# Transactions Letters

## A New Adaptive Two-Stage Maximum-Likelihood Decoding Algorithm for Linear Block Codes

Xianren Wu, *Student Member*, Hamid R. Sadjadpour, *Senior Member, IEEE*, and Zhi Tian, *Member, IEEE*

***Abstract*—In this paper, we propose a new two-stage (TS) structure for computationally efficient maximum-likelihood decoding (MLD) of linear block codes. With this structure, near optimal MLD performance can be achieved at low complexity through TS processing. The first stage of processing estimates a minimum sufficient set (MSS) of candidate codewords that contains the optimal codeword, while the second stage performs optimal or suboptimal decoding search within the estimated MSS of small size. Based on the new structure, we propose a decoding algorithm that systematically trades off between the decoding complexity and the bounded block error rate performance. A low-complexity complementary decoding algorithm is developed to estimate the MSS, followed by an ordered algebraic decoding (OAD) algorithm to achieve flexible system design. Since the size of the MSS changes with the signal-to-noise ratio, the overall decoding complexity adaptively scales with the quality of the communication link. Theoretical analysis is provided to evaluate the potential complexity reduction enabled by the proposed decoding structure.**

***Index Terms*—Adaptive decoding complementary decoding, maximum-likelihood decoding, ordered algebraic decoding.**

### I. Introduction

OPTIMAL decoding of linear block codes has been proven to be an NP-hard problem [1], whose complexity grows exponentially as the code length increases. Many research efforts have been attempted to develop optimal or suboptimal decoding algorithms with moderate decoding complexity [2]–[11]. In general, these algorithms can be classified into three major categories. The first category [2]–[4] utilizes algebraic decoders to perform list decoding. The tradeoff between decoding complexity and achievable performance is reflected in the size of the list. The second category of algorithms [7]–[9] converts the optimal decoding problem to a graph problem and regards the optimal decoding problem as finding the shortest path in a graph, using existing graph-based search algorithms. The third category of algorithms [5], [6] relies on the ordered statistics of the received data to attain good decoding performance without resorting to algebraic decoding. Ordered statistics decoding (OSD) algorithms generate codewords from the most reliable bits utilizing transformed encoding matrix. However, the decoding complexity of OSD grows exponentially in the order selected; thus it may still suffer from high decoding complexity as higher order decoding is favored over lower order decoding for performance consideration. Moreover, the complexity of OSD heavily depends on the information length and grows quickly with the increase of the code rate for a fixed block length, given a specific processing order.

There are several strategies to reduce the computational load of OSD algorithms. Complementary decoding [12] combines list decoding with OSD to achieve comparable performance with less complexity. Optimality test criterions (OTC) have been intensively investigated for the purpose of early termination or ruling out unnecessary iterations during decoding [10], [13]–[15]. Heuristic OTC-based search algorithms [10], [11] are proposed, in which optimal decoding is carried out adaptively to converge to the optimal codeword. Unfortunately, the convergence speed can be slow especially at low to moderate signal-to-noise ratio (SNR) regions, thus incurring high decoding complexity,

In this paper, we propose a new two-stage (TS) decoding structure that improves the convergence speed of optimal decoding, which in turn reduces the overall decoding complexity substantially. Based on this new structure, we develop a TS processing procedure to achieve optimal maximum-likelihood decoding (MLD) performance with high computational efficiency. The first stage of the processing aims at reducing the overall complexity by estimating a minimum sufficient set (MSS) of test codes. Albeit small in size, the MSS contains the optimal codeword, thus enabling ensuing optimal decoding at reduced complexity. Within the estimated MSS, constrained optimal or suboptimal decoding is then performed during the second stage of the processing to achieve the desired decoding performance. Specifically, we employ a low-complexity complementary decoding algorithm to estimate the MSS in the first stage of the processing and adopt an innovative ordered algebraic decoding (OAD) algorithm in the second stage of the processing to achieve flexible system design. Moreover, OTC is incorporated into both stages to further reduce the decoding complexity. The proposed decoding algorithm is able to provide a system tradeoff design between performance and complexity. The key distinction of our approach from OSD lies in the MSS we select. Our estimated MSS is the smallest possible codeword subset needed to retain optimality of ensuing decoding,

Paper approved by A. H. Banihashemi, the Editor for Coding and Communication Theory. Manuscript received October 18, 2003; revised August 11, 2004, November 17, 2004, and December 6, 2004. Parts of the work have been presented at the Asilomar'03 and ICC'04 conferences.

