AMS 10/10A, Homework 7 Solutions

Problem 1. Suppose $A$ is $m \times n$. Prove the following equality

$$\dim \text{Col}(A) + \dim \text{Nul}(A^T) = m$$

Proof: Matrix $A^T$ is $n \times m$. By the Rank Theorem,

$$\rank A^T + \dim \text{Nul}(A^T) = m$$

Also, $\rank A^T = \rank A = \dim \text{Col}(A)$. Therefore, we conclude

$$\dim \text{Col}(A) + \dim \text{Nul}(A^T) = m$$

Problem 2. Suppose $A$ is $m \times n$ and $b$ is in $\mathbb{R}^m$. Prove that if the equation $Ax = b$ is consistent, then $\rank [A, b] = \rank A$.

Proof: If the equation $Ax = b$ is consistent, then $b$ is a linear combination of the columns of $A$. That is, $b$ is a linear combination $\{a_1, a_2, \ldots, a_n\}$ where $a_1, a_2, \ldots, a_n$ are columns of $A$. Therefore, we have

$$\text{Col} [A, b] = \text{span}\{a_1, a_2, \ldots, a_n, b\} = \text{span}\{a_1, a_2, \ldots, a_n\} = \text{Col} A$$

It follows that

$$\rank [A, b] = \dim \text{Col} [A, b] = \dim \text{Col} A = \rank A$$

Problem 3:

- Yes. $Ax = 0$ has non-trivial solutions. $\rank A = 5$ means that matrix $A$ has 5 pivot columns. $Ax = 0$ has 8 variables. So $Ax = 0$ has 5 basic variables and 3 free variables.

- No. $A^T x = 0$ does not have a non-trivial solution. $\rank A = 5$ means that $\rank A^T = 5$ and matrix $A^T$ has 5 pivot columns. $A^T x = 0$ has 5 variables. So $A^T x = 0$ has 5 basic variables and no free variable.

Problem 4: \[
\det(A) = 0, \quad \det(B) = 107, \quad \det(C) = -48, \quad \det(D) = 48.\]
Problem 5:

\[
\begin{vmatrix}
    a & 0 & d & c \\
    b & 0 & -c & d \\
    0 & c & -b & a \\
    0 & d & a & b \\
\end{vmatrix}
= a \cdot \det \begin{vmatrix}
    0 & -c & d \\
    c & -b & a \\
    d & a & b \\
\end{vmatrix}
- b \cdot \det \begin{vmatrix}
    0 & d & c \\
    c & -b & a \\
    d & a & b \\
\end{vmatrix}
\]

\[
= a \left( -c \cdot \det \begin{vmatrix}
    -c & d \\
    a & b \\
\end{vmatrix} + d \cdot \begin{vmatrix}
    -c & d \\
    -b & a \\
\end{vmatrix} \right)
- b \left( -c \cdot \det \begin{vmatrix}
    d & c \\
    a & b \\
\end{vmatrix} + d \cdot \begin{vmatrix}
    d & c \\
    -b & a \\
\end{vmatrix} \right)
\]

\[
= a \left( -c(-cb - ad) + d(-ca + bd) \right)
- b \left( -c(db - ac) + d(da + bc) \right)
\]

\[
= ac^2b + a^2cd - a^2cd + abd^2 + b^2cd - ac^2b - abd^2 - b^2cd
\]

\[
= 0
\]

Problem 6:

\[
\begin{vmatrix}
    a & \sqrt{2} & 0 \\
    \sqrt{2} & a & \sqrt{2} \\
    0 & \sqrt{2} & a \\
\end{vmatrix}
= a \cdot \det \begin{vmatrix}
    a & \sqrt{2} \\
    \sqrt{2} & a \\
\end{vmatrix}
- \sqrt{2} \cdot \det \begin{vmatrix}
    \sqrt{2} & 0 \\
    \sqrt{2} & a \\
\end{vmatrix}
\]

\[
= a(a^2 - 2) - \sqrt{2}(\sqrt{2}a)
\]

\[
= a(a^2 - 4)
\]

Therefore, the determinant of this matrix is zero if \(a = 0\), \(a = 2\) or \(a = -2\).

Problem 7:

\[
\text{det}(2A) = 2^4 \text{det}(A) = 48
\]

\[
\text{det}(A^2) = (\text{det}(A))^2 = 27
\]

\[
\text{det}(A^{-1}) = (\text{det}(A))^{-1} = \frac{1}{3}
\]

\[
\text{det}(A^2B^3) = (\text{det}(A))^2(\text{det}(B))^3 = 3^2(-2)^3 = -72
\]

\[
\text{det}(A^2B^{-2}) = (\text{det}(A))^3(\text{det}(B))^{-2} = \frac{27}{4}
\]

Problem 8. Prove that \(\text{det}(AA^T)\) is nonnegative for any \(n \times n\) matrix \(A\).

Proof: \(\text{det}(AA^T) = \text{det}(A)\text{det}(A^T) = \text{det}(A)\text{det}(A) = (\text{det}(A))^2 \geq 0\).
Problem 9. Let $A$ be an $n \times n$ matrix and let $P$ be an $n \times n$ invertible matrix. Prove that $\det(P^{-1}AP) = \det(A)$.

Proof: $\det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) = \det(A)(\det(P))^{-1}\det(P) = \det(A)$.

Problem 10. Let $A$ be a $n \times n$ matrix such that $A^T = -A$. Prove that $A$ is not invertible if $n$ is odd.

Proof: $\det(A) = \det(A^T) = \det(-A) = (-1)^n\det(A)$. If $n$ is odd, $(-1)^n = -1$. We have, $\det(A) = -\det(A)$, which leads to $\det(A) = 0$. Therefore, $A$ is not invertible.

Problem 11. Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Show that i) $A^T = -A$ and ii) $A$ is invertible.

Does this result contradict the conclusion in Problem 10 above?

Proof:

$$A^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -A$$

$$\det A = 0 - (-1) = 1 \neq 0$$

So matrix $A$ is invertible.

This does not contradict the conclusion in Problem 10. In Problem 10, $n$ is odd and here $n = 2$ is even.

Problem 12. We use elementary row operations to calculate these determinants.

$$\det \begin{bmatrix} a & b & c \\ d + 2a & e + 2b & f + 2c \\ g & h & i \end{bmatrix} = \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 5$$

$$\det \begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix} = -\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 5$$

$$\det \left( 3 \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \right) = 3^3 \cdot \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 3^3 \cdot 5 = 135$$

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ a & b & c \end{bmatrix} = \det \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 0 \end{bmatrix} = 0$$