On the Capacity Improvement of Multicast Traffic with Network Coding

Zheng Wang†, Shirish Karande†, Hamid R. Sadjadpour†, J.J. Garcia-Luna-Aceves‡

Department of Electrical Engineering† and Computer Engineering‡
University of California, Santa Cruz, 1156 High Street, Santa Cruz, CA 95064, USA
‡ Palo Alto Research Center (PARC), 3333 Coyote Hill Road, Palo Alto, CA 94304, USA
Email:{wzgold, karandes, hamid, jj}@soe.ucsc.edu

***Abstract*—In this paper, we study the contribution of network coding (NC) in improving the multicast capacity of random wireless ad hoc networks when nodes are endowed with multi-packet transmission (MPT) and multi-packet reception (MPR) capabilities. We show that a per session throughput capacity of $\Theta\left(nT^3(n)\right)$,where n is the total number of nodes and $T(n)$ is the transmission range, can be achieved as a tight bound when each session contains a constant number of sinks. Surprisingly, an identical order capacity can be achieved when nodes have only MPR and MPT capabilities. This result proves that NC does not contribute to the order capacity of multicast traffic in wireless ad hoc networks. The result is in sharp contrast to the general belief (conjecture) that NC improves the order capacity of multicast. Furthermore, if the communication range is selected to guarantee the connectivity in the network, i.e., $T(n) \geq \Theta\left(\sqrt{\log n/n}\right)$, then the combination of MPR and MPT achieves a throughput capacity of $\Theta\left(\frac{\log^{\frac{3}{2}} n}{\sqrt{n}}\right)$ which provides an order capacity gain of $\Theta\left(\log^2 n\right)$ compared to the point-to-point communication capacity that was reported by Gupta and Kumar.**

I. Introduction

The seminal work by Gupta and Kumar [1] has sparked a growing amount of interest in understanding the fundamental capacity limits of wireless ad hoc networks. Several techniques [2]–[8] have been developed with the objective of improving the capacity of wireless ad hoc networks. Network coding (NC), which was originally proposed by Ahlswede et al. in [9], is one such technique. Unlike traditional store-and-forward routing algorithms, network coding schemes encode the messages received at intermediate nodes, prior to forwarding them to subsequent next-hop neighbors. In [9] it has been shown that for a single source, and under the assumptions of a directed graph, network coding can achieve a multicast flow equal to the min-cut.

The work in [9]–[11] has motivated a large number of researchers to investigate the impact of NC in increasing the throughput capacity of wireless ad hoc networks. However, Liu et al. [12] recently showed that NC does not increase the order of throughput capacity for multi-pair unicast traffic. Nevertheless, a number of efforts (analog network coding [13], physical network coding [14]) have continued the the quest for improving the multicast capacity of ad-hoc networks by using NC. Despite the claims of throughput improvement by such studies, the impact of NC on the multicast scaling law remains uncharacterized.

Approaches such as [13], [14] implicitly assume the combination of NC (transmitting multiple packets encoded in a single transmission) with Multi-packet Transmission (MPT) and Multi-packet Reception (MPR) [15]–[17](ability to transceive successfully multiple concurrent transmissions by employing physical-layer interference cancelation techniques). MPR has been shown to increase the capacity regions of ad hoc networks [18], and very recently Garcia-Luna-Aceves et al. [19] have shown that the order capacity in wireless ad hoc networks subject to multi-pair unicast traffic is increased with MPR. These prior efforts raise the following question: (a) What is the multicast throughput order achieved by the combination of NC with MPT and MPR ? (b) Does this combination provide us with a order gain over tractional techniques based on routing and point-to-point communication ? (c) If yes, what exactly leads to this gain? Is NC necessary or is the combination of MPT and MPR sufficient?

In this work we address the above questions. The answers can be summarized by our main results:

- When each multicast group consists of a constant number of sinks, the combination of NC, MPT and MPR provides a per session throughput capacity of $\Theta(nT^3(n))$, where $T(n)$ is the transmission range.
- This scaling law represents an order gain of $\Theta(n^2T^4(n))$ over a combination of routing and single packet transmission/reception.
- The combination of only MPT and MPR is sufficient to achieve a per-session multicast throughput order of $\Theta(nT^3(n))$. Consequently, NC DOES NOT ADD TO THE MULTICAST CAPACITY !!!!

The remainder of this paper is organized as follows. In Section II, we give an overview of capacity analysis for NC, MPT, MPR, and other existing techniques. In Section III, we introduce the models we used. In Section IV and V, we give our main results with MPT and MPR when network coding is not used and used respectively. We conclude our paper in Section VI.

II. LITERATURE REVIEWS

Gupta and Kumar in their seminal paper [1] proved that the throughput capacity in wireless ad hoc network is not scalable. Subsequently, many researchers have focused on identifying techniques that could alter this conclusion. It has been shown that changing physical layer assumptions, such as using multiple channels [2] or ultra wide band (UWB) technology [3], [4], can increase the capacity of wireless ad hoc networks. Recently, Ozgur et al. [5] proposed a hierarchical cooperation technique based on virtual MIMO to achieve linear per source-destination capacity. Cooperation can be extended to the simultaneous transmission and reception at the various nodes in the network, which is so called *many-to-many communication* and can result in significant improvement in capacity [6].

Most of the the research on network coding, since the original proposal in [9], has focused on the model of directed networks, where each communication link has a fixed direction. [20] were the first to study the benefits of network coding in undirected networks, where each communication link is bidirectional. The result in [20] shows that, for a single unicast or broadcast session, there are no improvement with respect to throughput due to network coding. In the case of a single multicast session, such an improvement is bounded by a factor of two. Meanwhile, [15]–[17] studied the throughput capacity of NC in wireless ad hoc networks. However [15]–[17] employ network models which are fundamentally inconsistent with the more commonly accepted assumptions of ad-hoc networks [1].Specifically, the model constraints of [15]–[17], [20], [21] differ as follows:(a) All the prior works assume a single source for unicast, multicast or even broadcast (b) [16], [17] differentiate the total nodes into source set, relay set and destination set. They don't allow all of the nodes to concurrently serve as sources, relays or destinations, as allowed in [1]. (b) An even bigger limitation of these works is that they do not consider the impact of interference in wireless ad hoc networks.

