Construction of OFDM M-QAM Sequences With Low Peak-to-Average Power Ratio

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***Abstract*—We present a technique to derive M-quadrature amplitude modulation (QAM) signals from quaternary phase-shift keying (QPSK) constellations when $M = 2^n$ and n is an even number. By utilizing QPSK Golay sequences, we have constructed M-QAM sequences with low peak-to-mean envelope power ratios. Several upper bounds for these M-QAM sequences were derived.**

***Index Terms*—Golay, multicarrier, orthogonal frequency-division multiplexing (OFDM), peak-to-mean envelope power ratio (PMEPR).**

I. Introduction

ORTHOGONAL frequency-division multiplexing (OFDM) [1] is a multicarrier modulation technique that has been adopted for many types of applications in wireless systems, such as wireless local-area networks [2] and digital video broadcasting [3]. One significant problem with OFDM systems is that when the channel is divided into N subchannels, the peak-to-average power ratio can be as high as N. This is a major constraint on the analog part of the modem. In some other applications, the peak envelope power has some restrictions due to regulatory or design constraints. Effectively, this characteristic of OFDM systems forces us to reduce the mean envelope power. Lowering the mean envelope power causes many limitations on the OFDM system and does not allow us to utilize the transmit power efficiently.

This is the motivation behind a lot of work to reduce the peak-to-mean envelope power ratio (PMEPR) of the signal in OFDM systems. In one approach [4], the authors recommended the use of block coding to transmit sequences with small PMEPR. These approaches consider phase-shift keying (PSK) signal constellations. However, there are many OFDM systems that utilize M-quadrature amplitude modulation (QAM) constellations. Recently, there was an approach to generalize these codes to a 16-QAM constellation [6]. We have built our approach based on this work to find the general solution for M-QAM $(M = 2^n)$ signal constellations when n has even values. The solution for odd values of n can be found in [7].

The paper is structured as follows. Section II briefly describes the mathematical model for OFDM systems. In Section III, we construct M-QAM signal constellations as the vector sum of quaternary phase-shift keying (QPSK) constellations. Section IV provides the upper bounds on the peak envelope power and PMEPR of these sequences.

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II. Problem Statements

If the M-QAM symbol assigned to the ith carrier of OFDM symbol is defined as $a_i, i = 0, 1, \cdots, N-1$, then the transmitted signal is represented as

$$S_a(t) = \sum_{i=0}^{N-1} a_i \exp(2\pi j\,(f_o + i f_s)\,t) \tag{1}$$

where f_o is the carrier frequency and f_s is the bandwidth of each subchannel. At the receiver, the inverse operation is conducted on the received data that contains the signal plus noise. The details are beyond the scope of this paper and can be found in [1] and [8].

The vector $\mathbf{a}$ represents the codeword of N symbols, i.e., $\mathbf{a} = (a_0, a_1, \ldots, a_{N-1})$, C the ensemble of all possible codewords ($\mathbf{a} \in C$), and $||\mathbf{a}||^2$ the power associated with each codeword $\mathbf{a}$. Let $p(\mathbf{a})$ denote the probability of codeword $\mathbf{a}$ being transmitted, then the mean envelope power of the transmitted signal is defined as

$$P_{\text{av}} = \sum_{a \in C} ||\mathbf{a}||^2 p(\mathbf{a}). \tag{2}$$

If the instantaneous envelope power is $P(t) = |S_a(t)|^2$, then the PMEPR of the codeword $\mathbf{a}$ is

$$\text{PMEPR}(\mathbf{a}) = \frac{max|S_a(t)|^2}{P_{\text{av}}}. \tag{3}$$

The maximization is during one OFDM symbol period. Our objective is to design codes C with small PMEPR. It can be seen [5] that the peak envelope power is upper bounded as N^2. The mean envelope power over one symbol period is N. Therefore, the PMEPR is equal to N for a sequence without any autocorrelation properties.

