AMS 10/10A, Homework 10 Solutions

Problem 1.

\[
\hat{y} = y^T u_1 u_1 + y^T u_2 u_2 = \frac{1}{3} u_1 + \frac{5}{7} u_2
\]

\[
= \begin{bmatrix}
\frac{-8}{21} \\
\frac{52}{21} \\
\frac{-23}{21}
\end{bmatrix}
\]

and

\[
z = y - \hat{y}
\]

\[
= \begin{bmatrix}
2 \\
2 \\
-3
\end{bmatrix} - \begin{bmatrix}
\frac{-8}{21} \\
\frac{52}{21} \\
\frac{-23}{21}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\frac{50}{21} \\
\frac{-10}{21} \\
\frac{-40}{21}
\end{bmatrix}
\]

They satisfy that \( y = \hat{y} + z \), where \( \hat{y} \) is a vector in \( H \) and \( z \) is a vector in \( H^\perp \).

Problem 2. By Best Approximation Theorem the closest point in \( \text{span}\{v_1, v_2\} \) to \( y \) is given by the projection of \( y \) onto \( \text{span}\{v_1, v_2\} \). Since \( v_1 \) and \( v_2 \) are orthogonal, this projection can be computed as

\[
y^T v_1 v_1 + y^T v_2 v_2 = \frac{1}{5} v_1 + \frac{1}{13} v_2 = \begin{bmatrix}
\frac{-7}{65} \\
\frac{-21}{65} \\
\frac{-1}{5} \\
\frac{41}{65}
\end{bmatrix}
\]

Problem 3-7.

- By applying elementary row operations on the augmented matrix \([A \mid b]\), we have

\[
[A \mid b] \sim \begin{bmatrix}
1 & 5 & 1 \\
0 & -14 & -3 \\
0 & 0 & 4
\end{bmatrix}
\]

Since the last column is a pivot column, the equation \( Ax = b \) is inconsistent.
• Since the columns of $A$ are an orthogonal set of non-zero vectors, they are a linearly independent set. Consequently, they form an orthogonal basis for $\text{Col}(A)$.

• The column of $A$ are an orthogonal basis for $\text{Col}(A)$. Hence, projection of $b$ onto $\text{Col}(A)$ is given by

\[
\begin{align*}
\hat{b} &= \frac{b^T a_1}{a_1^T a_1} a_1 + \frac{b^T a_2}{a_2^T a_2} a_2 \\
&= \frac{1}{2} a_1 - \frac{1}{6} a_2 \\
&= \begin{bmatrix}
-1/3 \\
4/3 \\
5/3
\end{bmatrix}
\end{align*}
\]

• The least square solution, $\hat{x}$, of $Ax = b$ is given by

\[
\hat{x} = (A^T A)^{-1} A^T b
\]

\[
\begin{bmatrix}
14 & 0 \\
0 & 42
\end{bmatrix}^{-1} \begin{bmatrix}
7 \\
-7
\end{bmatrix} = \begin{bmatrix}
1/2 \\
-1/6
\end{bmatrix}
\]

\[
A\hat{x} = \begin{bmatrix}
-1/3 \\
4/3 \\
5/3
\end{bmatrix} = \hat{b}.
\]

Problem 8-9. Let $A$ be an $m \times n$ matrix. Use the steps below to show that a vector $x$ in $\mathbb{R}^n$ satisfies $Ax = 0$ if and only if $A^T Ax = 0$.

• Show that if $Ax = 0$, then $A^T Ax = 0$.

Proof: Let $x$ be a vector such that $Ax = 0$. Multipling $A^T$ on both sides of the equation leads to $A^T Ax = A^T 0 = 0$.

• Suppose $A^T Ax = 0$. Show that $x^T A^T Ax = 0$, and use this to prove $Ax = 0$.

Proof: Let $x$ be a vector such that $A^T Ax = 0$. Multipling $x^T$ on both sides of the equation leads to $x^T A^T Ax = x^T 0 = 0$. Therefore, $x^T A^T Ax = (Ax)^T (Ax) = \|Ax\|^2 = 0$. Since the norm of a vector equals to zero if and only if the vector itself is the zero vector, we have $Ax = 0$.

Problem 10-11. Let $A$ be an $m \times n$ matrix. Problem 8-9 implies that $\text{Nul}(A) = \text{Nul}(A^T A)$. Use this result to prove that
• \( \text{rank}(A) = \text{rank}(A^T A) \).

Proof: Matrix \( A \) is \( m \times n \) and matrix \( A^T A \) is \( n \times n \). By the Rank Theorem, we have

\[
\text{rank}(A) + \text{dim}(\text{Nul} \; A) = n \\
\text{rank}(A^T A) + \text{dim}(\text{Nul} \; A^T A) = n
\]

Hence,

\[
\text{rank}(A) = n - \text{dim}(\text{Nul} \; A) \\
= n - \text{dim}(\text{Nul}(A^T A)) \quad (\text{since } \text{Nul}(A) = \text{Nul}(A^T A)) \\
= \text{rank}(A^T A) \quad (\text{By The Rank Theorem})
\]

• If \( \text{rank}(A) = n \), then \( A^T A \) is invertible.

Proof: If \( \text{rank}(A) = n \), from the result in Problem 10, \( \text{rank}(A^T A) = \text{rank}(A) = n \). Since matrix \( A^T A \) is a square matrix of \( n \times n \), by Invertible Matrix Theorem, \( \text{rank}(A^T A) = n \) implies that \( A^T A \) is invertible.