X. Wu and Z. Tian are with the Department of Electrical and Computer Engineering, Michigan Technological University, Houghton, MI 49931 USA (e-mail: xwu@mtu.edu; ztian@mtu.edu).

H. R. Sadjadpour is with the Department of Electrical Engineering, University of California, Santa Cruz, CA 95064 USA (e-mail: hamid@soe.ucsc.edu).

Digital Object Identifier 10.1109/TCOMM.2005.849790

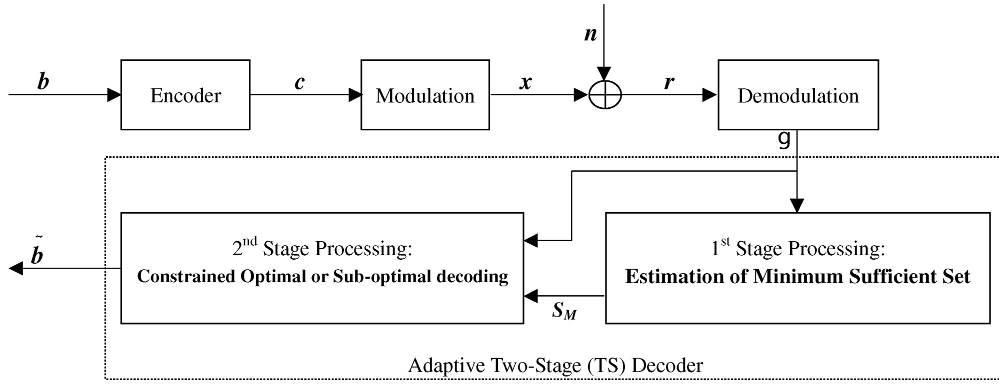


Fig. 1. System architecture.

which is naturally dictated by the receive SNR. In contrast, OSD relies on an *ad hoc* order number to trade performance for lower complexity and still incurs high complexity to achieve the optimal decoding performance.

The rest of this paper is organized as follows. Section II describes the new TS decoding structure and evaluates its capability in complexity reduction. Section III is dedicated to detailed description of the proposed TS ML decoding algorithm. Simulation results are presented in Section IV, followed by concluding remarks in Section V.

## II. TS Decoding Structure

### A. System Structure

Fig. 1 illustrates the overall architecture of a block-coded communication system employing our new adaptive TS decoder. A length-$K$ source information vector $\mathbf{b} = (b_1, \ldots, b_K)$ is first encoded into a length-$N$ codeword $\mathbf{c} = (c_1, \ldots, c_N)$, which is modulated as $\mathbf{x} = (x_1, \ldots, x_N)$ and then transmitted over an AWGN channel. The receiver signal vector $\mathbf{r} = (r_1, \ldots, r_N)$ is the sum of the signal vector $\mathbf{x}$ and the noise vector $\mathbf{n} = (n_1, \ldots, n_N)$, where each element of $\mathbf{n}$ is a Gaussian random variable with zero mean and variance $\sigma^2$, i.e., $\mathcal{N}(0, \sigma^2)$. Consider BPSK modulation. The demodulator calculates the log-likelihood vector $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_N)$ of $\mathbf{r}$ as

$$\gamma_i = \log \frac{\Pr(r_i \mid c_i = 1)}{\Pr(r_i \mid c_i = 0)} = \frac{2}{\sigma^2} \cdot r_i, \quad i = 1, \ldots, N. \qquad (1)$$

The vector $\mathbf{y} = (y_1, \ldots, y_N)$ is obtained by hard-decision rule. The corresponding reliability vector $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_N)$ is evaluated as $\beta_i = |\gamma_i|$, $i = 1, \ldots, N$. Based on their reliability values, the decisions can be reordered as $\beta_{\alpha_1} \leq \cdots \leq \beta_{\alpha_N}$. The vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N)$ records the positions of the decisions from the least reliable to the most reliable bits.