In the absence of interference, the communication scenario equates an ideal case where a node can simultaneously transmit and receive from multiple nodes. Interference cancelation techniques such as MPT and MPR indeed enable nodes with the ability of multi-point communication within a communication range of $T(n)$. Thus, the model assumptions in [15]–[17] at the very least assume that nodes are capable of MPT and MPR. Similarly works such as Physical-Layer Network Coding (PNC) [14] by Zhang et al. and Analog Network Coding [13] by Katti et al. also implicitly assume the ability of MPT and MPR.

III. NETWORK MODEL, DEFINITIONS, AND PRELIMINARIES

We assume a random wireless ad hoc network with n nodes distributed uniformly in a unit-square network area. Our capacity analysis is based on the protocol model for dense networks, introduced by Gupta and Kumar [1]. The case of what we call point-to-point communication corresponds to the original protocol model.

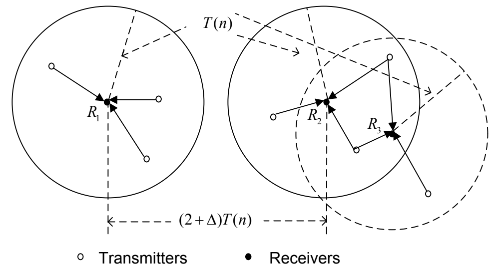

Fig. 1. MPT and MPR protocol model

Definition 3.1: The Protocol Model of Point-to-Point Communication: All nodes use a common transmission range $r(n)$ for all their communication. The network area is assumed to be a unit square area. Node X_i can successfully transmit to node X_j if for any node $X_k, k \neq i$, that transmits at the same time as X_i it is true that $|X_i - X_j| \leq T(n)$ and $|X_k - X_j| \geq (1 + \Delta)T(n)$.

We make the following extensions to account for MPT and MPR capabilities at the transmitters and receivers respectively. In wireless ad hoc networks with MPT capability, any transmitter node can transmit different information simultaneously to multiple nodes within the circle whose radius is $T(n)$. Similarly, in wireless ad hoc networks with MPR capability, any node can receive different information simultaneously from multiple transmitters within the circle whose radius is $T(n)$ [19]. We further assume that nodes cannot transmit and receive at the same time, which is equivalent to half-duplex communications [1]. From system point of view, MPT and MPR are dual if we consider the source and destination duality. The MPT and MPR model are shown in Fig. 1. It should be noted that R_3 is the another receiver in the reception circle of R_2. Since R_3 is already a receiver R_2 can't receive any information from R_3 in the same time slot because of the half-duplex communication.

Definition 3.2: Feasible throughput capacity: In a wireless ad hoc network of n nodes where each source transmits its packets to m destinations, a throughput of $C_m(n)$ bits per second for each node is feasible if there is a spatial and temporal scheme for scheduling transmissions, such that by operating the network in a multi-hop fashion and buffering at intermediate nodes when awaiting transmission, every node can send $C_m(n)$ bits per second on average to its m chosen destination nodes. That is, there is a $T < \infty$ such that in every time interval $[(i-1)T, iT]$ every node can send $TC_m(n)$ bits to its corresponding destination nodes.

Definition 3.3: Order of throughput capacity: $C_m(n)$ is said to be of order $\Theta(f(n))$ bits per second if there exist deterministic positive constants c and c' such that

$$\begin{cases} \lim_{n\to\infty} \text{ Prob } (C_m(n) = cf(n) \text{ is feasible}) = 1 \\ \lim_{n\to\infty} \text{ Prob } (C_m(n) = c'f(n) \text{ is feasible}) < 1. \end{cases} \quad (1)$$

Definition 3.4: Euclidean Minimum Spanning Tree (EMST): Consider a connected undirected graph $G = (V, E)$, where V and E are sets of vertices and edges in the graph G, respectively. The EMST of G is a spanning tree of G with the minimum sum of Euclidean distances between connected vertices of this tree.

Definition 3.5: Minimum Euclidean Multicast Tree (*MEMT*$(T(n))$)*:* The MEMT$(T(n))$ is a multicast tree in which the m destinations for each source receive information from the source and this multicast tree has the minimum total Euclidean distance.

In the rest of this paper, $\|T\|$ denotes the total Euclidean distance of a tree T; $\#T$ is used to denote the total number of vertices (nodes) in a tree T; and $\overline{\|T\|}$ is used for the statistical average of the total Euclidean distance of a tree.

Keshavarz et al. [22] used Maximum Independent Set MIS$(\Delta, r(n))$ to describe the maximum number of simultaneous transmitters. Similarly, to account for MPT and MPR ability, we define the Maximum MPT and MPR Independent Set (MMMIS$(\Delta, T(n))$.

Definition 3.6: Maximum MPT and MPR Independent Set (*MMMIS*$(\Delta, T(n))$)*:* An MPT and MPR independent set is a set of nodes in G that contains one transceiver and all its transceiver nodes within a distance of $T(n)$ from the transceiver node. A Maximum MPT and MPR Independent Set (MMMIS$(\Delta, T(n))$) consists of the maximum number channel links of MPT and MPR sets that simultaneously transceive packets while MPT and MPR protocol model is satisfied for all these MPT and MPR sets. If we add any transceiver node from G to MMMIS$(\Delta, T(n))$, there is at least one MPT and MPR set that violates the MPT and MPR protocol model.

