On the Capacity Improvement of Multicast Traffic with Network Coding

Zheng Wang[†], Shirish Karande[†], Hamid R. Sadjadpour[†], J.J. Garcia-Luna-Aceves[‡]
Department of Electrical Engineering[†] and Computer Engineering[‡]
University of California, Santa Cruz, 1156 High Street, Santa Cruz, CA 95064, USA
[‡] Palo Alto Research Center (PARC), 3333 Coyote Hill Road, Palo Alto, CA 94304, USA
Email:{wzgold, karandes, hamid, jj}@soe.ucsc.edu

***Abstract*—In this paper, we study the contribution of network coding (NC) in improving the multicast capacity of random wireless ad hoc networks when nodes are endowed with multi-packet transmission (MPT) and multi-packet reception (MPR) capabilities. We show that a per session throughput capacity of $\Theta\left(nT^3(n)\right)$, where n is the total number of nodes and $T(n)$ is the communication range, can be achieved as a tight bound when each session contains a constant number of sinks. Surprisingly, an identical order capacity can be achieved when nodes have only MPR and MPT capabilities. This result proves that NC does not contribute to the order capacity of multicast traffic in wireless ad hoc networks when MPR and MPT are used in the network. The result is in sharp contrast to the general belief (conjecture) that NC improves the order capacity of multicast. Furthermore, if the communication range is selected to guarantee the connectivity in the network, i.e., $T(n) \geq \Theta\left(\sqrt{\log n/n}\right)$, then the combination of MPR and MPT achieves a throughput capacity of $\Theta\left(\frac{\log^{\frac{3}{2}} n}{\sqrt{n}}\right)$ which provides an order capacity gain of $\Theta\left(\log^2 n\right)$ compared to the point-to-point multicast capacity with the same number of destinations.**

I. Introduction

The seminal work by Gupta and Kumar [1] has sparked a growing amount of interest in understanding the fundamental capacity limits of wireless ad hoc networks. Several techniques [2]–[4] have been developed with the objective of improving the capacity of wireless ad hoc networks. Network coding (NC), which was originally proposed by Ahlswede et al. in [5], is one such technique. Unlike traditional store-and-forward routing, network coding encodes the messages received at intermediate nodes, prior to forwarding them to subsequent next-hop neighbors. Ahlswede et al. [5] showed that network coding can achieve a multicast flow equal to the min-cut for a single source and under the assumptions of a directed graph. This and other work in NC [6], [7] has motivated a large number of researchers to investigate the impact of NC in increasing the throughput capacity of wireless ad hoc networks. However, Liu et al. [8] recently showed that NC does not increase the order of the throughput capacity for multi-pair unicast traffic. Nevertheless, a number of efforts (analog network coding [9], physical network coding [10]) have continued the quest for improving the multicast capacity of ad-hoc networks by using NC. Despite the claims of throughput improvement by such studies, the impact of NC on the multicast scaling law has remained uncharacterized.

Promising approaches [9], [10] implicitly assume the combination of NC with Multi-packet Transmission (MPT) and Multi-packet Reception (MPR) [11]–[13] (i.e., the ability to transceive successfully multiple concurrent transmissions by employing physical-layer interference cancelation techniques). MPR has been shown to increase the capacity regions of ad hoc networks [14], and very recently Garcia-Luna-Aceves et al. [15] have shown that the order capacity in wireless ad hoc networks subject to multi-pair unicast traffic is increased with MPR. These prior efforts raise three important following questions: (a) What is the multicast throughput order achieved by the combination of NC with MPT and MPR? (b) Does this combination provide us with an order gain over traditional techniques based on routing and point-to-point communication? (c) If yes, what exactly leads to this gain? Is NC necessary or does the combination of MPT and MPR suffice?

In this work, we address the above three questions. The answers can be summarized by our main results:

- When each multicast group consists of a constant number of sinks, the combination of NC, MPT and MPR provides a per session throughput capacity of $\Theta(nT^3(n))$, where $T(n)$ is the communication range.
- This scaling law represents an order gain of $\Theta(n^2T^4(n))$ over a combination of routing and single packet transmission/reception.
- The combination of only MPT and MPR is sufficient to achieve a per-session multicast throughput order of $\Theta(nT^3(n))$. Consequently, NC does not contribute to the multicast capacity when MPR and MPT are used in the network!

The remainder of this paper is organized as follows. In Section II, we give an overview of capacity analysis for NC, MPT, MPR, and other existing techniques. In Section III, we introduce the models we used. In Section IV and V, we give our main results with MPT and MPR when network coding is not used and used respectively. We conclude our paper in Section VI.

