On the (n, m, k)-Cast Capacity of Wireless Ad Hoc Networks

Hyunchul Kim, Hamid R. Sadjadpour, and Jose Joaquin Garcia-Luna-Aceves

Abstract: **The capacity of wireless ad-hoc networks is analyzed for all kinds of information dissemination based on single and multiple packet reception schemes under the physical model. To represent the general information dissemination scheme, we use (n, m, k)-cast model [1] where n, m, and k $(k \leq m)$ are the number of nodes, destinations and closest destinations that actually receive packets from the source in each (n, m, k)-cast group, respectively. We first consider point-to-point communication, which implies single packet reception between transmitter-receiver pairs and compute the (n, m, k)-cast communications. Next, the achievable throughput capacity is computed when receiver nodes are endowed with multipacket reception (MPR) capability. We adopt maximum likelihood decoding (MLD) and successive interference cancellation as optimal and suboptimal decoding schemes for MPR. We also demonstrate that physical and protocol models for MPR render the same capacity when we utilize MLD for decoding.**

Index Terms: **Capacity, maximum likelihood decoding (MLD), scaling law, wireless ad hoc network.**

I. INTRODUCTION

The seminal work by Gupta and Kumar [2] demonstrated that throughput capacity is bounded by $O\left(1/\sqrt{n}\right)$ and throughput capacity scaling as $\sqrt{1/(n \log n)}$ is achievable when the number of nodes n increases. Later this gap is closed by Franceschetti *et al.* [3] by utilizing the percolation theory. Zheng *et al.* [4] proved that the throughput capacity under physical model can be increased by a factor of $\Theta\left((\log n)^{\frac{\alpha-2}{2\alpha}}\right)$ compared to Gupta and Kumar's result when nodes are equipped with multipacket reception and a successive interference cancellation decoding scheme. Zheng [5] studied the behaviour of information dissemination in power-constrained wireless networks in terms of the broadcast capacity and information diffusion rate in both random extended and dense networks. Keshavarz *et al.* [6] extended Zheng's work by considering the interference effect in general wireless networks and proposed the most general case for broadcast capacity results with multihop routing under the protocol model. In [7], they extended the broadcast capacity for the physical model and the generalized physical model based on Shannon's formula [8]. Li *et al.* [9] unified the capacity of wireless ad hoc networks utilizing unicast, multicast, and broadcast routing schemes. Recent work [1] has shown that all forms of information dissemination in wireless ad hoc networks can be unified into a single (n, m, k)-cast model. (n, m, k)-cast is a general communication model where n is the number of nodes in the network, m is the number of destinations, and k $(k \leq m)$ is the actual number of the closest destinations that receive packets from source in each (n, m, k)-cast group. The (n, m, k)-cast communication was investigated under protocol model in [1]. Recent work [10] demonstrated that the throughput capacity can be improved by utilizing multipacket reception (MPR) at the receiver. However, all the results from prior work [1], [10] concentrated on the protocol model. This paper presents the throughput capacity of (n, m, k)-cast under the physical model when both single packet reception (SPR) and multipacket reception (MPR) schemes are considered.

As the first contribution of this paper, we study the throughput capacity of (n, m, k)-cast under the physical model when nodes are communicating based on point-to-point communication. As our second contribution, we propose the optimum decoding scheme at the receiver node and compute (n, m, k)-cast throughput capacity both under the proposed optimum decoding and successive interference cancellation schemes. The result corresponds to the throughput capacity result in [10] based on the protocol model with MPR which implies that protocol model is equivalent to physical model under the optimal maximum likelihood decoding (MLD) constraint.

II. NETWORK MODEL AND PRELIMINARIES

We consider a random wireless dense network where n static nodes are uniformly distributed according to the Poisson point process over a unit square area. In this model, the node density goes to infinity as the number of nodes n increases. As a channel model, path loss channel between transmitter-receiver pairs are considered along with addictive white Gaussian noise (AWGN) channel. In this paper, X_i denotes the location of node i and $|X_i - X_j|$ is the Euclidean distance between the node i and j. We assume all nodes are equipped with omni-directional antennas and transmit packets with same power level P (≥ 0). Also nodes are not allowed to simultaneously transmit/receive packets [2].

Definition 1 (Physical model with point to point communication). If the condition $\text{SINR}_{i \to j} \geq \beta$ between a pair of transmitting node i and receiving node j is satisfied, a constant data rate of W bits/second between transmitter-receiver pair is achieved [2].

$$\text{SINR}_{i \to j} = \frac{P g_{ij}}{N + \sum_{k \neq i, k=1}^{n} P g_{kj}} \geq \beta \tag{1}$$

where P is the common transmit power and $g_{ij} = 1/|X_i - X_j|^{\alpha}$

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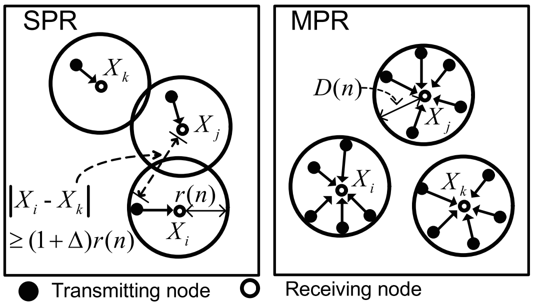

