AMS 10/10A Midterm Exam

Your Name: ____________________________________________

Your Student ID #: ______________________________________

The section you are attending: _____________________________

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Problem 1 (10 points):

a. Let \( z = \frac{6 + 7i}{1 + 4i} \). Find \( \text{Re}(z) \) and \( \text{Im}(z) \). (3 points)

\[
z = \frac{6 + 7i}{1 + 4i} = \frac{(6 + 7i)(1 - 4i)}{(1 + 4i)(1 - 4i)} = \frac{(6 - 28) + (7 - 24)i}{1 - 16} = \frac{34 - 17i}{17} = 2 - i
\]

b. Let \( z = 1 + i \). Write \( z \) in the exponential form \( z = r e^{\theta i} \). (3 points)

\[z \text{ is in the first quadrant. } |z| = \sqrt{2}, \quad \arg(z) = \tan^{-1}(1) = \frac{\pi}{4}\]

\[\implies z = \sqrt{2} e^{\frac{\pi}{4}i}\]

c. Let \( z = 1 + i \). Find the real and imaginary parts of \( z^{19} \). (4 points)

(Hint: Use the result of b.)

\[
z = \sqrt{2} e^{\frac{\pi}{4}i}
\]

\[\implies z^{19} = \left(\sqrt{2}\right)^{19} e^{19 \times \frac{\pi}{4}i} = 2^{9} \sqrt{2} e^{4\pi i + \frac{3\pi}{4}i} = 2^{9} \sqrt{2} \left(\cos \frac{3\pi}{4} + \sin \frac{3\pi}{4} i\right) = 2^{9} \sqrt{2} \left(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = -2^{9} + 2^{9}i\]

(or \( = -512 + 512i \))
Problem 2 (10 points):

Let \( z = 3i \). Write out **ALL 5-th roots** of \( z \) in the exponential form.

\[
z = 3i = 3 e^{\frac{\pi i}{2}}.
\]

The 5-th roots are

\[
z_k = 3^\frac{1}{5} e^{\frac{1}{5}\left(2k\pi + \frac{\pi}{2}\right)i}, \quad k = 0, 1, 2, 3, 4
\]

- \( z_0 = 3^\frac{1}{5} e^{\frac{\pi i}{10}} \)
- \( z_1 = 3^\frac{1}{5} e^{\frac{5\pi i}{10}} = 3^\frac{1}{5} e^{\frac{\pi i}{2}} \)
- \( z_2 = 3^\frac{1}{5} e^{\frac{9\pi i}{10}} \)
- \( z_3 = 3^\frac{1}{5} e^{\frac{13\pi i}{10}} \)
- \( z_4 = 3^\frac{1}{5} e^{\frac{17\pi i}{10}} \)
Problem 3 (10 points):

\[
A = \begin{bmatrix}
2 & -4 & 1 & 2 \\
0 & 0 & -2 & 4 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Let \( A \).

\[\text{a. Solve } A \vec{x} = \vec{0} \text{ and write the solution set in parametric form.} \quad \text{(5 points)}\]

Matrix A is already in an echelon form. Its reduced echelon form is

\[
A \sim \begin{bmatrix}
1 & -2 & 0 & 2 \\
0 & 0 & 1 & -2 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\( x_1 \) and \( x_3 \) are basic variables. \( x_2 \) and \( x_4 \) are free variables.

The solution set in parametric form is

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = \begin{bmatrix}
2x_2 - 2x_4 \\
x_2 \\
2x_4 \\
x_4
\end{bmatrix} = x_2 \begin{bmatrix}
2 \\
1 \\
0 \\
0
\end{bmatrix} + x_4 \begin{bmatrix}
-2 \\
0 \\
2 \\
1
\end{bmatrix}
\]

\[\text{b. Is } A \vec{x} = \vec{b} \text{ consistent for every } \vec{b} \text{ in } \mathbb{R}^3 ? \text{ Why?} \quad \text{(5 points)}\]

No. Row 3 of matrix A does not have a pivot position.
Problem 4 (10 points):

Let \( A = \begin{bmatrix} -7 & 5 & 1 & -4 & 2 & 6 \\ 2 & 1 & 3 & 5 & -7 & -2 \\ 4 & 3 & -5 & 3 & 1 & -3 \\ 3 & -2 & 7 & -2 & 4 & 5 \end{bmatrix} \)