II. Literature Reviews

Gupta and Kumar in their seminal paper [1] proved that the throughput capacity in wireless ad hoc network is not scalable. Subsequently, many researchers have focused on identifying techniques that could alter this conclusion. Recently, Ozgur et

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al. [2] proposed a hierarchical cooperation technique based on virtual MIMO to achieve linear per source-destination capacity. Cooperation can be extended to the simultaneous transmission and reception at the various nodes in the network, which is called *many-to-many communication* and can result in significant improvement in capacity [3].

Since the original work by Ahlswede et al. [5], most of the research on network coding has focused on directed networks, where each communication link has a fixed direction. Li and Li [16] were the first to study the benefits of network coding in undirected networks, where each communication link is bidirectional. Their result [16] shows that, for a single unicast or broadcast session, there are no improvement with respect to throughput due to network coding. In the case of a single multicast session, such an improvement is bounded by a factor of two. Meanwhile, [11]–[13] studied the throughput capacity of NC in wireless ad hoc networks. However [11]–[13] employ network models that are fundamentally inconsistent with the more commonly accepted assumptions of ad-hoc networks [1]. Specifically, the model constraints of [11]–[13], [16], [17] differ as follows: All the prior works assume a single source for unicast, multicast or even broadcast. Aly et al. [12] and Kong et al. [13] differentiate the total nodes into source set, relay set and destination set. They do not allow all of the nodes to concurrently serve as sources, relays or destinations, as allowed in the work by Gupta and Kumar [1]. An even bigger limitation of these results is that they do not consider the impact of interference in wireless ad hoc networks.

In the absence of interference, the communication scenario equates an ideal case where a node can simultaneously transmit and receive from multiple nodes. Interference cancelation techniques such as MPT and MPR indeed enable nodes with the ability of multi-point communication within a communication range of $T(n)$. Thus, the model assumptions in [11]–[13] at the very least assume that nodes are capable of MPT and MPR. Similarly, works such as Physical-Layer Network Coding (PNC) [10] and Analog Network Coding [9] also implicitly assume the ability of MPT and MPR.

III. Network Model, Definitions, and Preliminaries

We assume a random wireless ad hoc network with n nodes distributed uniformly in a unit-square network area. Our capacity analysis is based on the protocol model for dense networks, introduced by Gupta and Kumar [1]. The case of what we call point-to-point communication corresponds to the original protocol model.

Definition 3.1: The Protocol Model of Point-to-Point Communication: All nodes use a common transmission range $r(n)$ for all their communication. Node X_i can successfully transmit to node X_j if for any node $X_k, k \neq i$, that transmits at the same time as X_i it is true that $|X_i - X_j| \leq r(n)$ and $|X_k - X_j| \geq (1+\Delta)r(n)$.

We make the following extensions to account for MPT and MPR capabilities at the transmitters and receivers, respectively. In wireless ad hoc networks with MPT (MPR) capability, any transmitter (receiver) node can transmit (receive) different information simultaneously to (from) multiple nodes within the circle whose radius is $T(n)$ [15]. We further assume that nodes cannot transmit and receive at the same time, which is equivalent to half-duplex communications [1]. From system point of view, MPT and MPR are dual if we consider the source and destination duality.

Definition 3.2: Feasible throughput capacity
In a wireless ad hoc network of n nodes where each source transmits its packets to m destinations, a throughput of $C_m(n)$ bits per second for each node is feasible if there is a spatial and temporal scheme for scheduling transmissions, such that, by operating the network in a multi-hop fashion and buffering at intermediate nodes when awaiting transmission, every node can send $C_m(n)$ bits per second on average to its m chosen destination nodes. That is, there is a $T < \infty$ such that in every time interval $[(i-1)T, iT]$ every node can send $TC_m(n)$ bits to its corresponding destination nodes.

Definition 3.3: Euclidean Minimum Spanning Tree (EMST)
Consider a connected undirected graph $G = (V, E)$, where V and E are sets of vertices and edges in the graph G, respectively. The EMST of G is a spanning tree of G with the minimum sum of Euclidean distances between connected vertices of this tree.

Definition 3.4: Minimum Euclidean Multicast Tree (MEMT$(T(n))$): The MEMT$(T(n))$ is a multicast tree in which the m destinations receive information from the source and this multicast tree has the minimum total Euclidean distance.

Definition 3.5: Minimum Area Multicast Tree (MAMT$(T(n))$): The MAMT$(T(n))$ in a multicast tree with m destinations for each source is a multicast tree that has minimum total area. Area of a multicast tree is defined as the total area covered by circles centered around each source or relay with radius of $T(n)$ (see Fig. 1).