Fig. 1. Left figure describes the common successful transmission range $r(n)$ for the protocol model [2] and right figure shows the common decoding range $D(n)$ for MPR under the physical model. In this circular range, there are on average $\pi n D^2(n)$ nodes.

is the path loss channel gain between node i and j with an attenuation parameter $\alpha \geq 2$. Note in (1) that N denotes the ambient AWGN noise and simultaneously transmitted signals from nodes X_k $(i \neq k)$ are regarded as interference. In the physical model for MPR, we consider a common decoding range $D(n)$ that defines the area where the receiver is capable of decoding simultaneously received packets, which contrasts with point-to-point communication under the protocol model [2].

Proposition 1 (Encoding scheme for MPR with MLD). We consider a set of $\pi D^2(n)n$ transmitters inside a circle of radius $D(n)$. If the length of each codeword is L for any transmitter node i with a rate of R_i, then the code book for the ith user has a total of 2^{LR_i} codewords with power P. Each of the transmitters chooses an arbitrary codeword from its own codebook and send these vectors simultaneously to the node j centering at the decoding range $D(n)$. At the receiver end, these codewords are added constructively along with the Gaussian noise N and interference.

Proposition 2 (Decoding scheme for MPR with MLD). According to the encoding scheme in Proposition 1, the received signal at node j is defined as

$$Y_j = \sum_{i \neq j, i \in S(D(n))} \sqrt{g_{ij}} X_{T_i} + N + I \qquad (2)$$

where X_{T_i}, $S(D(n))$, and $I = \sum_{k \notin S(D(n)), k=1}^{n} P g_{kj}$ denote transmitted codeword from node i, set of nodes within the decoding range $D(n)$, and interference signal from the simultaneously transmitting nodes outside $D(n)$, respectively. Note that since the decoding in MLD is carried jointly, the average received power from each transmitter inside decoding range of $D(n)$ should satisfy physical model constraint. Hence, the physical model constraint should be modified for MPR under MLD.

Definition 2 (Physical model with MPR based on MLD). Successful decoding under physical model condition for a gaussian multiple access channel with received power Pg_{ij} at node j is defined as

$$\text{SINR}_{i \in S(D(n)) \to j} = \frac{\sum_{i \in S(D(n))} P g_{ij}}{N + \sum_{k \notin S(D(n))} P g_{kj}} \geq |S(D(n))|\beta \qquad (3)$$

where $S(D(n))$ is the set of nodes inside the decoding circle of radius $D(n)$ and $|S(D(n))|$ is the cardinality of this set. Note that in MPR with MLD, packets from the nodes inside decoding region are decoded jointly at a receiver node such that the total throughput capacity is equivalent to $\sum_{i \in S(D(n))} R_i = |S(D(n))| \times W$. It is feasible for a given decoding range $D(n)$ as long as the average signal-to-interference plus noise ratio (SINR) at each receiver node is greater than or equal to the minimum SINR constraint β in (1) multiplied by the total number of source-destination pairs in the decoding region. The following proposition states the decoding procedure for MPR using SIC and Definition 3 describes the condition for the minimum required SINR.

Proposition 3. The receivers decode the information from the nearest transmitters to farthest ones whose positions are the maximum distance inside of communication range $D(n)$ [4].

Definition 3 (Physical model with MPR based on SIC). The transmissions from all the transmitters centered around a receiver j with a distance smaller or equal to $D(n)$ occur successfully if the SINR of the transmitter $Z(D(n))$ at the edge of this receiver circle satisfies

$$SINR_{Z(D(n)) \to j} = \frac{P g_{Z(D(n))j}}{N + \sum_{k \notin I(D(n))} P g_{kj}} \geq \beta \qquad (4)$$

where $g_{Z(D(n))j} = |X_{Z(D(n))} - X_j|^{-\alpha}$ is the channel attenuation between nodes $Z(D(n))$ and j [4].

The definitions of feasible throughput capacity and order of throughput capacity in this paper are same as those defined in [2]. The following definitions and lemmas describe the basic notion of our analysis for the (n, m, k)-cast capacity.

Definition 4 (Feasible throughput capacity). In a dense random wireless ad hoc network with n nodes in which each source node transmits its packets to k out of m destinations, the per-node (n, m, k)-cast throughput capacity is defined as $C_{m,k}(n) = \frac{1}{n}\sum_{i=1}^{n} \lambda_{m,k}^{i}(n)$ where $\lambda_{m,k}^{i}(n)$ is the throughput capacity of source i transmitting packets to k out of its m chosen destinations with all such k nodes receiving the information within a finite time interval.