(a) Are the columns of matrix A linearly independent? Why? (5 points)

**No.** Matrix A has 6 columns and each column has 4 entries.

# of vectors in the set > # of entries of each vector.

\( \Rightarrow \) The vectors are linearly dependence

(b) Does \( A \bar{x} = \bar{0} \) have a non-trivial solution? Why? (5 points)

**Yes.** The columns of matrix A are linearly dependent.

\( \Rightarrow \) \( A \bar{x} = \bar{0} \) has a non-trivial solution.
Problem 5 (10 points):

Let 

\[
A = \begin{bmatrix}
2 & 9 & 5 & 1 \\
5 & -3 & 1 & 4 \\
7 & 1 & 3 & -2 \\
3 & 7 & 4 & 1 \\
4 & 2 & 7 & 3
\end{bmatrix}
\]

a. Do the columns of A span \( \mathbb{R}^5 \)? Why? \(\text{(5 points)}\)

\[
\text{No. Matrix A has 5 rows and only 4 columns.}
\]

\[
\Rightarrow \text{ At most, matrix A has 4 pivot positions.}
\]

\[
\Rightarrow \text{ Matrix A does not have a pivot position in every row.}
\]

\[
\Rightarrow \text{ The columns of A do not span } \mathbb{R}^5.
\]

b. Compute the \((1, 2)\) entry of \(A A^T\) \(\text{(2 points)}\)

\[
(A A^T)_{1,2} = 2 \times 5 + 9 \times (-3) + 5 \times 1 + 1 \times 4 = -8
\]

c. How many rows and how many columns does \(A^T (A A^T)\) have? \(\text{(3 points)}\)

A is \(5 \times 4\) \(\Rightarrow\) \(A^T\) is \(4 \times 5\)

\[
\Rightarrow \text{ AA}^T \text{ is } 5 \times 5 \Rightarrow A^T (A A^T) \text{ is } 4 \times 5\]
Problem 6 (10 points):

Let \( A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -2 \\ 2 & 1 & 1 \end{bmatrix} \)

a. Determine if matrix \( A \) is invertible. If invertible, find \( A^{-1} \). \(7 \text{ points}\)

\[
A \begin{bmatrix} 1 & -1 & 1 & | & 1 & 0 & 0 \\ -1 & 2 & -2 & | & 0 & 1 & 0 \\ 2 & 1 & 1 & | & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & | & 2 & 1 & 0 \\ 0 & 1 & 0 & | & -1.5 & -0.5 & 0.5 \\ 0 & 0 & 1 & | & -2.5 & -1.5 & 0.5 \end{bmatrix}
\]

Matrix \( A \) is invertible and

\[
A^{-1} = \begin{bmatrix} 2 & 1 & 0 \\ -1.5 & -0.5 & 0.5 \\ -2.5 & -1.5 & 0.5 \end{bmatrix}
\]

b. Does \( A\tilde{x} = \tilde{b} \) have a unique solution for every \( \tilde{b} \) in \( \mathbb{R}^3 \)? Why? \(3 \text{ points}\)

\[\text{Yes.} \quad \tilde{x} = A^{-1}\tilde{b}\]
Problem 7 (10 points): Let $A$ be an $n \times n$ matrix. Suppose $A^3$ is invertible.

Show that $A$ is invertible.

Based on the invertible matrix theorem, $A^3$ is invertible implies that there exists an $n \times n$ matrix $C$ such that $C \cdot A^3 = I$. We re-write it as

$$C \cdot (A^2 \cdot A) = I$$

$$\implies (C \cdot A^2) \cdot A = I$$

Using the invertible matrix theorem again, we conclude that matrix $A$ is invertible.
Problem 8 (10 points): Mark each statement TRUE or FALSE.

(2 points for each)

- If A and B are invertible, then A + B is invertible. **FALSE**

- If A and B are invertible, then \( A^{-1}B^{-1} \) is invertible. **TRUE**

- A square matrix with two identical rows cannot be invertible. **TRUE**

- A square matrix with two identical columns may be invertible. **FALSE**

- If matrix A is \( n \times n \) and columns of A are linearly independent, then the columns of A span \( \mathbb{R}^n \). **TRUE**