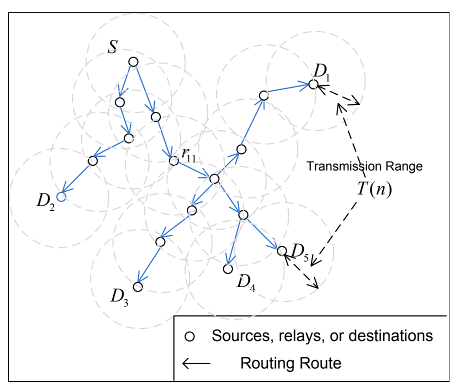

Fig. 1. Area Coverage By One Multicast Tree

Note that EMST and MEMT are spanning trees which includes only source and destinations, while MAMT is related to a real routing tree which includes relays.

Definition 3.6: Total Active Area (TAA$(\Delta, T(n))$):
The TAA$(\Delta, T(n))$ is the total area of the network multiplied by the average maximum number of simultaneous

transmissions and receptions inside a communication region of $\Theta(T^2(n))$.

It can be shown that this value has an upper bound of $O(1)$, $O(nT^2(n))$ and $O(n^2T^4(n))$ for point-to-point, MPR (or MPT) and MPR combined with MPT respectively.

In the rest of this paper, $\|T\|$ denotes the total Euclidean distance of a tree T; $\#T$ is used to denote the total number of vertices (nodes) in a tree T; $S(T)$ denotes the area of tree T covered; and $\overline{\|T\|}$ is used for the statistical average of the total Euclidean distance of a tree.

To compute the multicast capacity, we use the relationship between MAMT and EMST. Steele [18] determined a tight bound for $\overline{\|\text{EMST}\|}$ in the following lemma.

Lemma 3.7: Let $f(x)$ denote the node probability distribution function in the network area. Then, for large values of m and $d > 1$, the $\overline{\|\text{EMST}\|}$ is tight bounded as

$$\overline{\|\text{EMST}\|} = \Theta\left(c(d)m^{\frac{d-1}{d}}\int_{R^d} f(x)^{\frac{d-1}{d}}dx\right), \quad (1)$$

where d is the dimension of the network. Note that both $c(d)$ and the integral are constant values and not functions of m. When $d = 2$, then $\overline{\|\text{EMST}\|} = \Theta\left(\sqrt{m}\right)$.

Given that the distribution of nodes in a random network is uniform, if there are n nodes in a unit square, then the density of nodes equals n. Hence, if $|S|$ denotes the area of space region S, the expected number of the nodes, $E(N_S)$, in this area is given by $E(N_S) = n|S|$. Let N_j be a random variable defining the number of nodes in S_j . Then, for the family of variables N_j, we have the following standard results known as the Chernoff bounds [19]:

Lemma 3.8: Chernoff bound

For any $0 < \delta < 1$, we have

$$P[|N_j - n|S_j|| > \delta n|S_j|] < e^{-\theta n|S_j|}. \quad (2)$$

Therefore, for any $\theta > 0$, there exist constants such that deviations from the mean by more than these constants occur with probability approaching zero as $n \to \infty$. It follows that, w.h.p., we can get a very sharp concentration on the number of nodes in an area, so we can find the achievable lower bound w.h.p., provided that the upper bound (mean) is given. In the following sections, we first derive the upper bound, and then use the Chernoff bound to prove the achievable lower bound.

In [5] it was proved that the max-flow min-cut is equal to multicast capacity of a directed graph with single source. The directed graph model is more applicable for wired networks. However, in this work we wish to study the utility of NC in a wireless environment where links are bidirectional [11], [12].

In a single-source network, the cut capacity is equal to the maximum flow. Thus [12] provides an upper bound on the multicast capacity of a network with single source and NC+MPT+MPR capability. However, in [11]–[13], the source, relays and destinations are strictly different and information can not be transmitted directly towards the destinations. These two assumptions will be eventually relaxed in this paper.

IV. The Throughput Capacity with MPT and MPR

We now analyze the scaling laws in random geometric graphs with MPT and MPR abilities. Wang et al. [20] proved the unifying capacity with point-to-point communication, which resolves the general multicast case with m destinations for each source being a function of n. Here, we use a similar approach to prove the capacity with MPT and MPR when m is not a function of n but a constant.

A. Upper Bound

The following Lemma provides an upper bound for the per-session capacity as a function of $\overline{\text{TAA}(\Delta, T(n))}$ and $\overline{\text{MAMT}\,(T(n))}$. Essentially, $\overline{S\,(\text{MAMT}(T(n)))}$ equals the minimum area consumed to multicast a packet to m destinations (see Fig. 1), and $\overline{\text{TAA}(\Delta, T(n))}$ represents the maximum area which can be supported when MPT and MPR are used.

