AMS 10/10A, Homework 5 Solutions

Problem 1:

\[
A^{-1} = \begin{bmatrix}
-\frac{1}{n} & -\frac{3}{n} \\
\frac{3}{n} & -\frac{2}{n}
\end{bmatrix}
\]

\[
B^{-1} = \begin{bmatrix}
1 & -1 & -0.5 \\
2 & -1 & -1.5 \\
-0.5 & 0.5 & 0.5
\end{bmatrix}
\]

\[C : \text{not invertible}\]

Problem 2: When \(k = 0\), or \(k = 4\) or \(k = -4\) the matrix is not invertible.

Problem 3: \(X = CB - A\).

Problem 4:

\[
D^{-1} = \begin{bmatrix}
1/d_{11} & 0 & \cdots & 0 & 0 \\
0 & 1/d_{22} & \cdots & 0 & 0 \\
0 & 0 & \ddots & 1/d_{n-1,n-1} & 0 \\
0 & 0 & \cdots & 0 & 1/d_{nn}
\end{bmatrix}
\]

Problem 5: The second column of \(A^{-1}\) is given by the solution of the equation

\[
Ax = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}
\]

Solving this equation we get the second column of \(A^{-1}\), which is

\[
\begin{bmatrix}
0 \\
0 \\
0.5
\end{bmatrix}
\]

Problem 6: By The Invertible Matrix Theorem, \(A\) is invertible implies that \(A^T\) is invertible, which implies that the columns of \(A^T\) are linearly independent.

Problem 7: Since \(AB\) is invertible, there is a square matrix \(C\) such that \(C(AB) = I\). We re-write it as \((CA)B = I\), which implies that matrix \(B\) is invertible.
Similarly, since $AB$ is invertible, there is a square matrix $D$ such that $(AB)D = I$. We re-write it as $A(BD) = I$, which implies that matrix $A$ is invertible.

**Problem 8:** Since columns of $A_{n \times n}$ are linearly independent, matrix $A$ is invertible. Therefore, $A^2 = AA$ is invertible. By The Invertible Matrix Theorem, the columns of $A^2$ are linearly independent.

**Problem 9:** $A^{-1} = (A^T)^{-1} = (A^{-1})^T$. Therefore, $A^{-1}$ is symmetric.

**Problem 10:** Since $A$, $B$ and $(A + B)^{-1}$ are all invertible, matrix $(A(A + B)^{-1}B)$ is invertible and

$$
(A(A + B)^{-1}B)^{-1} = B^{-1}((A + B)^{-1})^{-1}A^{-1} = B^{-1}(A + B)A^{-1} = (B^{-1}A + I)A^{-1} = B^{-1} + A^{-1}
$$

Therefore, matrix $B^{-1} + A^{-1}$ is invertible and

$$(B^{-1} + A^{-1})^{-1} = A(A + B)^{-1}B$$

**Problem 11:** Mark each statement True or False

11.1. If $A$ and $B$ are invertible, then $A + B$ is invertible. $\text{F}$

11.2. If $A$ is $n \times n$ and not invertible, then the linear system $Ax = b$ is inconsistent. $\text{F}$

11.3. If $(A - I)$ is invertible, then the linear system $Ax = x$ has a nonzero solution for $x$. $\text{F}$

11.4. If a square matrix has nonzero entries on the diagonal, then $A$ is invertible. $\text{F}$

11.5. If $A$ is $n \times n$, and the columns of $A$ are linearly independent, then the columns of $A$ span $\mathbb{R}^n$. $\text{T}$

**Problem 12:** Mark each statement True or False

12.1. Let $A$ be a square matrix. If the equation $Ax = 0$ has a nontrivial solution, then $A$ is not invertible. $\text{T}$

12.2. A square matrix with two identical rows cannot be invertible. $\text{T}$

12.3. A square matrix with two identical columns cannot be invertible. $\text{T}$

12.4. A product of invertible matrices is invertible. $\text{T}$

12.5. If $A$ and $B$ are $n \times n$ invertible matrices, then $A^{-1}B^{-1}$ is the inverse of $AB$. $\text{F}$