Definition 5 (Transport capacity). The transport capacity [2] in a random wireless network is defined as the maximum bit-meters per second which can be achieved in aggregate by optimally operating the network. Therefore,

$$C_T = \sup \sum_{i \neq j} C_{ij} d_{ij} \qquad (5)$$

where C_{ij} is the data rate defined from each node i to each node j and $d_{ij} = |X_i - X_j|$ is the distance between node i and j.

Definition 6 (Euclidean minimum spanning tree (EMST)). Consider a connected undirected graph $G = (V, E)$ where V and E are sets of vertices and edges in the graph G, respectively. The EMST of G is a spanning tree of G with the minimum Euclidean distance in total between connected vertices of this tree.

Definition 7 ((n, m, k)-cast tree). The definition of (n, m, k)-cast tree is given in [1].

We can also define (n, m, m)-cast tree as a multicast tree.

Definition 8 (Minimum euclidean (n, m, k)-cast tree (MEMKT)). The definition of MEMKT for an (n, m, k)-cast is given in [1]. Especially when $k = m$, we use MEMT for an (n, m, m)-cast tree with a minimum Euclidean distance in total.

Lemma 1. Let $f(x)$ denote the node probability distribution function in the network area. Then, for large values of n and $d > 1$, the $\overline{\|\mathrm{EMST}\|}$ is tightly bounded as [11]

$$\overline{\|\mathrm{EMST}\|} = \Theta\left(c(d)n^{\frac{d-1}{d}}\int_{R^d} f(x)^{\frac{d-1}{d}}dx\right) \quad (6)$$

where d is the dimension of the network. Note that both $c(d)$ and the integration are constants and not functions of n.

III. CAPACITY ANALYSIS OF (n, m, k)-CAST WITH SPR

In this section, we first start from the throughput capacity of (n, m, m)-casting that corresponds to unicasting, multicasting, and broadcasting when $m = 1$, $m < n$, and $m = n$, respectively.

A. *The Capacity of* (n, m, m)*-Cast*

A.1 Upper Bound

At any arbitrary communication session of (n, m, m)-casting, when a node transmits a packet to the m destinations, there can be two different communication schemes [12] depending on the type of packet relayed. We can either assume that, for each transmission, only a single node receives the packet or multiple nodes within an area of transmission range. The former concept is called unicast concept communication while the latter approach corresponds to broadcast concept communication [12]. Keshavarz *et al.* used these two concepts to compute the multicast capacity in wireless ad hoc networks for both cases. In this paper, we compute the upper bound (n, m, m)-cast throughput capacity based on the unicast concept. Note that the part of the work can be found in [12] and [13] and stronger result exists for the one-to-many transmission scheme in [12].

Lemma 2. The per-node throughput capacity of (n, m, m)-cast in dense wireless ad-hoc networks is upper bounded by $O\left(\frac{1}{n}\frac{\sup \Sigma_{i\neq j} d_{ij}C_{ij}}{\overline{\|\mathrm{MEMT}\|}}\right)$ under the physical model.

Proof: The proof can be found in [14]. □

Lemma 3. The transport capacity for random networks under the physical model is $\Theta\left(\sqrt{n}\right)$ bit-meters per second.

Proof: The proof can be found in [2]. □

Theorem 1. In a dense wireless ad hoc network with (n, m, m)-cast, the upper bound of the per-node throughput capacity under the physical model is given by $C_{m,m}(n) = O\left(1/\sqrt{nm}\right)$

Proof: Assuming that there are $m+1$ nodes in (n, m, m)-cast tree, it is obvious that $\overline{\|\mathrm{MEMT}\|}$ is equal to $\Theta\left(\sqrt{m}\right)$ from (6). The proof is immediate by replacing $\overline{\|\mathrm{MEMT}\|}$ with $\Theta\left(\sqrt{m}\right)$ and combining Lemmas 2 and 3. □

Adopting the broadcast concept for the network, a transmitter can simultaneously deliver packets to multiple destinations or relays spread over an area where the successful communication occur. Thus, to find out the upper bound of the throughput capacity based on the broadcast concept, we have to consider the consumed area used to route packets from source to destinations instead of the $\overline{\|\mathrm{MEMT}\|}$. In [12], Keshavarz *et al.* showed that the per-node upper bound of the multicast throughput capacity is $C_{m,m}(n) = O\left(1/n\right)$ based on broadcast concept in the network and broadcast-based communication gives tighter upper bound than the unicast-based communication. Therefore, we conclude that the upper bound of the (n, m, m)-cast throughput capacity based on the unicast concept communication provides higher upper bound equal to $C_{m,m}(n) = O\left(1/\sqrt{nm}\right)$.