Lemma 4.1: In random dense wireless ad hoc networks, the per-node throughput capacity of multicast with MPT and MPR is given by $O\left(\frac{1}{n} \times \frac{\overline{\text{TAA}(\Delta,T(n))}}{\overline{S(\text{MAMT}(T(n)))}}\right)$.

Proof: With MPT and MPR, we observe that $\overline{S\,(\text{MAMT}(T(n)))}$ represents the total area required to transmit information from a multicast source to all its m destinations. The ratio between average total active area, $\overline{\text{TAA}(\Delta, T(n))}$, and $\overline{S\,(\text{MAMT}(T(n)))}$ represents the average number of simultaneous multicast communications that can occur in the network. Normalizing this ratio by n provides per-node capacity. ■

Lemma 4.1 provides the upper bound for the multicast throughput capacity with MPT and MPR as a function of $\overline{S\,(\text{MAMT}(T(n)))}$ and $\overline{\text{TAA}(\Delta, T(n))}$. In order to compute the upper bound, we derive the upper bound of $\overline{\text{TAA}(\Delta, T(n))}$ and the lower bound of $\overline{S\,(\text{MAMT}(T(n)))}$. Combining these results provides an upper bound for the multicast throughput capacity with MPT and MPR.

Lemma 4.2: The average area of a multicast tree with transmission range $T(n)$, $\overline{S\,(\text{MAMT}(T(n)))}$ is lower bounded by $\Omega\,(T(n))$, when m is a constant value.

Proof: From [21], it can be deduced that $\overline{S\,(\text{MAMT}(T(n)))}$ is lower bounded as $\Omega\left(\overline{\|\text{EMST}\|} \times T(n)\right)$. Even for the case of the minimum value for $T(n)$ to assure connectivity, this upper bound is guaranteed for constant values of m. Lemma 3.7 states that $\overline{\|\text{EMST}\|} = \Theta(\sqrt{m}) = \Theta(1)$. The proof follows immediately. ■

Lemma 4.3: The average total active area, $\overline{\text{TAA}(\Delta, T(n))}$, has the following upper bound in networks with MPT and MPR.

$$\overline{\text{TAA}(\Delta, T(n))} = O\left(n^2T^4(n)\right) \quad (3)$$

Proof: As discussed earlier, the $\overline{\text{TAA}(\Delta, T(n))}$ for point-to-point communication is equal to 1 since for each circle of radius $T(n)$, there is only a single pair of transmitter-receiver nodes (see Fig. 2). For the case of MPR and MPT, the number of nodes in a circle of radius $T(n)$ is upper bounded as $O(nT^2(n))$. This is also upper bound for the number of transmitters or receivers in this region. The upper bound for $\overline{\text{TAA}(\Delta, T(n))}$ is achieved when the maximum number of transmitter and receivers are employed in this circle. Figure 2 demonstrates an example that can achieve this upper bound simultaneously for transmitters and receivers. Let a circle of radius $\frac{T(n)}{2}$ located at the center of another circle of radius $T(n)$. Note that with this construction, any two nodes inside the small circle are connected. If we randomly assign half of the nodes inside the smaller circle as transmitters and the other half as receiver nodes, then the average number of transmitters and receivers in this circle are proportional to $\Theta(nT^2(n))$. Given the fact that this value also is the maximum possible number of transmitter and receiver nodes, the result follows immediately. ∎

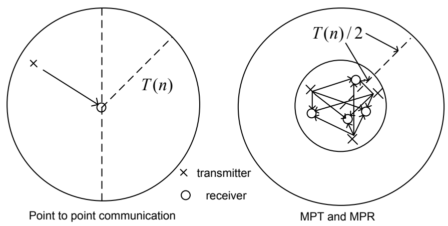

Fig. 2. Upper Bound of Total Available Area Based On Protocol Model

Combining Lemmas 4.1, 4.2 and 4.3, we can compute the upper bound for multicast capacity of MPT and MPR.

Theorem 4.4: In wireless ad hoc networks with MPT and MPR, the upper bound on the per-node throughput capacity of multicast with constant number of destinations is

$$C_m(n) = O\left(nT^3(n)\right) \tag{4}$$

B. Lower Bound

To derive an achievable lower bound, we use a TDMA scheme for random dense wireless ad hoc networks similar to the approach used in [22], [23].