III. Construction of M-QAM Signals From QPSK Constellations

The QPSK constellation can be realized as QPSK $= j^{x_i}$ where $x_i \in Z_4 = \{0, 1, 2, 3\}$. Thus, any QPSK sequence $\mathbf{a} = (a_0 a_1 \cdots a_{N-1})$ can be associated with another (unique) sequence $\mathbf{x}_i = (x_i^0 x_i^1 \cdots x_i^{N-1})$ where the elements of $\mathbf{x}_i$ are in Z_4. One can use shift and rotation operation to create M-QAM constellations from QPSK symbols. Fig. 1 shows this procedure

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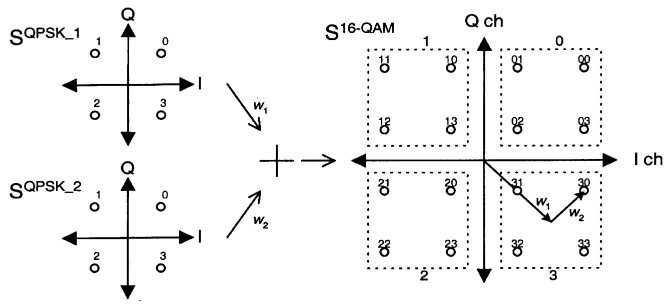

Fig. 1. Construction of a 16-QAM symbol by adding two QPSK symbols.

for a 16-QAM constellation and (4) describes the equivalent operation for general M-QAM symbols.

$$M-\mathrm{QAM}=\sum_{i_i=0}^{\frac{n}{2}-1}(2^{i_1})\left(\frac{\sqrt{2}}{2}\right)(j^{x_{i_1}})\exp\left(\frac{\pi j}{4}\right). \qquad (4)$$

Thus, any point of the general $M-QAM$ constellation at time k can be written as

$$\exp\left(\frac{\pi j}{4}\right)\left[\left(\frac{\sqrt{2}}{2}\right)\left(j^{x_0^k}\right)+(\sqrt{2})\left(j^{x_1^k}\right)+\cdots + 2^{\frac{n}{2}-1}\left(\frac{\sqrt{2}}{2}\right)\left(j^{x^k_{\frac{n}{2}-1}}\right)\right] \qquad (5)$$

for $x_0^k, x_1^k, x_2^k, \cdots, x^k_{(n/2)-1} \in Z_4$. In this way, one can associate with any M-QAM sequence $\mathbf{a} = a_0a_1\cdots a_{N-1}$ a unique sequence $x_0^0x_1^0\cdots x^0_{(n/2)-1}$, $x_0^1x_1^1\cdots x^1_{(n/2)-1}$, $x_0^{N-1}x_1^{N-1}\cdots x^{N-1}_{(n/2)-1} \in \underbrace{Z_4^N\times Z_4^N\times\cdots\times Z_4^N}_{n/2}$. In particular, the signal $\mathcal{S}_{\mathbf{a}}(t)$ can be written as

$$S_{\mathbf{x}_0,\mathbf{x}_1,\cdots,\mathbf{x}_{\frac{n}{2}-1}}(t)=\sum_{k=0}^{N-1}\sum_{i_1=0}^{\frac{n}{2}-1}\left(2^{i_1}\right)\left(\frac{\sqrt{2}}{2}\right)j^{x^k_{i_1}} \times \exp\left(2\pi j\left(f_0+kf_s\right)t+\frac{\pi j}{4}\right). \qquad (6)$$

Simplifying the above, the instantaneous envelope power is given by

$$P_{\mathbf{x}_0,\mathbf{x}_1,\cdots,\mathbf{x}_{\frac{n}{2}-1}}(t) =\left|\sum_{k=0}^{N-1}\sum_{i_1=0}^{\frac{n}{2}-1}\left(2^{i_1}\right)\left(\frac{\sqrt{2}}{2}\right)j^{\mathbf{x}^k_{i_1}}\exp\left(2\pi jkf_st\right)\right|^2. \qquad (7)$$

In other words

$$P_{\mathbf{x}_0,\mathbf{x},\cdots,\mathbf{x}_{\frac{n}{2}-1}}(t)=\left|\sum_{i=0}^{\frac{n}{2}-1}2^i\left(\frac{\sqrt{2}}{2}\right)S_{\mathbf{x}_i}(t)\right|^2 \qquad (8)$$

where $\mathcal{S}_{\mathbf{x}_i}(t)$ is defined as

$$S_{\mathbf{x}_i}(t)=\sum_{k=0}^{N-1}j^{x_i^k}\exp\left(2\pi j(f_0+kf_s)t+\frac{j\pi}{4}\right). \qquad (9)$$

We will use the above results to compute PMEPR bounds when the M-QAM sequence is constructed based on QPSK Golay sequences. Golay sequences were first introduced in [9].