### B. Adaptive TS Decoder

In decoding $\mathbf{b}$, a conventional optimal decoder needs to search through all the codewords $\mathbf{c}$, thus incurring insurmountable computational complexity. Our adaptive TS decoder structure proposed in Fig. 1 is motivated by the need for optimal decoding at affordable low complexity. The first stage aims at reducing the search complexity by identifying a minimum set of candidate codewords that contains the optimal one, while the second stage performs optimal decoding within the constrained codeword subset. In addition, OTCs [10], [13]–[15] can be embedded into the TS decoding procedure to further reduce the overall complexity. Adhering to this structure, the task of each stage can be fulfilled by various (existing) techniques respectively. For example, Chase-II [3] or Kaneko [10] algorithm can be used in the second stage subject to slight modifications. We will present new solutions to these two stages of processing in Section III.

To elaborate on the low-complexity feature of our optimal TS decoder, we now introduce the concept of MSS that is key to the first stage of processing. Let $E_m$ be the test set consisting of all length-$N$ binary vectors generated by applying all possible error patterns in the first $m$ least reliable positions $\{\alpha_1, \ldots, \alpha_m\}$ to the binary hard-decision vector $\mathbf{y}$. Using $E_m$ as the input, an algebraic decoder can generate a set of codewords, which we denote as $S_m$. Clearly, $E_1 \subseteq \cdots \subseteq E_N$ and $S_1 \subseteq \cdots \subseteq S_N$. Note that $S_m$ becomes the set of all codewords when $m \geq N - \tau$, where $\tau$ is the error correction capability of the algebraic decoder. At certain value of $m$, say $M$, $S_M$ contains the optimal codeword while $S_{M-1}$ does not. For any $m \geq M$, the set $S_m$ contains the optimal codeword and, hence, is sufficient for optimal decoding. We henceforth term such a set $S_m$ a sufficient set (SS) and $m$ the order of this SS. Among all SS, $S_M$ has the smallest size and is termed the minimum sufficient set (MSS). One important property of the MSS can be summarized as follows.

*Proposition 1:* The MSS is the set $S_M$ in which one error happens at position $\alpha_M$ and there are exactly $\tau$ errors outside of the first $M$ least reliable positions.

*Proof:* For any $k < M$, there will be more than $(\tau + 1)$ errors outside of the $k$ least positions. Since an algebraic decoder with error correction capability of $\tau$ can not correct all these errors, $S_k$ does not contain the optimal codeword, thus not being a SS. On the other hand, when $k \geq M$, there always exists a test code in the set $E_k$ that can be decoded into the optimal codeword. Therefore, $S_k$ contains the optimal codeword and is a sufficient set. ∎

### C. Complexity of Adaptive TS Decoding

The computational complexity of the TS decoder depends critically on the size of the MSS and its associated order $M$. Treating $M$ as a random variable subject to the noise effect, we denote $\Phi(M)$ as the probability of $S_M$ being the MSS for a given $M$, and $\Phi(M;e)$ as the joint probability associated with the events when $S_M$ is the MSS and there are exactly a total number of $e$ errors. The overall decoding complexity can be calibrated by the mean value of $M$, which we denote as $\bar{M}$. It follows immediately that $\bar{M} = \sum_{M=1}^{N} M \cdot \Phi(M)$ and $\Phi(M) = \sum_{e=0}^{N} \Phi(M;e)$.

When $e \leq \tau$, the hard-decision vector $\mathbf{y}$ can be directly mapped to the optimal codeword, resulting in $M = 0$. For $M \neq 0$, $\Phi(M)$ reduces to

$$\Phi(M) = \sum_{e=\tau+1}^{N} \Phi(M;e). \tag{2}$$

To evaluate $\Phi(M;e)$ for $e \geq \tau + 1$, we deduce from Proposition 1 that

$$\Phi(M;e) = \sum_{j_1=1}^{N-M-(\tau+1)} \cdots \sum_{j_\tau=j_{\tau-1}+1}^{N-M} P_e(M, M+j_1, \ldots, M+j_\tau; N) \tag{3}$$