We first divide the network area into square cells. Each square cell has an area of $T^2(n)/2$, which makes the diagonal length of square equal to $T(n)$, as shown in Fig. 3. Under this condition, connectivity inside all cells is guaranteed and all nodes inside a cell are within communication range of each other. We build a cell graph over the cells that are occupied with at least one vertex (node). Two cells are connected if there exist a pair of nodes, one in each cell, that are less than or equal to $T(n)$ distance apart. Because the whole network is connected when $T(n) = r(n) \geq \Theta\left(\sqrt{\log n/n}\right)$, it follows that the cell graph is connected [22], [23].

To satisfy the MPT and MPR protocol model, we organize cells in groups so that simultaneous transmissions within each group does not violate the conditions for successful communication in the MPT and MPR protocol model. Let L represent the minimum number of cell separations in each group of cells that communicate simultaneously. Utilizing the protocol model, L satisfies the following condition:

$$L = \left\lceil 1 + \frac{T(n) + (1+\Delta)T(n)}{T(n)/\sqrt{2}} \right\rceil = \lceil 1 + \sqrt{2}(2+\Delta) \rceil. \tag{5}$$

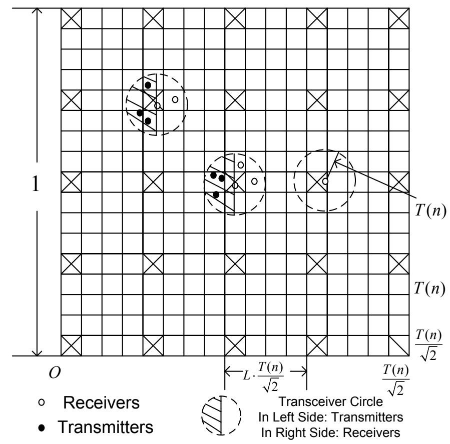

Fig. 3. Cell construction used to derive a lower bound on capacity

If we divide time into L^2 time slots and assign each time slot to a single group of cells, interference is avoided and the protocol model is satisfied. The separation example can be shown for the upper two receiver circles in Fig. 3. For the MPT and MPR protocol model, the distance between two adjacent receiving nodes is $(2+\Delta)T(n)$. Because this distance is smaller than $(L-1)T(n)$, this organization of cells guarantees that the MPT and MPR protocol model is satisfied. Fig. 3 represents one of these groups with a cross sign inside those cells for $L = 4$. We can derive an achievable multicast capacity for MPT and MPR by taking advantage of this cell arrangement and TDMA scheme. The capacity reduction caused by the TDMA scheme is a constant factor and does not change the order capacity of the network.

Next our objective is to find an achievable lower bound using the Chernoff bound, such that the distribution of the number of edges in this unit space is sharply concentrated around its mean, and hence the actual number of simultaneous transmissions occurring in the unit space in a randomly chosen network is indeed $\Theta(n^2T^2(n))$ w.h.p..

Lemma 4.5: The circular area of radius $T(n)$ corresponding to the transceiver range of any node j in the cross area in Fig. 3 contains $\Theta(nT^2(n))$ nodes w.h.p., and is uniformly distributed for all values of j, $1 \leq j \leq \frac{1}{(LT(n)/\sqrt{2})^2}$.

Proof: The statement of this lemma can be expressed as

$$\lim_{n\to\infty} P\left[\bigcap_{j=1}^{\frac{1}{(LT(n)/\sqrt{2})^2}} |N_j - E(N_j)| < \delta E(N_j)\right] = 1, \tag{6}$$

where N_j and $E\left(N_j\right)$ are the random variables that represent the number of transmitters in the receiver circle of radius $T(n)$ centered by the receiver j and the expected value of this random variable respectively, and δ is a positive arbitrarily small value close to zero.

From the Chernoff bound in Eq. (2), for any given $0 < \delta < 1$, we can find $\theta > 0$ such that $P\left[|N_j - E(N_j)| > \delta E(N_j)\right] < e^{-\theta E(N_j)}$. Thus, we can conclude that the probability that the value of the random variable N_j deviates by an arbitrarily small constant value from the mean tends to zero as $n \to \infty$. This is a key step in showing that when all the events $\bigcap_{j=1}^{\frac{1}{(LT(n)/\sqrt{2})^2}} |N_j - E(N_j)| < \delta E(N_j)$ occur simultaneously, then all N_j's converge uniformly to their expected values. Utilizing the union bound, we arrive at

$$\begin{aligned} & P\left[\bigcap_{j=1}^{\frac{1}{(LT(n)/\sqrt{2})^2}} |N_j - E(N_j)| < \delta E(N_j)\right] \\ \geq\ & 1 - \sum_{j=1}^{\frac{1}{(LT(n)/\sqrt{2})^2}} P\left[|N_j - E(N_j)| > \delta E(N_j)\right] \\ >\ & 1 - \frac{1}{(LT(n)/\sqrt{2})^2} e^{-\theta E(N_j)}. \end{aligned} \tag{7}$$