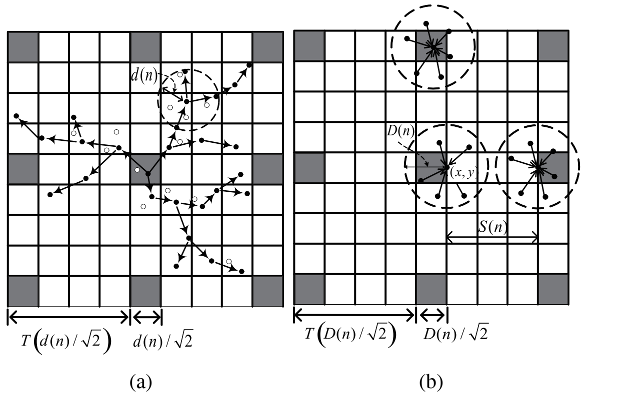

Fig. 2. a) Cell graph construction used to derive a lower bound on capacity. The solid dots are used to connect (n, m, m) cast tree while the blank dots are not part of the tree. b) TDMA scheme with decoding range D(n). The parameter S(n) is defined to separate simultaneously communicating cells for successful communication condition based on physical model

A.2 Lower Bound

The lower bound for (n, m, m)-cast is derived using a time division multiple access (TDMA) scheme as shown in Fig. 2(a). To construct the TDMA scheme, cells with the same side length of $d(n)/\sqrt{2}$ are grouped into T^2 non-interfering groups. By choosing a common value for $d(n)$, we derive a loose lower bound that can potentially be improved utilizing percolation theory [3]. The communication is divided into T^2 time slots. In each time slot, every node in the same group transmits packets with a common transmission power P.

Lemma 4. Under the physical model, by properly choosing TDMA parameter T, a particular node in a cell can successfully transmit to any other node within a distance of $d(n)$.

Proof: To use a common $d(n)$, we need to assure that the physical model condition is satisfied. We can achieve the lower bound for the capacity by computing the upper bound for interference at the receiver. Fig. 2(a) demonstrates the nodes that can simultaneously transmit in shaded cells while the physical model criterion is satisfied. Clearly, the interference is maximized when the interfering nodes have the closest distance to the receiver node, i.e., $\frac{d(n)}{\sqrt{2}}(iT-2)$ for $i \in I = \{1, 2, \cdots\}$. Since we already proved this part in [14], we only show the final condition for the TDMA parameter T.

$$T \geq \left(\frac{8c_1\left(\sqrt{2}\right)^{\alpha}}{\frac{1}{\beta} - \frac{N(d(n))^{\alpha}}{P}}\right)^{\frac{1}{\alpha}} \quad (7)$$

In this paper, the TDMA parameter that satisfies (7) is denoted as $T(\alpha, \beta, d(n))$. As mentioned earlier, we choose the transmit power as a function of transmission range, i.e., $P = k(d(n))^{\alpha}$ where k is a constant value. Under this assumption, the TDMA parameter is not a function of n. □

Since we already proved in [14] that a cell graph for any arbitrary $\overline{\|\mathrm{MEMT}\|}$ under the physical model is connected through the nodes on $\overline{\|\mathrm{MEMT}\|}$ and our TDMA scheme does not change the order of throughput capacity, we only show the following lemma without proof.

Lemma 5. The achievable lower bound of the (n,m,m)-cast capacity is

$$C_{m,m}(n) = \Omega\Bigg(\frac{1}{\overline{\#\text{MEMTC}(d(n))}} \times \frac{1}{T^2(\alpha,\beta,d(n))\,nd^2(n)}\Bigg). \quad (8)$$

Given the above lemma, to express the lower bound of $C_{m,m}(n)$ as a function of network parameters, we need to compute a tight bound for $\overline{\#\text{MEMTC}(d(n))}$, which we do next.

Lemma 6. The average number of the cells that belongs to a (n,m,m)-cast tree satisfies the following upper bound.

$$\overline{\#\text{MEMTC}(d(n))} = \min\left(\Theta\left(\sqrt{m}/d(n)\right), \Theta\left(d^{-2}(n)\right)\right). \quad (9)$$

Proof: The proof can be found in [14] □

By combining Lemmas 5 and 6, the achievable lower bound of the (n,m,m)-cast capacity when $d(n) = \Theta\left(\sqrt{\log n/n}\right)$ is

$$C_{m,m}(n) = \begin{cases} \Omega\left(\frac{1}{\sqrt{nm\log n}}\right), m = O\left(\frac{n}{\log n}\right) \\ \Omega\left(n^{-1}\right), m = \Omega\left(\frac{n}{\log n}\right). \end{cases} \quad (10)$$

B. Capacity Bounds of (n,m,k)-Cast

B.1 Upper Bound

In this subsection, we demonstrate the throughput capacity of (n,m,k)-cast in random wireless ad hoc networks. The proofs are very similar to those shown in the previous section. Thus, lemmas and theorems are only stated without proof for completeness of the paper.

Lemma 7. The per-node throughput capacity of (n,m,k)-cast in dense wireless ad-hoc networks is upper bounded by $O\left(\frac{1}{n} \times \frac{\sup \Sigma_{ij} d_{ij} C_{ij}}{\overline{\|\text{MEMKT}\|}}\right)$

Proof: This is similar to Lemma 1 except that we replace MEMT with MEMKT. □

Lemma 8. The average length of $\overline{\|\text{MEMKT}\|}$ has the lower bound of $\frac{\sqrt{mk}}{m}$.