Reed–Muller codes can be defined in terms of Boolean functions. These codes provide good error correction properties as long as the block length is not too large [10]. Their minimum distance is lower than that of Bose–Chaudhuri–Hocquengem (BCH) codes. The decoding of Reed–Muller codes are relatively simple using majority logic circuits. The rth-order binary Reed–Muller code of length 2^m RM(r,m) is constructed by the monomials in the Boolean function of degree r or less. For the binary data, Golay sequences are cosets of the first-order Reed–Muller code within the second-order Reed–Muller code. This was first introduced in [5]. For general nonbinary signals ($M=2^n$), these can be generalized for Reed–Muller codes [5] which is based on the results in [11].

IV. Computation of PMEPR Upper Bounds for M-QAM Sequences Constructed With QPSK Golay Sequences

For the QPSK constellation, it is proved that the PMEPR of a single Golay sequence is, at most, two [5]. For the M-QAM constellation, an analogous result is given below.

Theorem IV.1: For any sequence $\mathbf{x} \in \mathbf{Z_4^N}$, let $\mathbf{z}=\mathbf{x}+\mathbf{2}$ denote the sequence given by $z_i=x_i+2$ for $i=0,1,\cdots,N-1$. Then,

- **I.** If $\mathbf{x}$ is a Golay sequence, then $P_{\underbrace{\mathbf{x}+2,\cdots,\mathbf{x}+2}_{n/2-1},\mathbf{x}}(t) \le N$.
- **II.** If $\mathbf{x}_{(n/2)-1}$ and $\mathbf{x}_{(n/2)-2}$ form a Golay complementary pair and $\mathbf{x}_0,\cdots,\mathbf{x}_{(n/2)-3}$ are Golay sequences and not necessarily Golay complementary pairs, then $P_{\mathbf{x}_0,\mathbf{x}_1,\cdots,\mathbf{x}_{(n/2)-1}}(t) \le (\sqrt{5\times 2^{n-4}}+(2^{(n/2)-2}-1))^2N$.
- **III.** If $\mathbf{x}_0,\cdots,\mathbf{x}_{(n/2)-1}$ are Golay sequences and not necessarily Golay complementary pairs, then $P_{\mathbf{x}_0,\mathbf{x}_1,\cdots,\mathbf{x}_{(n/2)-1}}(t) \le (2^{n/2}-1)^2N$.

Proof: Recall from (8) that

$$P_{\underbrace{\mathbf{x}+2,\cdots,\mathbf{x}+2}_{\frac{n}{2}-1},\mathbf{x}}(t) =\left|2^{\frac{n}{2}-1}\left(\frac{\sqrt{2}}{2}\right)S_{\mathbf{x}}(t)+\sum_{i=0}^{\frac{n}{2}-2}2^{i_1}\left(\frac{\sqrt{2}}{2}\right)S_{\mathbf{x}+2}(t)\right|^2. \qquad (10)$$

By direct computation

$$S_{\mathbf{x}+2}(t)=-S_{\mathbf{x}}(t).$$

thus

$$P_{\underbrace{\mathbf{x}+2,\cdots,\mathbf{x}+2}_{\frac{n}{2}-1},\mathbf{x}}(t) =\left|2^{\frac{n}{2}-1}\left(\frac{\sqrt{2}}{2}\right)S_{\mathbf{x}}(t)+\sum_{i=0}^{\frac{n}{2}-2}2^{i_1}\times\left(\frac{\sqrt{2}}{2}\right)(-S_{\mathbf{x}}(t))\right|^2 =\left|\frac{\sqrt{2}}{2}S_{\mathbf{x}}(t)\right|^2. \qquad (11)$$

We now can use inequality $P_{\mathbf{a}}(t) = |S_{\mathbf{a}}(t)|^2 \le 2N$ to conclude that

$$P_{\underbrace{\mathbf{x}+2,\cdots,\mathbf{x}+2}_{\frac{n}{2}-1},\mathbf{x}}(t) \le N \tag{12}$$

which proves **I**.