Given that $E(N_j) = \pi n T^2(n)$, then we have

$$\begin{aligned} & \lim_{n\to\infty} P\left[\bigcap_{j=1}^{\frac{1}{(LT(n)/\sqrt{2})^2}} |N_j - E(N_j)| < \delta E(N_j)\right] \\ & \geq 1 - \lim_{n\to\infty} \frac{1}{(LT(n)/\sqrt{2})^2} e^{-\theta\pi n T^2(n)} \end{aligned} \tag{8}$$

Utilizing the connectivity criterion, $\lim_{n\to\infty} \frac{e^{-\theta\pi n T^2(n)}}{T^2(n)} \to 0$, which finishes the proof. ∎

Furthermore, we can arrange all of the nodes in the left side of the corresponding transceiver circle be the transmitters, and all of the nodes in the right side of the corresponding transceiver circle be the receivers. Thus, we arrive at the following lemma.

Lemma 4.6: In the unit square area for a wireless ad hoc network shown in Fig. 3, the total number of transmitter-receiver links (simultaneous transmissions) is $\Omega\left(n^2 T^2(n)\right)$.

Proof: From Lemma 4.5, for any node in the cross cell in the whole network shown in Fig. 3, there are $\Theta(nT^2(n))$ nodes in the transceiver circle. We divided the total nodes into two categories, transmitters in the left of the transceiver circles and receivers in the right of the transceiver circles. To guarantee all of the transmitters and receivers are in the transceiver range, we only consider the nodes in the circle with radius $T(n)/2$. Because of the MPT and MPR capabilities, so that every transmitter in the left of the transceiver circle with $T(n)/2$ radius can transmit successfully to every receiver in the right, then the total number of successful transmissions is $\pi^2 n^2 T^4(n)/16$ which is the achievable lower bound. The actual number of the transmissions can be much larger than this because we only consider $T(n)/2$ instead of $T(n)$. Using the Chernoff Bound in Eq. 2 and Lemma 4.5, we can get w.h.p. that the total number of successful transmissions is

$$\Omega\left(\frac{1}{\left(LT(n)/\sqrt{2}\right)^2} \times \frac{\pi^2 n^2 T^4(n)}{16}\right) = \Omega\left(n^2 T^2(n)\right) \tag{9}$$

∎

The above results enables us to obtain the following achievable lower bound.

Let us define $\overline{\#\text{MEMTC}(T(n))}$ as the total number of cells that contain all the nodes in a multicast group. The following lemma establishes the achievable lower bound for the multicast throughput capacity of MPT and MPR as a function of $\overline{\#\text{MEMTC}(T(n))}$.

Lemma 4.7: The achievable lower bound of the multicast capacity is given by

$$C_m(n) = \Omega\left(\frac{nT^2(n)}{\overline{\#\text{MEMTC}(T(n))}}\right). \tag{10}$$

Proof: There are $(T(n)/\sqrt{2})^{-2}$ cells in the unit square network area. From the definition of $\overline{\#\text{MEMTC}(T(n))}$ and the fact that our TDMA scheme does not change the order capacity, it is clear that there are at most in the order of $\overline{\#\text{MEMTC}(T(n))}$ interfering cells for multicast communication. Hence, from Lemma 4.6, there are a total of $\Theta\left(n^2T^2(n)\right)$ nodes transmitting simultaneously, which are distributed over all the $(T(n)/\sqrt{2})^{-2}$ cells. For each cell, the order of nodes in each cell is $\Omega\left(n^2T^4(n)\right)$. Accordingly, the total lower bound capacity is given by $\Omega\left(\left(T(n)/\sqrt{2}\right)^{-2} \times \left(n^2T^4(n)\right) \times \left(\overline{\#\text{MEMTC}(T(n))}\right)^{-1}\right)$. Normalizing this value by total number of nodes in the network, n, proves the lemma. ∎

Given the above lemma, to express the lower bound of $C_m(n)$ as a function of network parameters, we need to compute the upper bound of $\overline{\#\text{MEMTC}(T(n))}$, which we do next.

