AMS 10/10A, Homework 4 Solutions

Problem 1:
\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
\end{bmatrix}
= x_2 \begin{bmatrix}
  -2 \\
  1 \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
\end{bmatrix}
= x_3 \begin{bmatrix}
  1 \\
  -1 \\
  1 \\
  1 \\
\end{bmatrix}
+ x_4 \begin{bmatrix}
  -1 \\
  1 \\
  0 \\
  1 \\
\end{bmatrix}
\]

Problem 2: \( A(cv) = c(Av) = 0 \).

Problem 3: Since \( v_n \) is a linear combination of \( \{ v_1, \cdots, v_{n-1} \} \), there exist scalars \( c_1, c_2, \cdots, c_{n-1} \) such that
\[

v_n^\prime = c_1 v_1 + c_2 v_2 + \cdots + c_{n-1} v_{n-1}

\implies c_1 v_1 + \cdots + c_{n-1} v_{n-1} - v_n^\prime = 0

\implies \begin{bmatrix}
  v_1 & v_2 & \cdots & v_{n-1} & v_n
\end{bmatrix}
\begin{bmatrix}
  c_1 \\
  c_2 \\
  \vdots \\
  c_{n-1} \\
  1
\end{bmatrix}
= 0
\]

Therefore homogeneous equation \( Ax = 0 \) has a nontrivial solution which is \( \begin{bmatrix}
  c_1 \\
  c_2 \\
  \vdots \\
  c_{n-1} \\
  -1
\end{bmatrix} \). It implies that \( Ax = 0 \) has infinitely many solutions.

Problem 4: Yes, since \( A \) has a pivot position in every row.

Problem 5:
5.1. The equation \( Ax = b \) is homogeneous if the zero vector is a solution. \( \text{T} \)

5.2. The homogeneous equation \( Ax = 0 \) has the trivial solution if and only if the equation has at least one free variable. \( \text{F} \)
5.3. A homogeneous system of equations can be inconsistent. F

5.4. If \( v \) is a nontrivial solution of \( Ax = 0 \), then every entry in \( v \) is nonzero. F

5.5. If homogeneous equation \( Ax = 0 \) has a unique solution, then \( Ax = b \) cannot have infinitely many solutions. T

**Problem 6:**

\[
\begin{bmatrix}
2 & 6 & 1 & 5 \\
1 & 3 & 0 & 2 \\
3 & 9 & 0 & 6 \\
1 & 3 & 1 & 3
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 3 & 0 & 2 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

(2)

The set is linearly dependent.

**Problem 7:**

- Consider vector equation \( c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \). It is equivalent to matrix equation

\[
\begin{bmatrix}
2 & -1 & 8 \\
-3 & 4 & -2
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix}
= 0
\]

This homogeneous equation has infinitely many solutions given by

\[
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix} = c_3
\begin{bmatrix}
-6 \\
-4 \\
1
\end{bmatrix}
\]

Therefore, we have

\[-6v_1 - 4v_2 + v_3 = 0\]

That is, \( \{v_1, v_2, v_3\} \) is linearly dependent.

- \( v_2 = -\frac{3}{2}v_1 + \frac{1}{3}v_3 \)

**Problem 8:**

\[
\begin{bmatrix}
1 & -5 & 3 \\
3 & -8 & -5 \\
-1 & 2 & k
\end{bmatrix}
\sim
\begin{bmatrix}
1 & -5 & 3 \\
0 & 1 & -2 \\
0 & 0 & k - 3
\end{bmatrix}
\]

(3)

When \( k = 3 \) the columns of the matrix are linearly dependent.