To prove **II**, suppose that $\mathbf{x}_{(n/2)-2}$ and $\mathbf{x}_{(n/2)-1}$ are a Golay complementary pair. It follows from the proof of **I** that

$$S_{\mathbf{x}_{\frac{n}{2}-2}+2}(t) = -S_{\mathbf{x}_{\frac{n}{2}-2}}(t).$$

Furthermore, by (8) we have

$$\begin{aligned} &P_{\mathbf{x}_0,\mathbf{x}_1,\cdots,\mathbf{x}_{\frac{n}{2}-2},\mathbf{x}_{\frac{n}{2}-1}}(t) \\ &= \left|\sum_{i_1=0}^{\frac{n}{2}-1} 2^{i_1}\left(\frac{\sqrt{2}}{2}\right) S_{\mathbf{x}_{i_1}}(t)\right|^2 \\ &= \left|2^{\frac{n}{2}-2}\left(\frac{\sqrt{2}}{2}\right) S_{\mathbf{x}_{\frac{n}{2}-2}}(t) + 2^{\frac{n}{2}-1}\left(\frac{\sqrt{2}}{2}\right) S_{\mathbf{x}_{\frac{n}{2}-1}}(t)\right. \\ &\left. + \sum_{i_1=0}^{\frac{n}{2}-3} 2^{i_1}\left(\frac{\sqrt{2}}{2}\right) S_{\mathbf{x}_{i_1}}(t)\right|^2 \\ &= \left|2^{\frac{n}{2}-2}\left(\frac{\sqrt{2}}{2}\right) C + 2^{\frac{n}{2}-1}\left(\frac{\sqrt{2}}{2}\right) B + A\right|^2 \\ &\le \left(\left|2^{\frac{n}{2}-1}\left(\frac{\sqrt{2}}{2}\right) B + 2^{\frac{n}{2}-2}\left(\frac{\sqrt{2}}{2}\right) C\right| + |A|\right)^2 \end{aligned} \tag{13}$$

where $A = \sum_{i_1=0}^{(n/2)-3} 2^{i_1}(\sqrt{2}/2) S_{\mathbf{x}_{i_1}}(t)$, $B = S_{\mathbf{x}_{(n/2)-1}}(t)$, and $C = S_{\mathbf{x}_{(n/2)-2}}(t)$. We define the following positive real functions:

$$F1(B,C) = \left|2^{\frac{n}{2}-1}\left(\frac{\sqrt{2}}{2}\right) B + 2^{\frac{n}{2}-2}\left(\frac{\sqrt{2}}{2}\right) C\right|^2 \tag{14}$$

$$F2(B,C) = \left|2^{\frac{n}{2}-2}\left(\frac{\sqrt{2}}{2}\right) B - 2^{\frac{n}{2}-1}\left(\frac{\sqrt{2}}{2}\right) C\right|^2. \tag{15}$$

Combining the above, we arrive at

$$\begin{aligned} F1(B,C)+F2(B,C) &= \left(\frac{1}{2}2^{n-2} + \frac{1}{2}2^{n-4}\right) \times (|B|^2 + |C|^2) \\ &= 5N2^{n-4}. \end{aligned} \tag{16}$$

We used the fact that $S_{\mathbf{x}_{(n/2)-1}}(t)$ and $S_{\mathbf{x}_{(n/2)-2}}(t)$ are Golay complementary pairs. Since both $F1$ and $F2$ are real positive numbers, therefore

$$\begin{aligned} \sqrt{F1(B,C)} &= \left|2^{\frac{n}{2}-1}\left(\frac{\sqrt{2}}{2}\right) B + 2^{\frac{n}{2}-2}\left(\frac{\sqrt{2}}{2}\right) C\right| \\ &\le \sqrt{5N2^{n-4}}. \end{aligned} \tag{17}$$

All the elements in A are Golay sequences.

$$\begin{aligned} |A| &= \left|\sum_{i_1=0}^{\frac{n}{2}-3} 2^{i_1}\left(\frac{\sqrt{2}}{2}\right) S_{\mathbf{x}_{i_1}}(t)\right| \\ &\le \sqrt{2N}\left|\sum_{i_1=0}^{\frac{n}{2}-3} 2^{i_1}\left(\frac{\sqrt{2}}{2}\right)\right| \\ &= (2^{\frac{n}{2}-2} - 1)\sqrt{N}. \end{aligned} \tag{18}$$