where $P_e(M, M+j_1, \ldots, M+j_\tau; N)$ is the joint probability when $(\tau + 1)$ errors are located at positions $\{M, M+j_1, \ldots, M+j_\tau\}$ respectively in a length-$N$ ordered sequence. Each summand in (3) can be approximated as [16]

$$P_e(n_1, n_2, \ldots, n_j; N) \cong \prod_{l=1}^{j-1} \frac{N}{N - n_l} P_e(n_l; N) P_e(n_j; N) \tag{4}$$

where $P_e(i;N)$ is the error probability of having an error at location $i$ in an length-$N$ ordered sequence. It can be shown that

$$P_e(i;N) \approx \exp\left\{4 \times \frac{(1-m_i)}{N_0}\right\} \tag{5}$$

where $N_0 = 2\sigma^2$ is the one-sided noise power density, $m_i = f^{-1}(i/N)$ with $f(x) = Q(2-x) - Q(x)$, and $Q(x) = (\pi N_0)^{-1/2} \int_x^\infty \exp\{-y^2/N_0\} dy$ is the complementary error function.

Equations (2)–(5) provide the steps to numerically compute the decoding complexity measure $\bar{M}$. Simulation results in Section IV will show that $\bar{M}$ takes a fairly small value compared with the code-length $N$. As a result, the overall decoding complexity can be greatly reduced.

## III. TS ML Decoding Algorithm

Based on the TS structure, we propose a new TS ML decoding algorithm that is capable of not only reaching the optimal performance at low complexity, but also achieving bounded block error performance with bounded decoding complexity. The TS ML also consists of two stages: a low-complexity complementary decoding algorithm for estimating the MSS, and an innovative OAD algorithm for implementing constrained decoding.

### A. Stage 1: Estimation of the MSS

The first processing stage attempts to identify the MSS or obtain a SS that is as close to the MSS as possible. This goal should be accomplished at low complexity to avoid excessive overhead. Our strategy here is to apply Lemma 2 of [10] to efficiently estimate a SS from any available codeword. The closer the initial codeword is to the optimal one, the closer the SS estimate is to the MSS. The problem of estimating the MSS is thus reduced to finding a good codeword close to the optimal codeword. To this end, we propose a low-complexity complementary decoding algorithm by combining the Chase-III [3] and OSD-1 [5] algorithms. In contrast, the existing complementary decoding algorithm [12] combines the Chase-II [3] and OSD algorithms, which still incurs high complexity when close-to-optimal decoding performance is desired [12]. In our algorithm, we aims at simply finding a good codeword rather than the optimal codeword. With the relaxed performance requirement, our complementary decoding algorithm can afford to have lower complexity than that in [12], yet being capable of generating a good codeword sufficient for estimating the MSS $S_M$ via [10]. The estimated $S_M$, its set order $M$ and the corresponding error set $E_M$ are passed to the second stage for further decoding processing.

### B. Stage 2: Ordered Algebraic Decoding

The objective of the second processing stage is to search for the optimal codeword within the small set $S_M$. This problem belongs to constrained optimal decoding, for which we propose an ordered algebraic decoding (OAD) algorithm.

We note that the test set $E_M$ is generated from all possible error patterns in the first $M$ least reliable positions. Let $T_i$ represent the set comprised of all possible test codes in $E_M$ that are generated from the hard-decision vector $\mathbf{y}$ based on all possible error patterns that have exactly $i$ errors in the first $M$ least reliable positions $\{\alpha_1, \ldots, \alpha_M\}$. It is clear that $T_1 + \cdots + T_M = E_M$. Each test code can be decoded into a codeword and all test codes in $T_i$ yield a set of codewords. Among these codewords, the codeword that has the minimum Euclidean distance to $\mathbf{r}$ is chosen as the output, a procedure we term as order-$i$ algebraic processing (OAP-$i$).

Our order-$i$ algebraic decoding (OAD-$i$) algorithm consists of the following steps. In an ascending order, OAP-0 to OAP-$i$ are carried out successively. Among all the decoded codewords, the decoder selects its output to be the codeword with the minimum Euclidean distance to $\mathbf{r}$. Meanwhile, OTC [10] is embedded into the decoder to terminate decoding whenever the optimal codeword has been found. The estimated $S_M$ is also dynamically updated whenever a better estimate is obtained.