Definition 3.7: Minimum Connected Dominating Set (*MCDS*$(r(n))$)*:* A dominating set (DS$(r(n))$) of a graph G is defined as a set of nodes such that every node in the network either belongs to this set or it is within a transmission range of $r(n)$ of one of the elements of DS$(r(n))$. A Connected Dominating Set (CDS$(r(n))$) is a dominating set such that the subgraph induced by its nodes is connected. A Minimum Connected Dominating Set (MCDS$(r(n))$) is a CDS$(r(n))$ of G with the minimum number of nodes.

Given that the distribution of nodes in a random network is uniform, if there are n nodes in a unit square, then the density of nodes equals n. Hence, if $|S|$ denotes the area of space region S, the expected number of the nodes, $E(N_S)$, in this area is given by $E(N_S) = n|S|$. Let N_j be a random variable defining the number of nodes in S_j . Then, for the family of variables N_j, we have the following standard results known as the Chernoff bounds [23]:

Lemma 3.8: Chernoff bound

- For any $\delta > 0$, $P[N_j > (1+\delta)n|S_j|] < \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^{n|S_j|}$
- For any $0 < \delta < 1$, $P[N_j < (1-\delta)n|S_j|] < e^{-\frac{1}{2}n|S_j|\delta^2}$

Combining these two inequalities we have, for any $0 < \delta < 1$:

$$P[|N_j - n|S_j|| > \delta n|S_j|] < e^{-\theta n|S_j|}, \quad (2)$$

where $\theta = (1+\delta)\ln(1+\delta) - \delta$ in the case of the first bound, and $\theta = \frac{1}{2}\delta^2$ in the case of the second bound.

Therefore, for any $\theta > 0$, there exist constants such that deviations from the mean by more than these constants occur with probability approaching zero as $n \to \infty$. It follows that, w.h.p., we can get a very sharp concentration on the number of nodes in an area, so we can find the achievable lower bound w.h.p., provided that the upper bound (mean) is given. In the following sections, we first derive the upper bound, and then use the Chernoff bound to prove the achievable lower bound.

In [9] it was proved that the max-flow min-cut is equal to multicast capacity of a directed graph with single source. The directed graph model is more applicable for wired networks. However, in this work we wish to study the utility of NC in wireless environment, where the links are bidirectional. Hence,we utilize the following terminology, which builds upon the model set-up in [15], [16], for analysis of NC.

Definition 3.9: The connectivity graph $G = (V, E)$ is a subgraph of the random geometric graph with source s, a set T of destinations, and a set D of relay nodes such that $V = \{s\} \bigcup D \bigcup T$. In this paper, we only consider the case where $|T| = m$ is a constant and $|D| = n - m - 1$ for each multicast session.

The analysis in [16] can be summarized to obtain the following result for a connectivity graph.

Theorem 3.10: Let G be a connectivity graph with one source node s_i, $n - m - 1$ relay nodes, and a set T of destinations nodes. We have

$$\begin{cases} \lim\limits_{n\to\infty} \text{Prob } \left(C_{s,T}(n) \geq (1-\epsilon_1)(n-1-m)\pi T^2(n)\right) \\ \qquad\qquad = 1 - O(m/n^2) \\ \lim\limits_{n\to\infty} \text{Prob } \left(C_{s,T}(n) \leq (1+\epsilon_2)(n-1-m)\pi T^2(n)\right) \\ \qquad\qquad = 1 - O(1/n^{4/3}) \end{cases} \quad (3)$$

where, $\epsilon_1 = \sqrt{\frac{2^6 \log n}{n\pi c_1 T^2(n)}}$, $\epsilon_2 = \sqrt{\frac{4 \log n}{n\pi c_1 T^2(n)}}$, and $1/4 \leq c_1 \leq 1$. Thus for a constant number of sinks we have a tight bound on the cut-capacity

$$C_{s,T}(n) = \Theta\left(nT^2(n)\right) \quad (4)$$

In a single source network, the cut capacity is equal to the maximum flow. Thus the above theorem provides an upper bound on the multicast capacity of a network with single source and NC+MPT+MPR capability. However, in [15]–[17], the source, relays and destinations are strictly different and information can not be transmitted directly towards the destinations. These two assupmtions will be eventually relaxed in this paper.

IV. The Throughput Capacity with MPT and MPR

In this Section, we start to analyze the scaling law in random geometric graphs with MPT and MPR abilities. In [24], Wang et al. proved the unifying capacity with point-to-point communication, which resolves the general multicast case with m destinations for each source being a function of n. Here, we use the similar approach to prove the capacity with MPT and MPR when m is not a function of n but a constant.

A. Upper Bound

The following Lemma provides an upper bound for the per-session capacity in terms of the ratio of the size of $\overline{\#\text{MMMIS}(\Delta, T(n))}$ to the size of $\text{MEMT}(T(n))$. Essentially, $\overline{\#\text{MEMT}(T(n))}$ equals the minimum number of transmissions required to multicast a packet to m destinations, and $\overline{\#\text{MMMIS}(\Delta, T(n))}$ represents the maximum number of successful simultaneous transmissions when MPT and MPR are used.

Lemma 4.1: In random dense wireless ad hoc networks, the per-node throughput capacity of multicast with MPT and MPR is given by $O\left(\frac{1}{n} \times \frac{\overline{\#\text{MMMIS}(\Delta, T(n))}}{\overline{\#\text{MEMT}(T(n))}}\right)$.