Combining (18) and (17) with (13) will prove **II**

$$P_{\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_{\frac{n}{2}}}(t) \le \left(\sqrt{5\times 2^{n-4}} + \left(2^{\frac{n}{2}-2} - 1\right)\right)^2 N. \tag{19}$$

To prove **III**, we recall from (8) that

$$P_{\mathbf{x}_0,\mathbf{x}_1,\cdots,\mathbf{x}_{\frac{n}{2}-1}}(t) = \left|\sum_{i_1=0}^{\frac{n}{2}-1} 2^{i_1}\left(\frac{\sqrt{2}}{2}\right) S_{\mathbf{x}_{i_1}}(t)\right|^2.$$

By the triangle inequality, we have

$$\begin{aligned} &\left|2^{\frac{n}{2}-1}\left(\frac{\sqrt{2}}{2}\right) S_{\mathbf{x}_{\frac{n}{2}-1}}(t) + \cdots + \left(\frac{\sqrt{2}}{2}\right) S_{\mathbf{x}_0}(t)\right| \\ &\quad \le \left|2^{\frac{n}{2}-1}\left(\frac{\sqrt{2}}{2}\right) S_{\mathbf{x}_{\frac{n}{2}-1}}(t)\right| + \cdots + \left|\left(\frac{\sqrt{2}}{2}\right) S_{\mathbf{x}_0}(t)\right|. \end{aligned}$$

Considering the upper bound for $|S_{\mathbf{x}}(t)|^2$ is $2N$, we have

$$\begin{aligned} P_{\mathbf{x}_0,\mathbf{x}_1,\cdots,\mathbf{x}_{\frac{n}{2}-1}}(t) &\le \left|\sum_{i_1=0}^{\frac{n}{2}-1} 2^{i_1}(\sqrt{2N})\left(\frac{\sqrt{2}}{2}\right)\right|^2 \\ &= (2^{\frac{n}{2}} - 1)^2 N. \end{aligned} \tag{20}$$

This completes the proof.

In order to derive an upper bound on the PMEPR of M-QAM Golay sequences constructed above, we prove the following lemma.

Lemma IV.1: Let $\mathbf{x}_i^k$'s be independent sequences of length N and each element of them are equiprobable, such that $E(S_{\mathbf{x}_i}(t)S_{\mathbf{x}j}(t)) = 0$. Then the mean envelope power is $P_{\text{av}} = N/2 \times (2^{n/2} - 1)$.

Proof:

$$\begin{aligned} P_{\text{av}} &= E\left|\sum_{i=0}^{\frac{n}{2}-1} 2^{i}\left(\frac{\sqrt{2}}{2}\right) S_{\mathbf{x}_i}(t)\right|^2 \\ &= \sum_{i=0}^{\frac{n}{2}-1}\left|2^{i}\left(\frac{\sqrt{2}}{2}\right)\right|^2 E|S_{\mathbf{x}_i}(t)|^2 = \frac{N}{6}\left(2^n - 1\right). \end{aligned} \tag{21}$$

The second equality is based on the assumption that $S_{\mathbf{x}_i}(t)$ are independent sequences since they are based on statistically independent uncoded data, and for QPSK sequences, $E|S_{\mathbf{x}_i}(t)|^2 = N$.

Theorem IV.2: Let $A \subseteq Z_4^N$ be the set of Golay sequences. Then the PMEPR for $Z := A^1 \times \cdots \times A^{(n/2)-1}$ is bounded by $6(2^{n/2}-1)^2/2^n-1$, provided that $A^1 \times \cdots \times A^{(n/2)-1}$ is used for equiprobable M-QAM OFDM transmission.

Proof: By *Lemma IV.1*, the transmission requires a mean envelope power $P_{\rm av} = \frac{N}{6}(2^n-1)$. By **III** of *Theorem IV.1*, the peak transmitted envelope power is bounded above by $(2^{n/2}-1)^2N$. Thus, the PMEPR is bounded above by $6(2^{n/2}-1)^2/(2^n-1)$.

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