Lemma 4.8: The average number of cells covered by the nodes in MEMTC$(T(n))$, is upper bounded w.h.p. as follows:

$$\overline{\#\text{MEMTC}(T(n))} \leq \Theta\left(\frac{\sqrt{m}}{T(n)}\right) \tag{11}$$

Proof: Because $T(n)$ is the transceiver range of the network, the maximum number of cells for this multicast tree must be at most $\Theta\left(\sqrt{m}T^{-1}(n)\right)$, i.e., $\overline{\#\text{MEMTC}\left(T(n)\right)} \leq \Theta\left(\sqrt{m}T^{-1}(n)\right)$. This upper bound can be achieved only if every two adjacent nodes in the multicast tree belong to two different cells in the network. However, in practice, it is possible that some adjacent nodes in multicast tree locate in a single cell. Consequently, this value is upper bound as described in (11). ∎

Combining Lemmas 4.7 and 4.8, we arrive at the achievable lower bound of the multicast throughput capacity in dense random wireless ad hoc networks with MPT and MPR.

Theorem 4.9: When the number of the destinations m is a constant, the achievable lower bound of the m multicast throughput capacity with MPT and MPR is

$$C_m(n) = \Omega\left(\frac{nT^3(n)}{\sqrt{m}}\right) \tag{12}$$

In this paper, we have utilized an edge-counting argument to calculate the per-tree capacity when n multicast trees are packed in the network. Due to brevity we have focused on an average case analysis and are aware that a more rigorous deduction requires us to show that even in the worst case, appropriate load balancing is maintained so as to not create any bottlenecks that diminish network capacity. Details of such an argument are similar to Lemma in [21] and have been reserved for a longer version of the current document.

C. Tight Bound and Comparison with Point-to-Point Communication

From Theorems 4.4 and 4.9, we can provide a tight bound throughput capacity for multicasting when nodes have MPT and MPR capabilities in dense random wireless ad hoc networks as follows.

Theorem 4.10: The throughput capacity of multicast in random dense wireless ad hoc network with MPT and MPR is

$$C_m^{\text{MPT+MPR}}(n) = \Theta\left(\frac{nT^3(n)}{\sqrt{m}}\right) \tag{13}$$

The transceiver range of MPT and MPR should satisfy $T(n) \geq \Theta\left(\sqrt{\log n/n}\right)$.

The multicast throughput capacity with point-to-point communication is given by the following lemma [20].

Lemma 4.11: In multicast with a constant number m of destinations, without MPR or MPR ability, the capacity is

$$C_m^{\text{Routing}}(n) = \Theta\left(\frac{1}{\sqrt{m}nr(n)}\right) \tag{14}$$

where, $r(n) \geq \Theta\left(\sqrt{\log n/n}\right)$. When $r(n) = \Theta\left(\sqrt{\log n/n}\right)$ for the minimum transmission range to guarantee the connectivity, then we obtain the maximum capacity as $C_m^{\text{Routing-Max}}(n) = \Theta\left(\frac{1}{\sqrt{mn\log n}}\right)$.

Combining Theorem 4.10 with Lemma 4.11, the gain of throughput capacity with MPT and MPR capability in wireless ad hoc networks can be stated as follows.

Theorem 4.12: In multicast with a constant number m of destinations, with MPT and MPR ability, the gain of per-node throughput capacity compared with point-to-point communication is $\Theta\left(n^2T^4(n)\right)$, where, $T(n) = r(n) \geq \Theta\left(\sqrt{\log n/n}\right)$. When $T(n) = \Theta\left(\sqrt{\log n/n}\right)$, the gain of per-node capacity is at least $\Theta\left(\log^2 n\right)$.

V. Capacity with NC, MPT and MPR

We now study the multi-source multicast capacity of a wireless network *in the absence of interference* when nodes use NC. The results we present serve as an upper-bound for what can be achieved by combining NC, MPT and MPR in the presence of interference. Our arguments are generic and can be used to deduce upper bounds for the multicast capacity of other interesting cases where NC is used along with only one of MPT or MPR, or even the scenario where NC is used with traditional single packet transmission and reception.

We deduce the bounds for the case of multi-source multicasting by reducing it to a suitable unicast routing problem. Under the reduction, an upper bound for the unicast problem also serves for the original multicast routing problem. Thus consider the following simple yet powerful lemma

Lemma 5.1: Consider a network with n nodes $V = \{a_1, \ldots, a_n\}$ and k multicast sessions. Each session consists of one of the n nodes acting as a source with an arbitrary finite subset of V acting as the set of destinations. Let s_i be the source of the i^{th} session and let $D_i = \{d_{i1}, \ldots, d_{im_i}\}$ be the set of m_i destinations. Now, there exists a joint routing-coding-scheduling scheme that can realize a throughput of λ_i for the i^{th} session, i.e. $\lambda = [\lambda_1, \ldots \lambda_k]$ is a feasible rate vector. Then λ is also a feasible vector for any unicast routing problem in the same network such that the traffic consists of k unicast sessions with s_i being the source of the i^{th} session and the destination b_i is any arbitrary element of the set D_i.