Proof: We observe that $\overline{\#\text{MEMT}(T(n))}$ represents the total number of channel usage required to transmit information from a multicast source to all its m destinations. Denote by N_T the total number of multicast bits generated in $[0, T]$, then

$$nC_m(n) = \lim_{T\to\infty} \frac{N_T}{T}. \tag{5}$$

Since all multicast packets are received within a finite time $T_{\max}$, at time $T + T_{\max}$ all transmissions of N_T bits are finished. Therefore, with the definition of $\overline{\#\text{MMMIS}(\Delta, T(n))}$, we have

$$\overline{\#\text{MMMIS}(\Delta, T(n))}(T + T_{\max}) \geq N_T \overline{\#\text{MEMT}(T(n))}.$$

By combining the two previous equations we obtain

$$\begin{aligned} C_m(n) &= \frac{1}{n} \times \lim_{T\to\infty} \frac{N_T}{T} \\ &= \frac{1}{n} \times \lim_{T\to\infty} \frac{N_T}{T + T_{\max}} \\ &\leq \frac{1}{n} \times \frac{\overline{\#\text{MMMIS}(\Delta, T(n))}}{\overline{\#\text{MEMT}(T(n))}}, \end{aligned} \tag{6}$$

which proves the lemma. ■

Lemma 4.1 provides the upper bound for the multicast throughput capacity with MPT and MPR as a function of $\overline{\#\text{MMMIS}(\Delta, T(n))}$ and $\overline{\#\text{MEMT}(T(n))}$. We next compute the upper bound of $\overline{\#\text{MMMIS}(\Delta, T(n))}$ and the lower bound of $\overline{\#\text{MEMT}(T(n))}$. Combining these results provides an upper bound for the multicast throughput capacity with MPT and MPR.

Lemma 4.2: In multicast applications, the average number of nodes in $\text{MEMT}(T(n))$ has the following lower bound as a function of the transceiver range $T(n)$ when m is a constant:

$$\overline{\#\text{MEMT}(T(n))} \geq \Theta\left(\frac{1}{T(n)}\right) \tag{7}$$

Proof: Since the total number of destinations m is a constant, the total Euclidean length from the source s to all of the m destination is greater than 1 [24]. Since the transceiver range is $T(n)$ we need atleast $1/T(n)$ transmissions to cover the total length of the tree. ■

The next lemma states the upper bound for $\overline{\#\text{MMMIS}(\Delta, T(n))}$ with MPT and MPR.

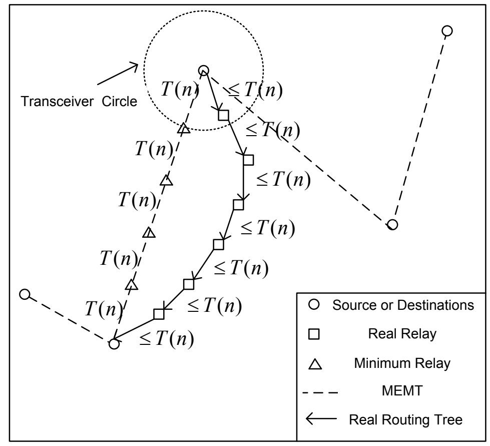

Fig. 2. The direct line between any two adjacent nodes in an multicast tree is equal to or smaller than the total Euclidean distance in the tree through multiple relays.

Lemma 4.3: The average number of channels that can transmit simultaneously, $\overline{\#\text{MMMIS}(\Delta, T(n))}$, has the following upper bound in networks with MPT and MPR.

$$\overline{\#\text{MMMIS}(\Delta, T(n))} \leq \Theta\left(n^2 T^2(n)\right) \tag{8}$$

Proof: We want to find out the maximum number of simultaneous transmissions in these dense networks. Thus, it is clear that, on average, there are $\pi T^2(n) n$ transmissions in one transceiver range $T(n)$ consuming an area of at least $\pi\left(T(n) + \frac{\Delta T(n)}{2}\right)^2$ in a dense network. Consider for each of the node in that area can transmit or receive at most $\pi n T^2(n)$ other transceivers. Using this argument, it is obvious that the upper bound of $\overline{\#\text{MMMIS}(\Delta, T(n))}$ is given by $\frac{n \times \pi n T^2(n)}{\left(1+\frac{\Delta}{2}\right)^2}$, which proves the lemma. ■

Combining Lemmas 4.1, 4.2, and 4.3, we can compute the upper bound of multicast capacity of MPT and MPR in the following theorem.

Theorem 4.4: In wireless ad hoc networks with MPT and MPR, the upper bound on the per-node throughput capacity of multicast with constant number of destinations is

$$C_m(n) = \Theta\left(n T^3(n)\right) \tag{9}$$

B. Lower Bound

To derive an achievable lower bound, we use a TDMA scheme for random dense wireless ad hoc networks similar to the approach used in [25], [26].

We first divide the network area into square cells. Each square cell has an area of $T^2(n)/2$, which makes the diagonal length of square equal to $T(n)$, as shown in Fig. 3. Under this condition, connectivity inside all cells is guaranteed and all nodes inside a cell are within communication range of each other. We build a cell graph over the cells that are occupied with at least one vertex (node). Two cells are connected if there exist a pair of nodes, one in each cell, that are less than or equal to $T(n)$ distance apart. Because the whole network is connected when $T(n) = r(n) \geq \Theta\left(\sqrt{\log n/n}\right)$, it follows that the cell graph is connected [25], [26].

To satisfy the MPT and MPR protocol model, we organize cells in groups so that simultaneous transmissions within each group does not violate the conditions for successful communication in the MPT and MPR protocol model. Let L represent the minimum number of cell separations in each group of cells that communicate simultaneously. Utilizing the protocol model, L satisfies the following condition:

$$L = \left\lceil 1 + \frac{T(n) + (1+\Delta)T(n)}{T(n)/\sqrt{2}} \right\rceil = \lceil 1 + \sqrt{2}(2+\Delta) \rceil \quad (10)$$

If we divide time into L^2 time slots and assign each time slot to a single group of cells, interference is avoided and the protocol model is satisfied. The separation example can be shown for the upper two receiver circles in Fig. 3. For the MPT and MPR protocol model, the distance between two adjacent receiving nodes is $(2+\Delta)T(n)$. Because this distance is smaller than $(L-1)T(n)$, this organization of cells guarantees that the MPT and MPR protocol model is satisfied. Fig. 3 represents one of these groups with a cross sign inside those cells for $L = 4$. We can derive an achievable multicast capacity for MPT and MPR by taking advantage of this cell arrangement and TDMA scheme. The capacity reduction caused by the TDMA scheme is a constant factor and does not change the order capacity of the network.

