \( A \)!!!)

**Question 23B:**

How to find a basis for \( \text{span} \{ \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n \} \)?

**Method:**

To answer this question, we consider matrix \( A = [\tilde{v}_1 \ \tilde{v}_2 \ \cdots \ \tilde{v}_n] \).

\( \text{span} \{ \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n \} = \text{Col} \ A \)

We find a basis for \( \text{Col} \ A \).

**Question 24:**

How to find \( \text{dim} \ (\text{Col} \ A) \)?

**Method:**

To answer this question, we find the number of pivot columns of \( A \).

**Question 25:**

How to find \( \text{dim} \ (\text{Nul} \ A) \)?

**Method:**

To answer this question, we find the number of free variables in \( A \bar{x} = \bar{0} \).

**The rank theorem:**

Suppose \( A \) is \( m \times n \). We have

- \( \text{rank} \ A^T = \text{rank} \ A \)
- \( \text{rank} \ A + \text{dim} \ (\text{Nul} \ A) = n \)

**Question 26:**
Suppose $A$ is $m \times n$.

Is it true that $\text{rank } A + \dim (\text{Nul } A^T) = m$?

**Question 26B:**

Suppose $A$ is $n \times n$ (square matrix).

Show that $\dim (\text{Nul } A) = \dim (\text{Nul } A^T)$

**Question 26C:**

Suppose $A$ is $7 \times 5$.

Calculate $\dim (\text{Nul } A) - \dim (\text{Nul } A^T)$

**Method:** Use the rank theorem to write out

$\text{rank } A + \dim (\text{Nul } A) = ?$

$\text{rank } A^T + \dim (\text{Nul } A^T) = ?$

**The invertible matrix theorem (extended in Section 2.9 and Section 5.2)**

**Question 27:**

Suppose an $n \times n$ matrix $A = [\bar{a}_1, \ldots, \bar{a}_n]$ is invertible.

Is $\{\bar{a}_1, \ldots, \bar{a}_n\}$ a basis for $\mathbb{R}^n$?

$\text{Col } A = \mathbb{R}^n$?

$\dim \text{Col } A = n$?

$\text{rank } A = n$?

$\dim \text{Nul } A = 0$?

$\det A = 0$?

Is 0 an eigenvalue of $A$?

**Determinant of a matrix (det A)**

- Co-factor expansion
  - Co-factor expansion along row $i$
  - Co-factor expansion along column $j$

**Determinant of a $2 \times 2$ matrix**
Determinant of a triangular matrix
Use row reduction to calculate $\det A$

Properties of determinants:

- $A: n \times n$
  - $\det A \neq 0$ if and only if $A$ is invertible
    - if and only if columns of $A$ are independent
    - if and only if rows of $A$ are independent
  - $\det A^T = \det A$
  - $\det (AB) = (\det A) (\det B)$
  - $\det A^{-1} = \frac{1}{\det A}$
    - $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ 0 & \ddots & \vdots \\ 0 & 0 & a_{nn} \end{bmatrix}$
    - $= a_{11} \cdots a_{nn}$
  - $\det (\alpha A) = \alpha^n \det A$

Question 28:
Find

$$\begin{bmatrix} 2 & 2 & 5 & 4 & 1 & 7 & 3 \\ 1 & 1 & 3 & 6 & 8 & 1 & 5 \\ 5 & 5 & 2 & 7 & 7 & 6 & 4 \\ 7 & 7 & 6 & 1 & 5 & 4 & 3 \\ 3 & 3 & 7 & 5 & 2 & 3 & 1 \\ 1 & 1 & 4 & 3 & 4 & 5 & 6 \\ 4 & 4 & 3 & 6 & 7 & 6 & 2 \end{bmatrix}$$

Method: Notice that column 1 and column 2 are the same.

Question 29:
Suppose matrix $A$ is $5 \times 5$ and $\det A = 3$.
Find
$$\det A^{-1}, \quad \det A^T, \quad \det A^2, \quad \det (A^T A), \quad \det (A A^T),$$
\[
\det(A^3A^T), \quad \det\left(\frac{1}{3}A\right), \quad \det(A+A), \quad \det(A^T+A^T)
\]

**Method:** Use properties of determinants.

Eigenvalues and eigenvectors of A

\(\lambda\) is an eigenvalue of A if and only if \(\det(A - \lambda I) = 0\)

Characteristic polynomial: \(\det(A - \lambda I)\)

Characteristic equation: \(\det(A - \lambda I) = 0\)

Algebraic multiplicity of an eigenvalue

Eigenspace of an eigenvalue

A basis for the eigenspace

Dimension of the eigenspace

Geometric multiplicity of an eigenvalue

Eigenvalues of a triangular matrix

Row reduction of matrix A is not a tool for calculating eigenvalues of matrix A.