If a multicast capacity from a source to multiple destinations is feasible, then clearly it is feasible to achieve the same capacity to only any single node from this set of destinations.

Lemma 5.2: Consider a random geometric network with n nodes distributed uniformly in a unit square. Consider a decomposition of the unit-square into two disjoint regions R and R^c such that the area of each region is of order $\Theta(1)$. Now consider a multicast traffic scenario consisting of n sessions with each node being the source of a session and m randomly chosen nodes being the destination of the session. We say that a source satisfies property P if the source belongs to region R and at least one of its destination belongs to R^c OR if the source belongs to region R^c and at least one of its destination belongs to R. It can be shown that w.h.p the number of sources satisfying property P are $\Theta(n)$

Theorem 5.3: In a wireless ad hoc network formed by n nodes distributed randomly in a unit square with traffic formed by each node acting as source for a multicast sessions with $m = \Theta(1)$ randomly chosen nodes as destinations, the per-session multicast capacities are

$$C_m^{\text{NC+PTP}} = \Theta\left(\frac{1}{nT(n)}\right) \tag{15}$$

$$C_m^{\text{NC+MPT}} = C_m^{\text{NC+MPR}} = \Theta\left(T(n)\right) \tag{16}$$

$$C_m^{\text{NC+MPT+MPR}} = \Theta\left(nT^3(n)\right) \tag{17}$$

where NC + PTP denotes the use of NC with point-to-point communication (no MPT or MPR), i.e., a node can only transmit or receive at most one packet at a time.

Proof: (Sketch) For any sparsity cut of the unit area as illustrated in Fig. 4, the middle line induces a sparsity cut, lemmas 5.2. 5.1 tell us that we can construct a unicast routing problem satisfying the property that any rate for the unicast problem is feasible for the original multicast problem and we have $\Theta(n)$ source-destination pairs across the cut. Thus, the capacity of the sparsity cut provides an upper bound for the unicast problem, which can in turn be used to provide a bound for the multicast problem. Liu et al. [8] showed that the maximum number of packets that can be simultaneously transmitted across the cut is $\Theta\left(\frac{1}{T(n)}\right)$ for the case of unicast with point-to-point communication and NC. With similar arguments, we can show that the combination of NC+MPT or NC+MPR allows us to transmit a maximum of $\Theta\left(nT(n)\right)$ packets across the cut. Finally, we can extend such arguments

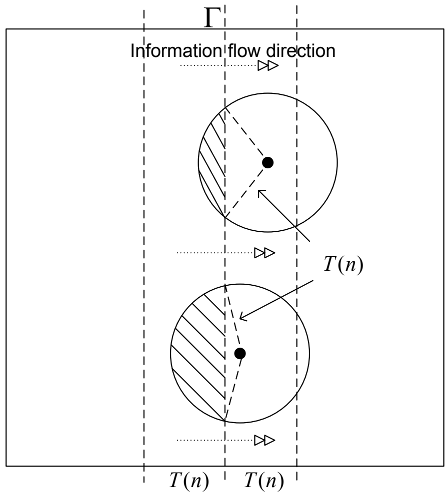

Fig. 4. All the nodes in the shaded region can send a message simultaneously

to show that the combination of NC+MPT+MPR allows us to simultaneously transmit a maximum of $\Theta\left(n^2T^3(n)\right)$ packets across the cut. The result of the theorem then follows from the fact that the cut capacity has to be divided among the $\Theta(n)$ source-destination pairs across the cut. ∎

VI. Discussion

By combining the results from theorems 4.10 and 5.3, the main contribution of this paper is stated in the following theorem.

Theorem 6.1: In wireless ad hoc networks with multi-pair multicast sessions and with a finite number of destinations for each source (m), the throughput capacity utilizing NC, MPT and MPR capabilities for all nodes is the same order as when the nodes are endowed only with MPT and MPR.

$$C_m^{\text{MPT+MPR+NC}}(n) = C_m^{\text{MPT+MPR}}(n) \tag{18}$$

It is also important to emphasize that, as Theorem 5.3 shows, NC does not provide any order capacity gain for multi-source multicasting when the size of receiver groups is $m = \Theta(1)$ and nodes use point-to-point communication. Hence, the result in Theorem 6.1 implies that NC does not provide an order capacity gain when either MPT and MPR are used, or point-to-point communication is used, and that MPT and MPR are the real contributing factor for order capacity increases in wireless networks.

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