Obviously, OAD-$M$ achieves the optimal decoding performance since $S_M$ is a SS. The following proposition summarizes the error correction capability of the OAD-$i$ technique.

*Proposition 2:* For a linear block code with the minimum Hamming distance $d$, the OAD-$i$ algorithm is able to correct up to $(i+\tau)$ errors. In particular, the OAD-($d-1-\tau$) algorithm can correct up to $d-1$ errors to achieve the same performance as the Chase-I algorithm [3].

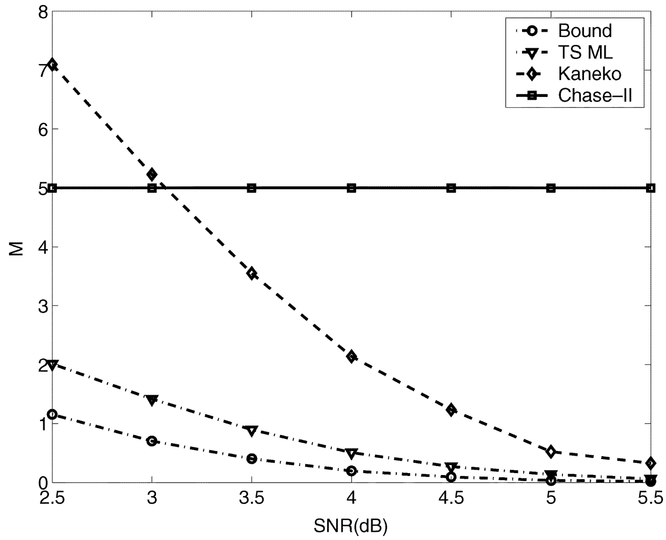


Fig. 2. Complexity ($\bar{M}$) versus SNR: BCH(31,16).

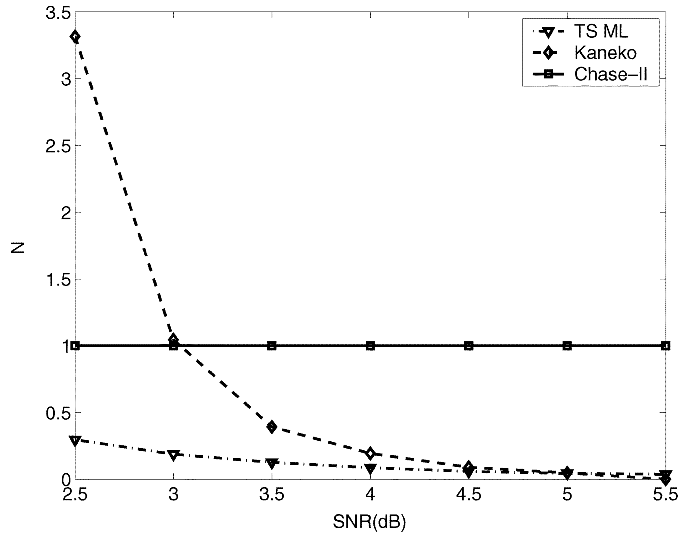


Fig. 3. Complexity ($\bar{N}$) versus SNR: BCH(31,16).

*Proof:* As a sufficient set, $S_M$ includes the optimal codeword and there are at most $\tau$ errors outside of the first $M$ least reliable positions. OAD-$i$ considers all possible (up to $i$) errors in the first $M$ least reliable positions. When the total number of errors $e \leq (\tau + i)$, it is guaranteed that at least $e - \tau \leq i$ errors fall within those $M$ positions. OAD-$i$ is then capable of correcting up to $(\tau + i)$ errors. In particular, OAD-$(d - 1 - \tau)$ corrects up to $(\tau + (d - 1 - \tau)) = (d - 1)$ errors, thus achieving the near-optimal performance of Chase-I algorithm. By properly selecting the decoding order $i$, the OAD algorithm also effects flexible design tradeoff between decoding complexity and performance. To guide such a design, we derive the union bound of the block error rate of OAD-$i$ as follows:

$$P_{e,\text{block}}(i) \leq 1 - \sum_{m=0}^{\tau+i} \binom{n}{m} \times P_b^m \times (1 - P_b)^{(n-m)} \quad (6)$$

where $P_b$ is the uncoded bit-error rate. Based on (6), the decoding order $i$ of OAD can be determined based on the desired block error rate performance, thus avoiding unnecessary computation spent on higher order decoding.