**Question 30:**

How to find eigenvalues of A?

**Method:**

To answer this question, we solve the characteristic equation \(\det(A - \lambda I) = 0\).

**Question 31:**

Find the eigenvalues of

\[
\begin{bmatrix}
a_{11} & \cdots & a_{1n} \\
0 & \ddots & \vdots \\
0 & 0 & a_{nn}
\end{bmatrix}
\]

**Question 32:**

How to find a basis for the eigenspace of eigenvalue \(\lambda\)?

**Method:**

To answer this question, we solve the homogeneous linear system \((A - \lambda I)\vec{x} = \vec{0}\) and write the solution set in parametric form.
Question 32B:
How to find the geometric multiplicity of eigenvalue $\lambda$?

Method:
To answer this question, we find the number of free variables in \((A - \lambda I)\vec{x} = \vec{0}\).

Question 33:
How to find the algebraic multiplicity of eigenvalue $\lambda$?

Method:
To answer this question, we factor the characteristic polynomial \(\det(A - \lambda I)\).

Theorem 2: Eigenvectors for distinct eigenvalues are linearly independent.

Similarity between two matrices:
\[ B = P^{-1}AP \]
Diagonalizable:
\[ P^{-1}AP = \text{diagonal} \]

Theorem 5 of chapter 5: (The diagonalization theorem)
An \(n \times n\) matrix is diagonalizable if and only if A has \(n\) linearly independent eigenvectors.

Theorem 6 of chapter 5:
If an \(n \times n\) matrix A has \(n\) distinct eigenvalues, then A is diagonalizable.

Question 34:
How to read out eigenvalues and eigenvectors from the given diagonalization
\[ P^{-1}AP = D \quad \text{where matrix D is diagonal?} \]

Question 35:
How to construct diagonalization of A?
Is the construction guaranteed to be successful for all matrices?
Inner product of two vectors
norm of a vector
distance between two vectors
orthogonal vectors
orthogonal complement of a subspace
orthogonal set
orthogonal basis

**Theorem 3 of chapter 6:**
A is an \( m \times n \) matrix. We have
\[
(\text{Row } A)^\perp = \text{Nul } A \\
(\text{Col } A)^\perp = \text{Nul } A^T
\]

**Question 36:**
How to find a basis for \( (\text{Col } A)^\perp \)?

**Method:**
To answer this question, we use \( (\text{Col } A)^\perp = \text{Nul } A^T \).

To find a basis for \( \text{Nul } A^T \), we solve \( A^T \vec{x} = \vec{0} \) and write the solution set in parametric form.

**Question 37:**
Let \( W = \text{span} \{ \vec{v}_1, \ldots, \vec{v}_p \} \).

How to find a basis for \( W^\perp \)?

**Method:**
We set \( A = [\vec{v}_1 \cdots \vec{v}_p] \). We have
\[
W = \text{Col } A \\
W^\perp = (\text{Col } A)^\perp = \text{Nul } A^T
\]
Then we use the method in the previous question.

**Question 38:**
Suppose $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a set of $n$ non-zero vectors in $\mathbb{R}^n$.

Suppose $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is also an orthogonal set.

Is $\{\vec{v}_1, \ldots, \vec{v}_n\}$ linearly independent?

Is $\{\vec{v}_1, \ldots, \vec{v}_n\}$ a basis for $\mathbb{R}^n$?

Is $\{\vec{v}_1, \ldots, \vec{v}_n\}$ an orthogonal basis for $\mathbb{R}^n$?

**Question 39:**

Let $W$ be a subspace in $\mathbb{R}^n$.

Let $\{\vec{u}_1, \ldots, \vec{u}_p\}$ be an orthogonal basis for $W$.

Express

$\vec{x}$ in $W$ as a linear combination of $\{\vec{u}_1, \ldots, \vec{u}_p\}$.

**Orthogonal projection of a vector onto a subspace**

**Question 40:**

Let $W$ be a subspace in $\mathbb{R}^n$.

Let $\{\vec{u}_1, \ldots, \vec{u}_p\}$ be an orthogonal basis for $W$.

Find

$\overline{\text{proj}_W \vec{y}} = ?$

Projection of $\vec{y}$ onto $W$