**Problem 9:** Consider vector equation

\[c_1(v_1 + v_3) + c_2(v_1 - 2v_2) + c_3(-4v_1 + v_2 + 3v_3) = 0\]
which can be rewritten as
\[
(c_1 + c_2 - 4c_3)v_1 + (-2c_2 + c_3)v_2 + (c_1 + 3c_3)v_3 = 0
\]
Since \(\{v_1, v_2, v_3\}\) is linearly independent, \(\{c_1, c_2, c_3\}\) must satisfy
\[
\begin{align*}
    c_1 + c_2 - 4c_3 &= 0 \\
    -2c_2 + c_3 &= 0 \\
    c_1 + 3c_3 &= 0
\end{align*}
\]  
\[
(4)
\]
We do row reduction on the coefficient matrix.
\[
\begin{bmatrix}
1 & 1 & -4 \\
0 & -2 & 1 \\
1 & 0 & 3
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 1 & -4 \\
0 & -2 & 1 \\
0 & 0 & 13/2
\end{bmatrix}
\]
\[
(5)
\]
Every column is a pivot column. There is no free variable. Equation (4) has only the trivial solution. Therefore, by definition, \(\{v_1 + v_3, v_1 - 2v_2, -4v_1 + v_2 + 3v_3\}\) is also linearly independent.

**Problem 10:** Mark each statement True or False

10.1. The set \(\{0, v_1, v_2, \ldots, v_k\}\) is always linearly dependent. **T**

10.2. Let \(v_1, v_2, v_3\) and \(v_4\) be vectors in \(\mathbb{R}^n\) such that \(v_1 - v_2 = v_3 - v_4\). Then the set \(\{v_1, v_2, v_3, v_4\}\) is linearly dependent. **T**

10.3. If \(u\) and \(v\) are linearly independent, and if \(w\) is in \(\text{span}\{u, v\}\), then \(\{u, v, w\}\) is linearly dependent. **T**

10.4. If a set in \(\mathbb{R}^n\) is linearly dependent, then the set contains more than \(n\) vectors. **F**

10.5. If \(\{v_1, v_2, v_3, v_4\}\) is a set of vectors in \(\mathbb{R}^4\) and \(\{v_1, v_2, v_3\}\) is linearly dependent, then \(\{v_1, v_2, v_3, v_4\}\) is also linearly dependent. **T**

**Problem 11:**

\[
AB = \begin{bmatrix}
-3 & 9 \\
11 & -5 \\
-23 & 6
\end{bmatrix}, \quad A^T B = \begin{bmatrix}
3 & 5 \\
36 & -3 \\
-14 & 0
\end{bmatrix}
\]

\[
BC = \begin{bmatrix}
-1 & 4 & -2 & 5 \\
-9 & 1 & 3 & -4 \\
7 & 7 & -7 & 14
\end{bmatrix}, \quad CD = \begin{bmatrix}
-1 & 5 \\
-1 & 2
\end{bmatrix}, \quad (CD)^2 = \begin{bmatrix}
-4 & 5 \\
-1 & -1
\end{bmatrix}
\]

**Problem 12:** B has 6 rows.
Problem 13: The second column of $AB$ is a zero column.

Problem 14: When $k = -2$, $AB = BA$.

Problem 15: Let $b_1, b_2, \ldots, b_n$ be the columns of $B$. Since $\{b_1, \ldots, b_n\}$ are linearly dependent, there exist $c_1, c_2, \ldots, c_n$, not all zero, such that
\[
c_1 b_1 + c_2 b_2 + \cdots + c_n b_n = 0
\]

\[
\Rightarrow A(c_1 b_1 + c_2 b_2 + \cdots + c_n b_n) = 0
\]

\[
\Rightarrow c_1 (Ab_1) + c_2 (Ab_2) + \cdots + c_n (Ab_n) = 0
\]

Therefore, $\{Ab_1, Ab_2, \ldots, Ab_n\}$ are linearly dependent.

Problem 16:
\[
w_{2,1} = \begin{bmatrix} 2 & 1 & -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 & 5 \\ 3 & -4 \\ 2 & 2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix}
\]
\[
= \begin{bmatrix} 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix}
\]
\[
= 6
\]

Problem 17:
\[
A^2 - 3A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}^2 - \begin{bmatrix} 6 & 3 \\ -3 & 9 \end{bmatrix}
\]
\[
= \begin{bmatrix} 3 & 5 \\ -5 & 8 \end{bmatrix} - \begin{bmatrix} 6 & 3 \\ -3 & 9 \end{bmatrix}
\]
\[
= \begin{bmatrix} -3 & 2 \\ -2 & -1 \end{bmatrix}
\]

Problem 18:
\[
B^3 = B^2 \cdot B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\]
\[
= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]
Therefore, for all $n \geq 3$, $B^n = B^{n-3}B^3 = B^{n-3} \cdot 0 = 0$. 