Proof: The proof can be found in [1]. □

Theorem 2. The per-node upper bound throughput capacity of the (n,m,k)-cast in dense wireless ad hoc network under the physical model is given by $C_{m,k}(n) = O\left(\frac{\sqrt{m}}{\sqrt{nk}}\right)$.

Proof: The proof is similar to the proof of Theorem 1. □

B.2 Lower Bound

In this subsection, we demonstrate the lower bound for (n,m,k)-cast based on the same approach used in subsection III-A.2.

Lemma 9. The achievable lower bound of the (n,m,k)-cast capacity is given by

$$C_{m,m}(n) = \Omega\Bigg(\frac{1}{\overline{\#\text{MEMKTC}(d(n))}} \times \frac{1}{T(\alpha,\beta,d(n))^2\,nd^2(n)}\Bigg). \quad (11)$$

where $\overline{\#\text{MEMKTC}(d(n))}$ is the mean number of cells in MEMKT$(d(n))$.

Proof: The proof is similar to Lemma 5 except that $\overline{\#\text{MEMTC}(d(n))}$ is replaced with $\overline{\#\text{MEMKTC}(d(n))}$. □

Lemma 10. The average number of cells in MEMKT$(d(n))$ tree is upper bounded as

$$\overline{\#\text{MEMKTC}(d(n))} = \begin{cases} \Theta\left(k(\sqrt{m}d(n))^{-1}\right), m = O\left(d^{-2}(n)\right) \\ \Theta\left(k\right), \Omega(k) = \left(d^{-2}(n)\right) = O(m) \\ \Theta\left(d^{-2}(n)\right), k = \Omega\left(d^{-2}(n)\right). \end{cases} \quad (12)$$

Proof: The proof is similar to the proof of Lemma 6. □

The maximum attainable lower bound capacity is achieved when $d_{\min}(n) = \Omega\left(\sqrt{\log n/n}\right)$ is applied for $d(n)$.

Theorem 3. The maximum achievable lower bound for the (n,m,k)-cast capacity is

$$C_{m,k}(n) = \begin{cases} \Omega\left(\frac{\sqrt{m}}{k\sqrt{n\log n}}\right), m = O\left(\frac{n}{\log n}\right) \\ \Omega\left(\frac{1}{k\log n}\right), \Omega(k) = \frac{n}{\log n} = O(m) \\ \Omega\left(1/n\right), k = \Omega\left(\frac{n}{\log n}\right). \end{cases} \quad (13)$$

Proof: Combining Lemmas 9 and 10 with the minimum distance parameter for $d(n)$ provides us with the result. □

IV. ACHIEVABLE CAPACITY OF (n,m,k)-CAST WITH MPR

In this section, we present the achievable throughput capacity of (n,m,k)-casting by allowing MPR capability for each node. In order to compare the results with protocol model, refer to the throughput capacity under protocol model in [1]. Note that the decoding scheme for MPR under the physical model, both MLD and SIC decoding schemes are considered.

A. Lower Bound under the Physical Model

The lower bound with MPR is derived using a TDMA scheme shown in Fig. 2(b). The side length of the square cell is replaced by $D(n)/\sqrt{2}$ while T^2 non-interfering groups of cells and data packet relaying scheme remain the same as SPR. As introduced in [14], the longest edge of NNG follows $M_n \le \sqrt{(\log n + a)/n\pi}$ with high probability as n tends to infinity. Similar to the lower bound analysis for SPR, we consider $D(n) \ge M_n = (\sqrt{\log n/n})$ by choosing $a = \log n\,(\pi - 1)$. As long as the successful communication condition in the physical model is satisfied, a particular node can successfully relay packets to its adjacent nodes within $D(n)$ distance away and the connectivity of cell graph on $\overline{\|\text{MEMKT}\|}$ is guaranteed. The TDMA parameter T for minimum cell separation guaranteeing the successful transmission is defined according to the MPR physical model. Note that in physical model, there is no common communication range and in order for this scheme to work, we need to derive the condition under which the SINR condition is satisfies.

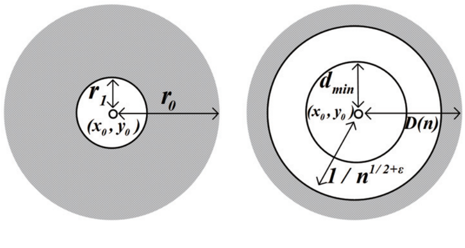

Fig. 3. In the left figure, received signal power at node (x_0, y_0) from nodes transmitting simultaneously in a shaded region is represented by (16). Right figure shows relationship between minimum distance between any two nodes and $1/n^{1/2+\epsilon}$centering at node (x_0, y_0).