## IV. Performance Evaluation and Simulations

The proposed TS ML algorithm attains low-complexity optimal decoding by exploiting the MSS estimated during the first stage of processing. In contrast, Kaneko's algorithm [10] directly starts from the entire set of codewords for optimal decoding, while the Chase-II algorithm [3] narrows down to a subset $S_m$ with a prescribed value for $m$ regardless of the sufficiency of the set $S_m$, resulting in suboptimal decoding performance.

Computer simulations are conducted to compare our TS ML algorithm with Kaneko [10], Chase-II [3], and OSD-2 [5] algorithms. The (31,16) BCH code is used in all simulations. For a good tradeoff design, we choose OAD-2 of order 2 in the second stage of processing and correspondingly refer this version of our algorithm as 2-TS ML. Fig. 2 depicts the average decoding order $\bar{M}$ of all the algorithms, along with the theoretical lower bound of $\bar{M}$ derived in Section II-C. Several observations can be made.

- The theoretical bound of $\bar{M}$ indicates the potential complexity reduction enabled by the first stage of the new TS structure, had an optimal decoder been used in the second stage. Apparently, the TS structure could lead to optimal decoding with very low complexity.
- The low-complexity complementary decoding algorithm in Section III-A yields very good estimate of the MSS.
- Much lower complexity is expected from the 2-TS ML algorithm than Kaneko and Chase-II algorithms, with better or comparable performance.

Fig. 3 depicts the complexity of 2-TS ML, Kaneko, and Chase-II algorithms with respect to the algebraic decoder usage $\bar{N}$. The decoding complexity of Chase-II algorithm is normalized to 1, while the relative complexity of other algorithms are plotted in the normalized coordinate. Chase-II algorithm has fixed complexity regardless of the SNR value. The decoding complexity of our TS ML algorithm, on the other hand, drops monotonically as SNR increases, and stays well below that of Chase-II algorithm in all the simulated SNR regions. The decoding complexity of our TS ML is substantially less than that of Kaneko algorithm in low to moderate SNR regions, because the TS structure considerably speeds up the convergence rate for decoding.

Fig. 4 shows the error performance of our 2-TS ML as compared to Kaneko, Chase-II, and OSD-2 algorithms. Our algorithm attains word error rate (WER) performance comparable to that of Kaneko and OSD-2 algorithms. In all the simulated SNR regions, our algorithm performs consistently better than the Chase-II algorithm at much lower complexity.

## V. Conclusion

In this paper, we propose a new decoding structure along with practical implementation for optimal decoding of linear block codes. The decoding is divided into two stages with distinct objectives. The first stage substantially decreases the decoding complexity without affecting the decoding performance, while

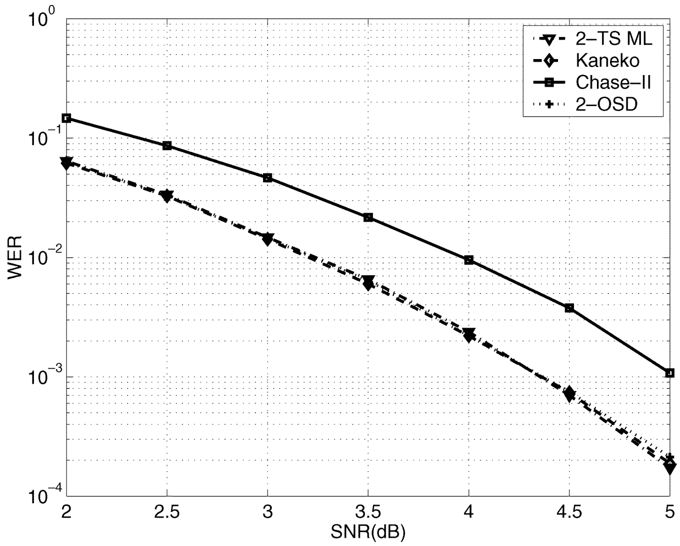


Fig. 4. Performance comparisons: BCH(31,16).

the second stage attains the optimal performance. Through the TS processing, the algorithm achieves the optimum performance at low average complexity. Furthermore, bounded suboptimal performance can be achieved in the second stage at further reduced complexity.