Next our objective is to find an achievable lower bound using the Chernoff bound, such that the distribution of the number of edges in this unit space is sharply concentrated around its mean, and hence the actual number of simultaneous transmissions occurring in the unit space in a randomly chosen network is indeed $\Theta(n^2T^2(n))$ w.h.p..

Lemma 4.5: The circular area of radius $T(n)$ corresponding to the transceiver range of any node j in the cross area in Fig. 3 contains $\Theta(nT^2(n))$ nodes w.h.p., and is uniformly distributed for all values of j, $1 \leq j \leq \frac{1}{(LT(n)/\sqrt{2})^2}$.

Proof: The statement of this lemma can be expressed as

$$\lim_{n\to\infty} P\left[\bigcap_{j=1}^{\frac{1}{(LT(n)/\sqrt{2})^2}} |N_j - E(N_j)| < \delta E(N_j)\right] = 1, \quad (11)$$

where N_j and $E(N_j)$ are the random variables that represent the number of transmitters in the receiver circle of radius $T(n)$ centered by the receiver j and the expected value of this random variable respectively, and δ is a positive arbitrarily small value close to zero.

From the Chernoff bound in Eq. (2), for any given $0 < \delta < 1$, we can find $\theta > 0$ such that $P\left[|N_j - E(N_j)| > \delta E(N_j)\right] < e^{-\theta E(N_j)}$. Thus, we can conclude that the probability that the value of the random variable N_j deviates by an arbitrarily small constant value from the mean tends to zero as $n \to \infty$. This is a key step in showing that when all the events $\bigcap_{j=1}^{\frac{1}{(LT(n)/\sqrt{2})^2}} |N_j - E(N_j)| < \delta E(N_j)$ occur simultaneously, then all N_j's converge uniformly to their expected values. Utilizing the union bound, we arrive at

$$\begin{aligned} & P\left[\bigcap_{j=1}^{\frac{1}{(LT(n)/\sqrt{2})^2}} |N_j - E(N_j)| < \delta E(N_j)\right] \\ = \;& 1 - P\left[\bigcup_{j=1}^{\frac{1}{(LT(n)/\sqrt{2})^2}} |N_j - E(N_j)| > \delta E(N_j)\right] \\ \geq \;& 1 - \sum_{j=1}^{\frac{1}{(LT(n)/\sqrt{2})^2}} P\left[|N_j - E(N_j)| > \delta E(N_j)\right] \\ > \;& 1 - \frac{1}{(LT(n)/\sqrt{2})^2} e^{-\theta E(N_j)}. \end{aligned} \quad (12)$$

Given that $E(N_j) = \pi n T^2(n)$, then we have

$$\begin{aligned} & \lim_{n\to\infty} P\left[\bigcap_{j=1}^{\frac{1}{(LT(n)/\sqrt{2})^2}} |N_j - E(N_j)| < \delta E(N_j)\right] \\ & \geq 1 - \lim_{n\to\infty} \frac{1}{(LT(n)/\sqrt{2})^2} e^{-\theta\pi n T^2(n)} \end{aligned} \quad (13)$$

Utilizing the connectivity criterion in Eq. (**??**), $\lim_{n\to\infty} \frac{e^{-\theta\pi n T^2(n)}}{T^2(n)} \to 0$, which finishes the proof. ■

Furthermore, we can arrange all of the nodes in the left side of the corresponding transceiver circle be the transmitters, and all of the nodes in the right side of the corresponding transceiver circle be the receivers. Thus, we can get the following lemma.

Lemma 4.6: In the unit square area for a wireless ad hoc network shown in Fig. 3, the total number of transmitter-receiver links (simultaneous transmissions) is $\Omega\left(n^2T^2(n)\right)$.

Proof: From the Lemma 4.5, for any node in the cross cell in the whole network shown in Fig. 3, there are $\Theta(nT^2(n)$ nodes in the transceiver circle. We divided the total node into two categories, transmitters in the left of the transceiver circles and receivers in the right of the transceiver circles. To guarantee all of the transmitters and receivers are in the transceiver range, we only consider the nodes in the circle with radius $T(n)/2$. Because of the MPT and MPR capabilities, so that every transmitter in the left of the transceiver circle with $T(n)/2$ radius can transmit successfully to every receiver in the right, then the total number of successful transmissions is $\pi^2n^2T^4(n)/16$ which is the achievable lower bound. The actual number of the transmissions can be much larger than this because we only consider $T(n)/2$ instead of $T(n)$. Using the Chernoff Bound in Eq. 2 and Lemma 4.5, we can get w.h.p. that the total number successful transmissions is

$$\Omega\left(\frac{1}{\left(LT(n)/\sqrt{2}\right)^2} \times \frac{\pi^2n^2T^4(n)}{16}\right) = \Omega\left(n^2T^2(n)\right) \quad (14)$$

■

With Lemmas **??** and 4.6, we have done the preparation for the following achievable lower bound.