A.1 MPR with ML Optimum Decoding

Lemma 11. Consider nodes equipped with MPR capability utilizing MLD decoder in a given TDMA scheme. Multiple packets are successfully decoded if T satisfies

$$T \geq \sqrt{2}\left(\frac{8\pi c_1(\alpha-2)D^{(4-\alpha)}(n)\beta}{\delta n^{\frac{(\alpha-4)}{2}}}\right)^{\frac{1}{\alpha}}. \tag{14}$$

Proof: For any time slot only one receiver node in each shaded region of Fig. 2(b) is activated and all other nodes within a radius of $D(n)$ around the receiver node act as transmitters. Given that the physical model criterion in Definition 2 is satisfied at the receiver node, each node in shaded cells can successfully and simultaneously decode packets transmitted from nodes within the decoding region $D(n)$. To achieve the lower bound of the throughput capacity, we need to compute the upper bound of interference at the receiver and the lower bound of the signal power. We first compute the lower bound of the received signal power. In [15], it is proven that with the nodes transmitting outside of the circular region with radius r_0, the received signal power at coordinate (x_0, y_0) is given by

$$P_r(x_0, y_0, r_0) = \frac{2\pi\delta P n}{(\alpha-2)\, r_0^{(\alpha-2)}}\left[1 - \frac{r_0^{(\alpha-2)}}{2\pi}C(x_0, y_0)\right] \tag{15}$$

where $C(x_0, y_0)$ is a constant value related to the receiver location (x_0, y_0). Assuming that the coordinate of the receiver node j is (x_0, y_0) and nodes are uniformly distributed in the network area, then (as shown in the left side of Fig. 3) the received signal power at node j from the nodes in an area of $\pi(r_0 - r_1)^2$ is approximated as

$$\sum_{i\in I(D(n))} P g_{ij} = P_r(x_0, y_0, r_1) - P_r(x_0, y_0, r_0). \tag{16}$$

Let's define d_{min} as the minimum distance between any two nodes in the network with high probability. Then, the signal power at the receiver node j is approximated based on (16). In other words, if transmitting nodes are from distance $d_{\min}$ to $D(n)$ away, then the received signal power at node j is computed as

$$\sum_{i\in I(D(n))} P g_{ij} = P_r(x_0, y_0, d_{\min}) - P_r(x_0, y_0, D(n)). \tag{17}$$

Let's define P_a as the probability of having no node inside the circular region of radius $\frac{1}{n^{1/2+\epsilon}}$ around node j. It is easy to prove P_a goes to zero for large n, for $\epsilon \geq 0$. At any arbitrary node in the network, we can draw circular region with radius of $\frac{1}{n^{1/2+\epsilon}}$. Then, the P_a is computed as $(1 - \frac{\pi}{(n^{1/2+\epsilon})^2}n)^{n-1}$. It can be proved that this probability goes to zero as n tends to infinity. This implies that the minimum distance $d_{\min}$ between any two nodes in the network is larger than $\frac{1}{n^{1/2+\epsilon}}$ as n goes to infinity. Based on this fact, we arrive at

$$P_r(x_0, y_0, d_{\min}) \geq P_r(x_0, y_0, \frac{1}{n^{\frac{1}{2}+\epsilon}}) \tag{18}$$

By combining (16) and (18), we have

$$\begin{aligned}\sum_{i\in I(D(n))} P g_{ij} &\geq P_r(x_0, y_0, \tfrac{1}{n^{\frac{1}{2}+\epsilon}}) - P_r(x_0, y_0, D(n)) \\ &= \frac{2\pi\delta P n}{(\alpha-2)}\left(\frac{n^{(\frac{1}{2}+\epsilon)(\alpha-2)}D(n)^{(\alpha-2)} - 1}{D(n)^{(\alpha-2)}}\right).\end{aligned} \tag{19}$$

The interference is maximized when the interfering nodes have the closest distance to the receiver node, i.e., $\frac{D(n)}{\sqrt{2}}(iT-2)$ for $i \in I = \{1, 2, \cdots\}$. Therefore, the total interference experienced by each node is given by

$$\sum_{k\notin I(D(n))} P g_{kj} \leq \sum_{i=1}^{\infty} 8i\frac{\pi D^2(n) n P}{\{iS(n) - D(n)\}^{\alpha}} \tag{20}$$

where $S(n)$ is the number of cells required for separating simultaneously communicating cells. Since we do not allow overlapping different decoding regions, it is obvious from Fig. 2(b) that $S(n) \geq 2D(n)$. By applying this to (20), it can be upper bounded as

$$\sum_{i=1}^{\infty} 8i\frac{\pi D^2(n) n P}{\{iS(n) - D(n)\}^{\alpha}} \leq \frac{8\pi D^2(n) n P}{S^{\alpha}(n)}\sum_{i=1}^{\infty}\frac{i}{\{i - \frac{1}{2}\}^{\alpha}}. \tag{21}$$

Note that $\sum_{i=1}^{\infty}\frac{i}{(i-\frac{2}{T})^{\alpha}}$ converges into a bounded value of c_1 when $\alpha \geq 2$. By applying this value to (21), we arrive at

$$\sum_{k\notin I(D(n))} P g_{kj} \leq \frac{8\pi D^2(n) n P}{S^{\alpha}(n)}c_1. \tag{22}$$