## REFERENCES


[1] S. G. Wilson, *Digital Modulation and Coding*. Englewood Cliffs, NJ: Prentice-Hall, 1996.

[2] G. D. Forney, "Generalized minimum distance decoding," *IEEE Trans. Inf. Theory*, vol. IT–12, no. 2, pp. 125–131, Apr. 1966.

[3] D. Chase, "A class of algorithms for decoding block codes with channel measurement information," *IEEE Trans. Inf. Theory*, vol. IT–18, no. 1, pp. 170–182, Jan. 1972.

[4] H. Tanaka and K. Kakigahara, "Simplified correlation decoding by selecting codewords using erasure information," *IEEE Trans. Inf. Theory*, vol. IT–29, no. 5, pp. 743–748, Sep. 1983.

[5] M. P. C. Fossorier and S. Lin, "Soft-decision decoding of linear block codes based on ordered statistics," *IEEE Trans. Inf. Theory*, vol. 41, no. 5, pp. 1379–1396, Sep. 1995.

[6] M. P. C. Fossorier, "Reliability-based soft-decision decoding with iterative information set reduction," *IEEE Trans. Inf. Theory*, vol. 48, no. 12, pp. 3101–3106, Dec. 2002.

[7] Y. S. Han, C. R. P. Hartmann, and C. C. Chen, "Efficient priority-first search maximum-likelihood soft-decision decoding of linear block codes," *IEEE Trans. Inf. Theory*, vol. 39, no. 5, pp. 1514–1523, Sep. 1993.

[8] Y. S. Han, C. R. P. Hartmann, and K. G. Mehrotra, "Decoding linear block codes using a priority-first search: Performance analysis and suboptimal version," *IEEE Trans. Inf. Theory*, vol. 44, no. 3, pp. 1233–1246, May 1998.

[9] M. Kokkonen and K. Kalliojärvi, "Soft-decision decoding of binary linear block codes using reduced breadth-first search algorithm," *IEEE Trans. Commun.*, vol. 48, no. 6, pp. 905–907, Jun. 2000.

[10] T. Kaneko, T. Nishijimam, H. Inazume, and S. Hirasawa, "An efficient maximum-likelihood-decoding algorithm for linear block codes with algebraic decoder," *IEEE Trans. Inf. Theory*, vol. 40, no. 2, pp. 320–327, Mar. 1994.

[11] T. Kaneko, T. Nishijima, and S. Hirasawa, "An improvement of soft-decision maximum-likelihood decoding algorithm using hard-decision bounded-distance decoding," *IEEE Trans. Inf. Theory*, vol. 43, no. 4, pp. 1314–1319, Jul. 1997.

[12] M. Fossorier and S. Lin, "Complementary reliability-based decoding of binary linear block codes," *IEEE Trans. Inf. Theory*, vol. 43, no. 5, pp. 1667–1672, Sep. 1997.

[13] D. J. Taipale and M. B. Pursley, "An improvement to generalized-minimum distance decoding," *IEEE Trans. Inf. Theory*, vol. 37, no. 1, pp. 167–172, Jan. 1991.

[14] T. Kasami, Y. Tang, T. Koumoto, and T. Fujiwara, "Sufficient conditions for ruling-out useless iterative steps in a class of iterative decoding algorithms," *IEICE Trans. Fundam.*, vol. E82-A, pp. 2061–2073, Oct. 1999.

[15] P. F. Swaszek and W. Jones, "How often is hard-decision decoding enough?," *IEEE Trans. Inf. Theory*, vol. 44, no. 3, pp. 1187–1193, May 1998.

[16] M. P. C. Fossorier and S. Lin, "First-order approximation of the ordered binary-symmetric channel," *IEEE Trans. Inf. Theory*, vol. 42, no. 5, pp. 1381–1387, Sep. 1996.