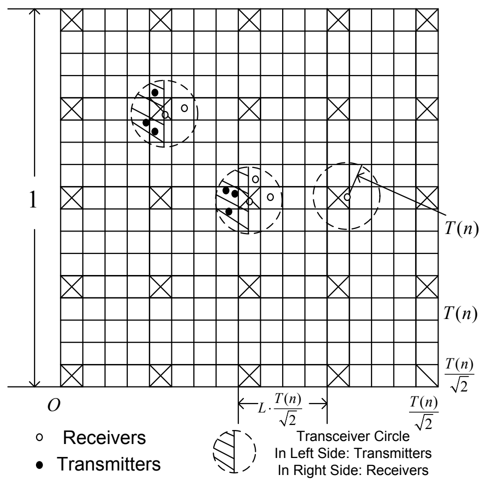

Fig. 3. Cell construction used to derive a lower bound on capacity

Let us define $\overline{\#\text{MEMTC}(T(n))}$ as the total number of cells that contain all the nodes in a multicast group. The following lemma establishes the achievable lower bound for the multicast throughput capacity of MPT and MPR as a function of $\overline{\#\text{MEMTC}(T(n))}$.

Lemma 4.7: The achievable lower bound of the multicast capacity is given by

$$C_m(n) = \Omega\left(\frac{nT^2(n)}{\overline{\#\text{MEMTC}(T(n))}}\right). \tag{15}$$

Proof: There are $(T(n)/\sqrt{2})^{-2}$ cells in the unit square network area. From the definition of $\overline{\#\text{MEMTC}(T(n))}$ and the fact that our TDMA scheme does not change the order capacity (Lemma **??**), it is clear that there are at most in the order of $\overline{\#\text{MEMTC}(T(n))}$ interfering cells for multicast communication. Hence, from Lemma 4.6, there are a total of $\Theta\left(n^2T^2(n)\right)$ nodes transmitting simultaneously, which are distributed over all the $(T(n)/\sqrt{2})^{-2}$ cells. For each cell, the order of nodes in each cell is $\Omega\left(n^2T^4(n)\right)$. Accordingly, the total lower bound capacity is given by $\Omega\left(\left(T(n)/\sqrt{2}\right)^{-2} \times \left(n^2T^4(n)\right) \times \left(\overline{\#\text{MEMTC}(T(n))}\right)^{-1}\right)$. Normalizing this value by total number of nodes in the network, n, proves the lemma. ■

Given the above lemma, to express the lower bound of $C_m(n)$ as a function of network parameters, we need to compute the upper bound of $\overline{\#\text{MEMTC}(T(n))}$, which we do next.

Lemma 4.8: The average number of cells covered by the nodes in MEMTC$(T(n))$, is upper bounded w.h.p. as follows:

$$\overline{\#\text{MEMTC}(T(n))} \leq \Theta\left(\frac{1}{T(n)}\right) \tag{16}$$

Proof: Because $T(n)$ is the transceiver range of the network, the maximum number of cells for this multicast tree must be at most $\Theta\left(mT^{-1}(n)\right)$, i.e., $\overline{\#\text{MEMTC}\left(T(n)\right)} \leq \Theta\left(T^{-1}(n)\right)$. This upper bound can be achieved only if every two adjacent nodes in the multicast tree belong to two different cells in the network. However, in practice, it is possible that some adjacent nodes in multicast tree locate in a single cell. Consequently, this value is upper bound which is Eq. (16). ■

Combining Lemmas 4.7 and 4.8, we arrive at the achievable lower bound of the multicast throughput capacity in dense random wireless ad hoc networks with MPT and MPR.

Theorem 4.9: When the number of the destinations m is a constant, the achievable lower bound of the m multicast throughput capacity with MPT and MPR is

$$C_m(n) = \Omega\left(nT^3(n)\right) \tag{17}$$

C. Tight Bound and Comparison with Point-to-Point Communication

From Theorems 4.4 and 4.9, we can provide the tight bound throughput capacity for the multicast when the node have MPT and MPR capability in dense random wireless ad hoc networks as follows.

Theorem 4.10: The throughput capacity of multicast in a random dense wireless ad hoc network with MPT and MPR is

$$C_m^{\text{MPT+MPR}}(n) = \Theta\left(nT^3(n)\right) \tag{18}$$

The transceiver range of MPT and MPR should satisfy $T(n) \geq \Theta\left(\sqrt{\log n/n}\right)$.

The throughput capacity with point-to-point communication can be get from [24] as following lemma.

Lemma 4.11: In multicast with a constant number m of destinations, without MPR or MPR ability, The capacity is

$$C_m^{\text{Routing}}(n) = \Theta\left(\frac{1}{nr(n)}\right) \tag{19}$$

where, $r(n) \geq \Theta\left(\sqrt{\log n/n}\right)$. When $r(n) = \Theta\left(\sqrt{\log n/n}\right)$ for the minimum transmission range to guarantee the connectivity, then we get the maximum capacity as $C_m^{\text{Routing-Max}}(n) = \Theta\left(\frac{1}{\sqrt{n\log n}}\right)$.

Combine the Theorem 4.13 and Lemma 4.14, the gain of throughput capacity with MPT and MPR capability in wireless ad hoc networks can be get as followings.

Theorem 4.12: In multicast with a constant number m of destinations, with MPT and MPR ability, the gain of per-node throughput capacity compared with point-to-point communication is $\Theta\left(n^2T^4(n)\right)$, where, $T(n) = r(n) \geq \Theta\left(\sqrt{\log n/n}\right)$. When $T(n) = \Theta\left(\sqrt{\log n/n}\right)$, the gain of per-node capacity is at least $\Theta\left(\log^2 n\right)$.

D. Tight Bound and Comparison with Point-to-Point Communication

From Theorems 4.4 and 4.9, we can provide the tight bound throughput capacity for the multicast when the node have MPT and MPR capability in dense random wireless ad hoc networks as follows.

Theorem 4.13: The throughput capacity of multicast in a random dense wireless ad hoc network with MPT and MPR is

$$C_m^{\text{MPT+MPR}}(n) = \Theta\left(nT^3(n)\right) \tag{20}$$

The transceiver range of MPT and MPR should satisfy $T(n) \geq \Theta\left(\sqrt{\log n/n}\right)$.