Let's assume that AWGN noise is negligible compared to the interference. Combining (3), (16), and (22) yields the lower bound of the $\mathrm{SINR}_{i\in S(D(n))\to j}$ which is

$$\frac{\sum_{i\in S(D(n))} P g_{ij}}{N + \sum_{k\notin S(D(n))} P g_{kj}} \geq \frac{S^{\alpha}(n)\delta(D^{(\alpha-2)}(n) n^{(\frac{1}{2}+\epsilon)(\alpha-2)} - 1)}{8c_1(\alpha-2)D^{\alpha}(n)}. \tag{23}$$

Since we considered nodes in the range of $[\frac{1}{n^{1/2+\epsilon}}, D(n)]$, we further assume $D(n) \gg \frac{1}{1/2+\epsilon}$, which implies $D^{(\alpha-2)}(n)n^{(1/2+\epsilon)(\alpha-2)} \gg 1$ in (23). Then, by approximating $D^{(\alpha-2)}(n)n^{(1/2+\epsilon)(\alpha-2)} - 1 \approx D^{(\alpha-2)}(n)n^{(1/2+\epsilon)(\alpha-2)}$, the lower bound of $\mathrm{SINR}_{i\in S(D(n))\to j}$ can be further reduced into

$$\begin{aligned}\frac{\sum_{i\in S(D(n))} P g_{ij}}{N + \sum_{k\notin S(D(n))} P g_{kj}} &\geq \frac{\delta S^{\alpha}(n) n^{(\frac{1}{2}+\epsilon)(\alpha-2)}}{8c_1(\alpha-2)D^2(n)} \\ &\geq \frac{\delta S^{\alpha}(n) n^{\frac{(\alpha-2)}{2}}}{8c_1(\alpha-2)D^2(n)}\end{aligned} \tag{24}$$

The second inequality in (24) is due to $\epsilon \geq 0$.

According to the physical model condition in (3) and the fact that there are on average $\pi D^2(n)n$ nodes in $S(D(n))$, the following condition should be satisfied for the successful communication.

$$\frac{\delta S^{\alpha}(n) n^{\frac{(\alpha-2)}{2}}}{8c_1(\alpha-2)D^2(n)} \geq \pi D^2(n) n\beta. \quad (25)$$

Based on (25), we establish following condition for $S(n)$.

$$S(n) \geq \left(\frac{8\pi c_1(\alpha-2)D^4(n)\beta}{\delta n^{\frac{(\alpha-4)}{2}}}\right)^{\frac{1}{\alpha}}. \quad (26)$$

Finally from Fig. 2(b), we know $S(n) = (T-1)\frac{D(n)}{\sqrt{2}}$. By replacing this with the left side of (26) we have

$$T \geq \sqrt{2}\left(\frac{8\pi c_1(\alpha-2)D^{(4-\alpha)}(n)\beta}{\delta n^{\frac{(\alpha-4)}{2}}}\right)^{\frac{1}{\alpha}} \quad (27)$$

which proves the lemma. □

Lemma 12. The TDMA parameter T is a constant value and does not change the order capacity of the network when the following condition is satisfied.

$$D^{(4-\alpha)}(n) = \Omega(n^{(\alpha-4)/2}). \quad (28)$$

Proof: The proof is immediate by combining (27) and (28) and showing that the lower bound of T is

$$T \geq \sqrt{2}\left(\frac{8\pi c_1(\alpha-2)\beta}{\delta}\right)^{\frac{1}{\alpha}}. \quad (29)$$

□

Next, let us define $\overline{\#\text{MEMKTC}(D(n))}$ as the total number of cells that contain all the nodes in an (n,m,k)-cast group. The following lemma establishes the achievable lower bound for the (n,m,k)-cast throughput capacity of MPR as a function of $\overline{\#\text{MEMKTC}(D(n))}$.

Lemma 13. The achievable lower bound of the (n,m,k)-cast capacity is given by $C_{m,k}(n) = \Omega\left(1/\overline{\#\text{MEMKTC}(D(n))}\right)$

Proof: There are $(D(n)/\sqrt{2})^{-2}$ cells in the unit square network area. From the definition of $\overline{\#\text{MEMKTC}(D(n))}$ and the fact that our TDMA scheme does not change the order capacity (Lemma 12), it is clear that there are at most in the order of $\overline{\#\text{MEMKTC}(D(n))}$ interfering cells for any (n,m,k)-cast communication. For each cell, the order of nodes in each cell is $\Theta\left(\pi D^2(n)n\right)$. Accordingly, the total lower bound capacity is given by

$$C_{m,k}(n) = \Omega\left(\frac{1}{\overline{\#\text{MEMKTC}(D(n))}} \frac{\left(\pi D^2(n)n\right)}{(\frac{D(n)}{\sqrt{2}})^2}\right). \quad (30)$$

Normalizing this value by total number of nodes in the network n, proves the lemma. □

Given the above lemma, to express the lower bound of $C_{m,k}(n)$ as a function of network parameters, we need to compute the upper bound of $\overline{\#\text{MEMKTC}(D(n))}$, which we do next.

Lemma 14. The average number of cells covered by the nodes in MEMKTC$(D(n))$, is upper bounded w.h.p. as follows:

$$\overline{\#\text{MEMKTC}(D(n))} = \begin{cases} O\left(\frac{k}{\sqrt{m}D(n)}\right), m = O\left(D^{-2}(n)\right) \\ O\left(k\right), \Omega(k) = D^{-2}(n) = O(m) \\ O\left(D^{-2}(n)\right), k = \Omega(D^{-2}(n)). \end{cases} \quad (31)$$

Proof: The proof can be found in [1]. □

Combining Lemmas 13 and 14, we arrive at the achievable lower bound of the (n,m,k)-cast throughput capacity in dense random wireless ad hoc networks with MPR based on MLD.

Theorem 4. The achievable lower bound of the (n,m,k)-cast throughput capacity with MPR based on MLD is

$$C_{m,k}(n) = \begin{cases} \Omega\left((k)^{-1}\sqrt{m}D(n)\right), m = O\left(D^{-2}(n)\right) \\ \Omega\left(k^{-1}\right), \Omega(k) = D^{-2}(n) = O(m) \\ \Omega\left(D^2(n)\right), k = \Omega(D^{-2}(n)). \end{cases} \quad (32)$$

A.2 MPR with SIC Decoding

In this subsection we prove the lower bound of the throughput capacity for MPR with SIC.

Lemma 15. Under the physical model based on MPR with SIC, a receiver node can decode successfully all the nodes within a range of $D(n)$ by the proper choice of TDMA parameter T.

Proof: Each node in shaded cells can successfully and simultaneously decode packets transmitted from the decoding region $D(n)$ if (4) is satisfied at the circumference of $D(n)$.

Utilizing the result from (22) and considering signal power at the circumference of the circle of radius $D(n)$, the SINR at node X_j is computed as

$$\text{SINR}_{Z(D(n))\to j} \geq \frac{\frac{P}{D^{\alpha}(n)}}{N + \frac{8\pi D^2(n)nP}{S^{\alpha}(n)}c_1} \geq \beta. \quad (33)$$

Since $D(n) = \Omega(\sqrt{\log n/n})$ and the second term in the denominator goes to infinity by increasing n, we can ignore the noise term. Thus, the following relationship between $D(n)$ and $S(n)$ is derived as $S(n) \geq (8\pi c_1\beta)^{\frac{1}{\alpha}} n^{\frac{1}{\alpha}} D^{1+\frac{2}{\alpha}}(n)$.

From Fig. 2(b), we have $S(n) = (T(n)-1)\frac{D(n)}{\sqrt{2}}$. Following similar steps as before we arrive at

$$\sqrt{2}\{nD^{\alpha+2}(n)(8\pi c_1\beta)\}^{\frac{1}{\alpha}} + 1 \leq T(n). \quad (34)$$

Note that the TDMA parameter is a function of n and it changes the order of throughput capacity. □

Lemma 16. The achievable lower bound with SIC decoding for (n,m,k)-cast capacity is given by

$$C_{m,k}(n) = \Omega\left(\left(\overline{\#\text{MEMKTC}(D(n))}T(n)\right)^{-1}\right) \quad (35)$$

Proof: The proof is similar to before except that the TDMA parameter $T(n)$ should be considered. □

By combining Lemmas 6 and 16, we arrive at

Theorem 5. The achievable lower bound of the (n, m, k)-cast throughput capacity with MPR based on SIC is

$$C_{m,k}(n) = \begin{cases} \Omega\left((kT(n))^{-1}\sqrt{m}D(n)\right) \ , m = O\left(D^{-2}(n)\right) \\ \Omega\left((kT(n))^{-1}\right) , \ \Omega(k) = D^{-2}(n) = O(m) \\ \Omega\left((T(n))^{-1}D^{2}(n)\right) , \ k = \Omega(D^{-2}(n)). \end{cases} \tag{36}$$

From the result so far, it can be deduced that MLD in physical model is equivalent to MPR in protocol model.

V. CONCLUSION

We presented the throughput capacity of (n, m, k)-casting model when nodes are communicating based on point to point packet transmission. A new upper bound of $O\left(\sqrt{m}/\sqrt{n}k\right)$ and similar lower bound results to [1] were derived. The lower bound capacity consists of three different regions with values of $\Omega(\sqrt{m}/k\sqrt{n \log n})$, $\Omega(1/k \log n)$, and $\Omega(1/n)$ when $m = O(n/\log n)$, $\Omega(k) = (n/\log n) = O(m)$, and $\Omega(n/\log n) = k$, respectively. It is worth investigating as future work, if the gap in the physical model for (n, m, k)-cast can be closed using percolation theory. The achievable throughput capacity assuming MPR scheme is also provided for (n, m, k)-casting. Based on MLD, we proved that in case of $D(n) = R(n)$, the protocol and physical models actually render the same achievable throughput capacity. On the other hand, the suboptimum SIC decoding reduces the capacity under the physical model by a factor of $T(n)$. As a future work, we investigate the upper bound of the throughput capacity for both MLD and SIC.

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