The throughput capacity with point-to-point communication can be get from [24] as following lemma.

Lemma 4.14: In multicast with a constant number m of destinations, without MPR or MPR ability, The capacity is

$$C_m^{\text{Routing}}(n) = \Theta\left(\frac{1}{nr(n)}\right) \tag{21}$$

where, $r(n) \geq \Theta\left(\sqrt{\log n/n}\right)$. When $r(n) = \Theta\left(\sqrt{\log n/n}\right)$ for the minimum transmission range to guarantee the connectivity, then we get the maximum capacity as $C_m^{\text{Routing-Max}}(n) = \Theta\left(\frac{1}{\sqrt{n\log n}}\right)$.

Combine the Theorem 4.13 and Lemma 4.14, the gain of throughput capacity with MPT and MPR capability in wireless ad hoc networks can be get as followings.

Theorem 4.15: In multicast with a constant number m of destinations, with MPT and MPR ability, the gain of per-node throughput capacity compared with point-to-point communication is $\Theta\left(n^2T^4(n)\right)$, where, $T(n) = r(n) \geq \Theta\left(\sqrt{\log n/n}\right)$. When $T(n) = \Theta\left(\sqrt{\log n/n}\right)$, the gain of per-node capacity is at least $\Theta\left(\log^2 n\right)$.

The multicast throughput capacity with MPT and MPR can be extended to 3-D easily.

V. Capacity with NC, MPT and MPR

In this section, we study network coding capacity for multiple sources and multiple destinations transmission in the absence of interference. This capacity serves as an upper-bound for that achieved by combining NC,MPT and MPR in the presence of interference. Consider the following scheme:

Scheme 5.1 (Hierarchical Scheme): Separate the routing in the network into n disjoint levels, each corresponding to a connectivity graph associated with a distinct multicast session. The hierarchical scheme distributes the time slots equally in activating each of the connectivity graphs. The scheme can be described by Fig. 4. The right of that figure is the model as [15]–[17]. S_i, $T_i = \{t_1^i, t_2^i, \cdots, t_m^i\}$ are the source and the m destinations in i-th level.

From Scheme 5.1, for each level, the network model is combined with NC, MPT and MPR, and we assume the nodes in the source set $\{s\}$, relay set D and destination set T can't be interchanged with each other. In the followings, we will prove that this effect won't affect the scaling law of the capacity.

Lemma 5.2: For one layer, the minimum number of links needed to transmit $C_{s,T}(n) = \Theta\left(nT^2(n)\right)$ information is $\Theta\left(nT(n)\right)$, when the number of the destination for each source m is a constant.

Proof: There should be $\Theta\left(nT^2(n)\right)$ different information units (or packets)which will be transmitted to m destinations

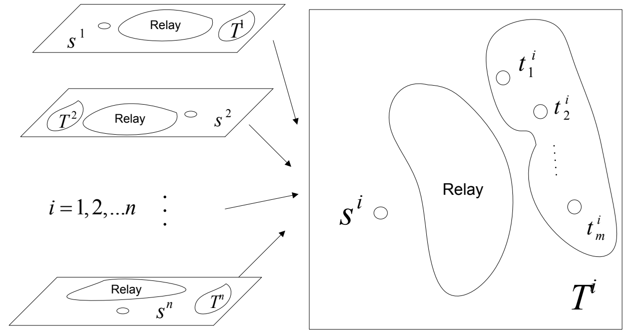

Fig. 4. Hierarchical transmissions scheme

$t_1^i, t_2^i, \cdots, t_m^i$ from the source s. Thus we require $\Theta\left(nT^2(n)\right)$ edge-disjoint paths from the source to each sink. Since the sinks are chosen at random, each information unit has to travel atleast $\Theta(1)$ distance. Furthurmore, since the transmission range is $T(n)$ each path consists of at least $\Theta\left(1/T(n)\right)$ edges with high probability. Therefore we should consume at least $\Theta\left(nT(n)\right)$ links to transmit $\Theta\left(nT^2(n)\right)$ information units. ■

Now, let's consider the whole networks and use our Scheme 5.1 to analyze, then we can get our result as followings.

Theorem 5.3: In a wireless ad hoc networks with NC plus MPT and MPR ability, the throughput capacity per node is upper bounded by $\Theta\left(nT^3(n)\right)$.

Proof: From the Lemma **??**, the total possible number of links from any source/relay to any relay/destinations is upper bounded by $\Theta\left(n^2T^2(n)\right)$. Combining Theorem 3.10 and Lemma 5.2, we can get the capacity for the whole networks is upper bounded by the capacity for one level times the links available for the total networks, which gets the $\frac{\Theta\left(nT^2(n)\right)\times\Theta\left(n^2T^2(n)\right)}{\Theta\left(nT(n)\right)}$ for total capacity, which proved this theorem by dividing n for per node capacity. ■

VI. Discussion

From the above analysis, we can get some implications for the design of protocols in wireless ad hoc networks. Combining our result Theorem 4.13 in Section IV and Theorem 5.3 in Section V, we can get the final result which is the primary contribution of this paper.

Theorem 6.1: In wireless ad hoc networks, with multicast application with constant m destinations for each source, the throughput capacity when all of the nodes are endowed with NC, MPT and MPR capabilities is the same order as the one when all of the nodes are endowed with only MPT and MPR abilities.

$$C_m^{\text{MPT+MPR+NC}}(n) = C_m^{\text{MPT+MPR}}(n) \tag{22}$$

The result of this paper implies that, in the wireless ad hoc networks, the interference is the real bottleneck. The reason MPT and MPR increase the capacity is because they do combat the interference by allowing multiple transmission at a time. Once the interference is canceled NC does not add anything to the capacity and in an order sense behaves identical to traditional routing. Since, NC adds significantly to the complexity of the network, we believe that future designs of wireless networks need not deviate from using traditional routing schemes and rather concentrate on realizing interference cancelation that enables MPT + MPR.

REFERENCES

[1] P. Gupta and P. R. Kumar, "The capacity of wireless networks," *IEEE Transactions on Information Theory*, vol. 46, no. 2, pp. 388–404, 2000.

[2] P. Kyasanur and N. Vaidya, "Capacity of multi-channel wireless networks: Impact of number of channels and interfaces," in *Proc. of ACM MobiCom 2005*, Cologne, Germany, August 28-September 2 2005.

[3] R. Negi and A. Rajeswaran, "Capacity of power constrained ad-hoc networks," in *Proc. of IEEE INFOCOM 2004*, Hong Kong, March 7-11 2004.

[4] H. Zhang and J. Hou, "Capacity of wireless ad-hoc networks under ultra wide band with power constraint," in *Proc. of IEEE INFOCOM 2005*, Miami, Florida, USA., March 13-17 2005.

[5] A. Ozgur, O. Leveque, and D. Tse, "Hierarchical cooperation achieves optimal capacity scaling in ad hoc networks," *IEEE Transactions on Information Theory*, vol. 53, no. 10, pp. 2549–3572, 2007.

[6] R. M. de Moraes, H. R. Sadjadpour, and J. J. Garcia-Luna-Aceves, "Many-to-many communication: A new approach for collaboration in manets," in *Proc. of IEEE INFOCOM 2007*, Anchorage, Alaska, USA., May 6-12 2007.

[7] M. Grossglauser and D. Tse, "Mobility increases the capacity of ad hoc wireless networks," *IEEE/ACM Transactions on Networking*, vol. 10, no. 4, pp. 477–486, 2002.

[8] A. El Gamal, J. Mammen, B. Prabhakar, and D. Shah, "Optimal throughput-delay scaling in wireless networks part i: The fluid model," *IEEE Transactions on Information Theory*, vol. 52, no. 6, pp. 2568–2592, 2007.

[9] R. Ahlswede, C. Ning, S.-Y. R. Li, and R. W. Yeung, "Network information flow," *IEEE Transactions on Information Theory*, vol. 46, no. 4, pp. 1204–1216, 2000.

[10] S.-Y. R. Li, R. W. Yeung, and N. Cai, "Linear network coding," *IEEE Transactions on Information Theory*, vol. 49, no. 2, pp. 371–381, 2003.

[11] R. Koetter and M. Medard, "An algebraic approach to network coding," *IEEE/ACM Transactions on Networking*, vol. 11, no. 5, pp. 782–795, 2003.

[12] J. Liu, D. Goeckel, and D. Towsley, "Bounds on the gain of network coding and broadcasting in wireless networks," in *Proc. of IEEE INFOCOM 2007*, Anchorage, Alaska, USA., May 6-12 2007.

[13] S. Katti, S. Gollakota, and D. Katabi, "Embracing wireless interference: Analog network coding," in *Proc. of ACM SIGCOMM 2007*, Kyoto, Japan, August 27-31 2007.

[14] S. Zhang, S. Liew, and P. P. Lam, "Hot topic: Physical-layer network coding," in *Proc. of ACM MobiCom 2006*, Los Angeles, California, USA., September 23-29 2006.

[15] A. Ramamoorthy, J. Shi, and R. Wesel, "On the capacity of network coding for random networks," *IEEE Transactions on Information Theory*, vol. 51, no. 8, pp. 2878–2885, 2005.

[16] S. A. Aly, V. Kapoor, J. Meng, and A. Klappenecker, "Bounds on the network coding capacity for wireless random networks," in *Proc. of Third Workshop on Network Coding, Theory, and Applications (NetCod 2007*, San Diego, California, USA, January 29 2007.

[17] Z. Kong, S. A. Aly, E. Soljanin, E. M. Yeh, and A. Klappenecker, "Network coding capacity of random wireless networks under a signal-to-interference-and noise model," *Submitted to IEEE Transactions on Information Theory*, 2007.

[18] S. Toumpis and A. J. Goldsmith, "Capacity regions for wireless ad hoc networks," *IEEE Transactions on Wireless Communications*, vol. 2, no. 4, pp. 736–748, 2003.

[19] J. J. Garcia-Luna-Aceves, H. R. Sadjadpour, and Z. Wang, "Challenges: Towards truly scalable ad hoc networks," in *Proc. of ACM MobiCom 2007*, Montreal, Quebec, Canada, September 9-14 2007.

[20] Z. Li and B. Li, "Network coding in undirected networks," in *Proc. of CISS 2004*, Princeton, NJ, USA., March 17-19 2004.

[21] Z. Li, B. Li, and L. Lau, "On achieving maximum multicast throughput in undirected networks," *IEEE/ACM Transactions on Special issue on Networking and Information theory*, vol. 52, pp. 2467–2485, 2006.

[22] A. Keshavarz, V. Ribeiro, and R. Riedi, "Broadcast capacity in multihop wireless networks," in *Proc. of ACM MobiCom 2006*, Los Angeles, California, USA., September 23-29 2006.

[23] R. Motwani and P. Raghavan, *Randomized Algorithms*. Cambridge University Press, 1995.

[24] Z. Wang, H. R. Sadjadpour, and J. J. Garcia-Luna-Aceves, "A unifying perspective on the capacity of wireless ad hoc networks," in *IEEE INFOCOM 2008*, Phoenix, Arizona, USA., April 13-18 2008.

[25] A. Giridhar and P. R. Kumar, "Computing and communicating functions over sensor networks," *IEEE Journal on Selected Areas in Communications*, vol. 23, no. 4, pp. 755–764, 2005.

[26] S. Kulkarni and P. Viswanath, "A deterministic approach to throughput scaling wireless networks," *IEEE Transactions on Information Theory*, vol. 50, no. 6, pp. 1041–1049, 2004.